Department of Physics, UCSD
Physics 225B, General Relativity
Winter 2015
Homework 4, solutions

1. (i) and (ii) at once. Recall from our discussion of SBH geodesics,

$$\ddot{r} = -V'(r) = -\frac{d}{dr}\left[\frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}\right]$$

Using $r = 1/u$ and $\dot{u} = u' \cdot \frac{d\phi}{dt} = Lu^2u'$, we have

$$-L^2u^2u'' = -\left[GMu^2 - L^2u^3 + 3GML^2u^4\right]$$

or, dividing through by $-L^2u^2$,

$$u'' + u = \frac{GM}{L^2} + 3GMu^2.$$

Neglecting the last term on the right we get the result for part (i). A particular solution is $u = GM/L^2$ to which we add a solution to the homogeneous equation $u'' + u = 0$ which is the harmonic oscillator. So we have the general solution $u = GM/L^2(1 + e\cos(\phi - \phi_0))$ in terms of two constants of integration, e and ϕ_0. Clearly ϕ_0 is trivial (we can shift ϕ by a constant to define where we count from) and we have then

$$r(\phi) = \frac{\frac{L^2}{GM}}{1 + e\cos\phi} = \frac{a(1-e^2)}{1 + e\cos\phi}$$

where a is the semi-major axis of an ellipse with eccentricity e. The *semi-latus rectum*, $a(1-e^2)$ is L^2/GM.

In part (ii) we estimate the size of the relativistic term relative the Newtonian term,

$$\frac{3GMu^2}{\frac{GM}{L^2}} = 3\frac{L^2}{r^2} = 3\left(\frac{r\dot{\phi}}{c}\right)^2$$

where I have used unit mass and restored units in the last step by inserting appropriate powers of speed of light. Then in MKS, for mercury we can use $r \lesssim 10^{11}$ m, and $\dot{\phi} = 2\pi/T$ with orbital period $T \approx \frac{1}{4}$ yr $\approx \frac{1}{4}\pi \times 10^7$ s. Plugging in I obtain less than 10^{-6} for the ratio above.

(iii) We approximate the solution by $u = u^{(0)} + u^{(1)}$ where $u^{(0)}$ is the solution of the unperturbed equation:

$$u = GM/L^2(1 + e\cos\phi) + f(\phi)$$

and $f = u^{(1)}$ satisfies

$$f'' + f = \kappa(1 + e\cos\phi)^2 \quad \text{where} \quad \kappa = \frac{3(GM)^3}{L^4}$$

Being a bit lazy I put into Mathematica and obtain

$$f = \kappa + \tfrac{1}{2}\kappa e^2 + \kappa e\phi\sin\phi - \tfrac{1}{6}\kappa e^2\cos(2\phi) + C_1\cos\phi + C_2\sin\phi$$

with $C_{1,2}$ constants of integration (and of course, this added to the above gives the solution for $u(\phi)$ to this order). Note that C_1 can be absorbed into e of the unperturbed solution. This changes the meaning of e in the perturbation by something that is of higher order than what we are retaining, so we are free to set $C_1 = 0$. Similarly, the C_2 term just rotates the orientation of the major axes and changes slightly the eccentricity, so we may also set it to zero. Finally, the constant terms just shift the constant in the unperturbed solution.

(iv) Perihelion (Min distance to focal point) of the unperturbed trajectory is at $\phi = 0$. We look for maxima of the function

$$u = GM/L^2(1 + e\cos\phi) + \kappa e\phi\sin\phi - \tfrac{1}{6}\kappa e^2\cos(2\phi)$$

Taking one derivative we find a solution (to lowest order) at $\phi = 0$ and another at

$$-GM/L^2 e\phi + 2\pi\kappa e = 0$$

So the perihelion has shifted in one revolution by

$$\Delta\phi = \frac{2\pi\kappa e}{eGM/L^2} = 6\pi\frac{(GM)^2}{L^2} = 3\pi(2GM)\frac{1}{a(1-e^2)}$$

Numerics (in SI/MKS units):

$$\Delta\phi = \frac{6\pi(6.7\times10^{-11})(2.0\times10^{30})}{(5.8\times10^{10}(1-(0.21)^2)c^2} = 5.1\times10^{-7}\ \text{rad}$$

To change this into per century we have to multiply the number of revolutions in a century, and since the orbital period is 0.24 yr, we have

$$\text{revs per century} = 100/0.24 = 417$$

So the shift per century is 2.1×10^{-4} radians, or (times $360\times60^2/2\pi$) 43 arcsec per century.

2. From class we have that for a null geodesic in the SBH metric

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\left(1 - \frac{2GM}{r^2}\right)\frac{L^2}{r^2} = \frac{1}{2}E^2$$

where $L = r^2 d\phi/d\lambda$. As in the previous problem introduce $u = 1/r$ and $dr/d\lambda = -u^{-2}u' d\phi/d\lambda = -Lu'$, so that

$$(Lu')^2 + L^2u^2(1 - 2GMu) = E^2$$

It is convenient to differentiate this, to put it in the same form as in the previous problem (you could of course just integrate the above):

$$u'' + u = 3GMu^2$$

The right hand side is a small quantity for the sun:

$$\frac{3GMu^2}{u} \approx \frac{GM}{R_{\text{sun}}} \sim 10^{-6}$$

Neglecting the right hand side we have a straight path,

$$u^{(0)} = b^{-1}\sin\phi$$

where b is the impact parameter. Using this as the zeroth order approximation in teh perturbative solution gives

$$f'' + f = \frac{3GM}{b^2}\sin^2\phi$$

which has a solution $f = (3GM/2b^2)(1 + 1/3\cos 2\phi)$ giving the solution

$$u = b^{-1}\sin\phi + \frac{3GM}{b^2}\left(1 + \frac{1}{3}\cos 2\phi\right)$$

For large r (small u) ϕ is small modulo π. Taking the small solution, as $u \to 0$ we have $\phi \to \phi_{\text{in}}$ with

$$\phi_{\text{in}} = -\frac{2GM}{b}$$

and the total deflection Δ is twice (the absolute value of) this,

$$\Delta = \frac{4GM}{b}$$

Closest approach is at $\phi = \pi/2$ or

$$1/r_0 = u_0 = \frac{1}{b} + \frac{GM}{b^2} \qquad \text{or} \qquad r_0 \approx b - GM.$$

The largest deflection is for $b \approx r_0 = r_{\text{sun}}$, the radius of the sun. Plugging in

$$\Delta_{\text{max}} = 4\frac{(6.7 \times 10^{-11})(2.0 \times 10^{30})}{(7.0 \times 10^8)c^2} = 8.51 \times 10^{-6} \text{ rad} = 1.75 \text{ arc sec}$$

3. We want to solve the equation

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}$$

or equivalently

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Moreover we are instructed to assume a spherical symmetric static metric. That is

$$ds^2 = -T(r)dt^2 + R(r)dr^2 + r^2 d\Omega_2^2$$

On chap 5, p.3 of my class notes you will find the Ricci tensor for this metric (this was used in connection with the RN BH). The R_{tt} equation is

$$\frac{1}{2}\frac{T''}{R} - \frac{1}{4}\frac{T'R'}{R^2} - \frac{1}{4}\frac{T'^2}{RT} + \frac{1}{r}\frac{T'}{R} = -\Lambda T$$

As in every static case we try a solution with $RT = 1$. This gives

$$T'' + \frac{2}{r}T' - 2\Lambda = 0$$

This is easy to solve (using $r^{-2}\frac{d}{dr}(r^2 T) = T'' + \frac{2}{r}T'$ and integrating twice):

$$T = C_1 + C_2\frac{1}{r} - \frac{1}{3}\Lambda r^2$$

Now we can fix the constants of integration as follows. $C_1 = 1$ is necessary for solving the $R_{\theta\theta}$ equation. You could guess it from setting $\Lambda = 0$ and requiring this gives a SBH. In fact this is what we do to interpret (cheaply) the second constant, which we set to $C_2 = -2GM$. So we have

$$T = \frac{1}{R} = 1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2$$

It is straightforward to check that the other component of Einstein's equations are satisfied.

(ii) For $M = 0$ the metric is

$$ds^2 = -(1 - \tfrac{1}{3}\Lambda r^2)dt^2 + (1 - \tfrac{1}{3}\Lambda r^2)^{-1}dr^2 + r^2 d\Omega_2^2$$

We know this should be dS or AdS, depending on the sign of Λ, but looks unfamiliar. We could simply check that the Riemann tensor satisfies the condition ofr maximal symmetry. But more interesting is to display a change of coordinates that relates this to one of our known versions of the metric. Or beter yet, to derive this static metric form the embedding in 5D. Recall, dS is the 4D hypersurface

$$-u^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2$$

in 5D flat Lorentzian space, $ds^2 = -du^2 + dw^2 + dx^2 + dy^2 + dz^2$. The obvious thing to try in going to spherical symmetry is spherical coordinates for x, y, z, so that

$$-u^2 + w^2 = \alpha^2 - r^2$$

Then we need to choose one more coordinate form u and w and solve for the last. But we do not want square roots. So we go to lightcone coordinates, $w_\pm = w \pm u$. We choose then

$$w_+ = \frac{\alpha^2 - r^2}{w_-}$$

Finally, since w_- will now appear in the metric as dw_-/w_- we introduce $\xi = \ln w_-$ so that

$$\begin{aligned} ds^2 &= -(\alpha^2 - r^2)d\xi^2 - 2r\,dr\,d\xi + dr^2 + r^2 d\Omega_2^2 \\ &= -(\alpha^2 - r^2)\left(d\xi^2 - 2\frac{r}{\alpha^2 - r^2}dr\,d\xi\right) + dr^2 + r^2 d\Omega_2^2 \end{aligned}$$

Now, let

$$dR = \frac{r}{\alpha^2 - r^2}dr$$

which can be obtained explicitly by integration, but we will not need. Then

$$\begin{aligned} ds^2 &= -(\alpha^2 - r^2)\left(d\xi^2 - 2dR\,d\xi\right) + dr^2 + r^2 d\Omega_2^2 \\ &= -(\alpha^2 - r^2)\left(d\xi - dR\right)^2 + (\alpha^2 - r^2)dR^2 + dr^2 + r^2 d\Omega_2^2 \\ &= -(\alpha^2 - r^2)\frac{1}{\alpha^2}dt^2 + (\alpha^2 - r^2)\left(\frac{r}{\alpha^2 - r^2}dr\right)^2 + dr^2 + r^2 d\Omega_2^2 \\ &= -(1 - r^2/\alpha^2)dt^2 + \frac{\alpha^2}{\alpha^2 - r^2}dr^2 + r^2 d\Omega_2^2 \end{aligned}$$

In going from the second to the third line we have defined $t = \alpha(\xi - R)$. Identifying $\Lambda = 3/\alpha^2$ we see this is the $M = 0$ limit of the BH metric above. Note that $\Lambda > 0$ necessarily, since $\alpha^2 > 0$. A similar computation can be made for the AdS case.

Killing horizon: we are looking for a solution to $g_{tt} = 0$. That is $T(r_\Lambda) = 0$ or

$$1 - \frac{2GM}{r_\Lambda} - \frac{1}{3}\Lambda r_\Lambda^2 = 0$$

For $\Lambda > 0$ there are generically zero or two solutions, while for $\Lambda < 0$ there is always one solution.

(iii) As instructed

$$ds^2 = T(r)d\tau^2 + T(r)^{-1}dr^2 + r^2 d\Omega_2^2$$

Now, let $r = r_\Lambda + q$, so that $T(r) = T'(r_\Lambda)q + \cdots$. Now the point is that

$$T^{-1}dr^2 \sim \frac{dq^2}{q} = 4(d\sqrt{q})^2 = du^2$$

where $u = 2\sqrt{q} = 2\sqrt{r - r_\Lambda}$. The metric near r_Λ is now

$$ds^2 = u^2 d\tau^2 + du^2 + r_H^2 d\Omega^2$$

The first two terms tell the story. u is only defined for $u > 0$ so it is a radial coordinate and $du^2 + u^2 d\tau^2$ is precisely of the form of polar coordinates, and defines a regular space if τ (the polar coordinate) is periodic mod 2π.

4. First generalities/review: The observer moves on a time-like geodesic, $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$. For the RN-BH with mass M, electric charge Q and zero magnetic charge,

$$ds^2 = -T(r)dt^2 + T(r)^{-1}dr^2 + r^2 d\Omega_2^2, \quad T(r) = 1 - \frac{2GM}{r} + \frac{4\pi G Q^2}{r^2}$$

we can take the geodesic to be on the $\theta = \pi/2$ (equatorial) plane with conserved quantities (from the two Killing vectors $\vec{\partial}_t$ and $\vec{\partial}_\phi$):

$$E = -g_{tt}\frac{dt}{d\tau} = T(r)\frac{dt}{d\tau} \qquad \text{and} \qquad L = g_{\phi\phi}\frac{d\phi}{d\tau} = r^2\frac{d\phi}{d\tau}$$

Using this and the fact that on a circular orbit $r =$ constant we have

$$-1 = -\frac{E^2}{T(r)} + \frac{L^2}{r^2}$$

.

The circumference, a given, $C = 2\pi r$ gives r and therefore we have a relation between E and L. To fix them both we need in addition to require that $\ddot{r} = 0$, that is, that the effective potential energy is minimized at this value of r. This is just as in classical mechanics. I will not carry out the computation, and give only implicit results in what follows. The instantaneous velocity of the observer has magnitude

$$\beta = r\frac{d\phi}{dt} = \frac{L/r}{E/T(r)}$$

and obviously is tangential to the circle.

At any point on the circular trajectory the observer has some tangential velocity $\vec{\beta}$ and observes, instantaneously, the charged BH moving with velocity $-\vec{\beta}$. Therefore the observer sees a current in the direction of $-\vec{\beta}$ which produces a magnetic field transverse to the plane of the orbit: if the circulation is couter-clockwise as seen from above the plane of the orbit then the magnetic field points up.

To compute the field we take the field of the BH, $E_r = Q/r^2$ and $B_r = 0$ and boost it to a frame for which the observer is instantaneously at rest. It is simplest to use cartesian coordinates: instantaneously we can take the motion of the observer (in the rest frame of the BH) to be in the x-direction, the direction towards the BH to be $-z$, so that z is the radial direction. Then y points up from the plane of the orbit. Boosting the electric field strength $F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$ along the x-axis, $\Lambda_0{}^0 = \Lambda_1{}^1 = \gamma = 1/\sqrt{1-\beta^2}$, $\Lambda_0{}^1 = \Lambda_1{}^0 = -\beta\gamma$, we have

$$F'_{13} = \Lambda_1{}^\mu \Lambda_3{}^\nu F_{\mu\nu} = -\beta\gamma F_{03}$$

so that the magnitude of the magnetic field pointing up is

$$\beta\gamma E_r$$

with β, γ and E_r given above.