Hawking radiation: baby version

In QFT, vacuum fluctuations: vac $\to \gamma + \gamma \to$ vac
are virtual processes $E^{(1)} + E^{(2)} = 0$ so $E^{(1)} > 0$ ~~an~~ $E^{(2)} < 0$
and photon's cannot propagate freely. But in ~~curved background~~
vicinity of horizon, imagine 2nd γ crosses horizon. Then
even if $E^{(2)} < 0$, it must fall into singularity. The 1st photon
escapes, taking energy away to asymptotically flat region.

Take a freely falling observer with 4-velocity $\vec{U}$. ~~In his locally flat system~~ {He is inertial, so he sees normal laws of quantum electrodynamics. He sees vacuum fluctuations. A fluctuation of energy E lasts no longer that

$$\Delta t \sim \frac{\hbar}{E}$$

{This is a bit backwards. To measure energy E need at least time Δt, $\Delta t \cdot E \geq \hbar$}. If he is close to horizon, then he has limited time to measure and this sets the scale for energy of photon's he sees are pair created (virtual photons).

Say he starts free falling from $r = R + \epsilon$ $(R = 2GM)$.
then,

$$-1 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2$$

and $\hat{E} = -\vec{\partial}_t \cdot \vec{U} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}$ so

$$\left(\frac{dr}{d\tau}\right) = -\sqrt{\hat{E}^2 - \left(1 - \frac{2GM}{r}\right)}$$

Since $\frac{dr}{d\tau} = 0$ at $r = R + \epsilon$, we have

$$\hat{E}^2 = 1 - \frac{2GM}{R+\epsilon} = 1 - \frac{2GM}{2GM+\epsilon} = \frac{\epsilon}{2GM+\epsilon} \approx \frac{\epsilon}{2GM}$$

How much time does he have to observe photon creation before crossing $r=R$?

$$\Delta\tau = \int_0^{\Delta\tau} d\tau = \int_{R+\epsilon}^{R} dr \frac{1}{\frac{dr}{d\tau}} = + \int_{R+\epsilon}^{R} \frac{dr}{\sqrt{E^2 - \left(1-\frac{2GM}{r}\right)}}$$

$$= + \int_{2GM+\epsilon}^{2GM} \frac{dr}{\sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{R+\epsilon}}}$$

Mathematica can do full integral. We are happy with approximation: with $\epsilon \ll 2GM$, change variables to $\xi = r - 2GM$

$$\frac{1}{2GM+\xi} - \frac{1}{2GM+\epsilon} = \frac{\epsilon - \xi}{(2GM+\xi)(2GM+\epsilon)} \simeq \frac{\epsilon-\xi}{(2GM)^2}$$

$$\Delta\tau \approx \sqrt{2GM} \int_{\epsilon}^{0} \frac{d\xi}{\sqrt{\epsilon - \xi}} = 2\sqrt{2GM\epsilon}$$

So the energy of the photon created which escapes to ∞ is

$$\mathcal{E} = \frac{\hbar}{\Delta\tau} = \frac{\hbar}{2\sqrt{2GM\epsilon}}$$ as observed in his frame.

Now

$$\mathcal{E} = -\vec{p}\cdot\vec{U}$$

where $\vec{U}$ is for our falling observer. The energy E of the photon as observed at ∞ is

$$E = -\vec{p}\cdot\vec{\partial}_t$$

Or $\cancel{\frac{E}{\mathcal{E}} = \frac{\vec{p}\cdot\vec{U}}{\vec{p}\cdot\vec{\partial}_t} \Big| = \frac{g_{tt}p^tU^t}{g_{tt}p^t} = \frac{\hat{E}}{1-\frac{2GM}{R+\epsilon}} = \frac{1}{\hat{E}} = \sqrt{\frac{2GM}{\epsilon}}}$

$\cancel{\text{at } r=R+\epsilon \Rightarrow E = \sqrt{\frac{2GM}{\epsilon}}}$.

Computing the ratio at $r = R + \epsilon$

$$\frac{E}{\mathcal{E}} = \frac{\vec{p}\cdot\vec{\partial}_t}{\vec{p}\cdot\vec{U}} = \frac{g_{tt}p^t}{g_{tt}p^t U^t} = \frac{1}{U^t} = \frac{1-\frac{2GM}{R+\epsilon}}{\hat{E}} = \hat{E} = \sqrt{\frac{\epsilon}{2GM}}$$

$$\Rightarrow \quad E = \sqrt{\frac{\epsilon}{2GM}} \cdot \frac{\hbar}{2\sqrt{2GM\epsilon}} = \frac{\hbar}{4GM}$$

An observer at ∞ sees a ~~an ener~~ photon with energy

$$E \approx \frac{\hbar}{4GM}$$

This is independent of ϵ in the argument above! So we don't know exactly where the photon was emitted, but it does not matter.

A complete calculation shows the spectrum of photons is thermal with temperature $T = E/k_B$ with E as above.

Recall we had before that for black hole thermodynamics

$$T = \frac{\kappa}{2\pi} \quad \text{where } \kappa \text{ for Schwarzschild is } \kappa = \frac{1}{4GM}$$

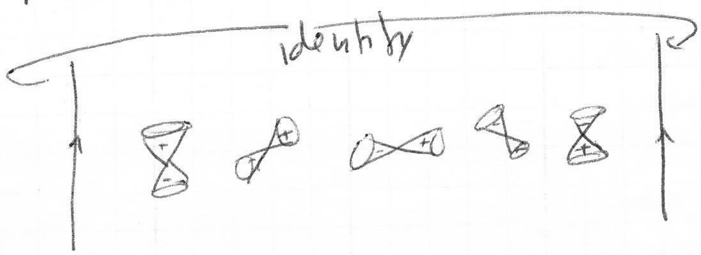
In units of $\hbar = 1 = $ ~~...~~ we see that this agrees with the previous!

Causal Structure

The following definitions are for any spacetime (M, g_{ab})

(M, g_{ab}) is <u>time orientable</u> if as you vary continuously $p \in M$ the future lightcone at p can be continuously defined

Example: non-time-orientable

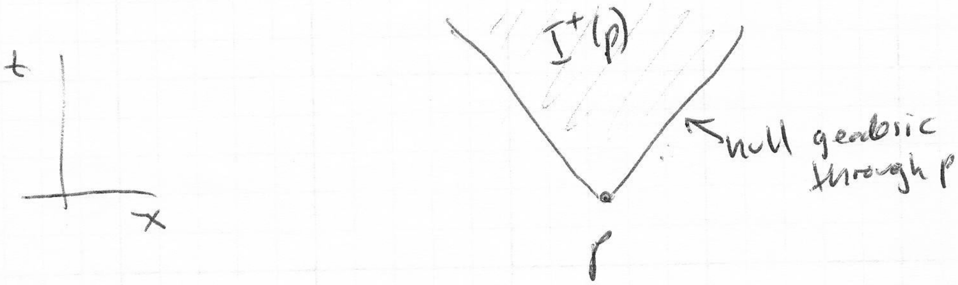


We assume (M, g_{ab}) is time orientable from here on.

time orientable $\Leftrightarrow$ there is a continuous timelike vector field

$I^+(p)$: Chronological future of p

set of points that can be reached from p by a future directed timelike curve

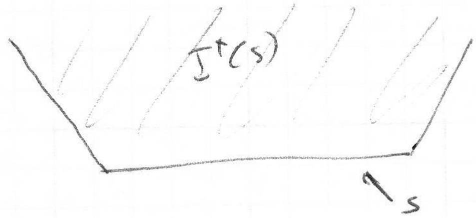


Notes

- $I^+(p)$ is open (the null geodesic cone is not in $I^+(p)$).
- generally $p \notin I^+(p)$, but p may be in $I^+(p)$ if there is a closed timelike curve from p to p.

For any set S define $I^+(S) = \bigcup_{p \in S} I^+(p)$

~~Particularly not~~ $I^+(S)$ is also open. ~~and $\dot{I}^+(S)$~~ we'll be interested in surfaces S



Define also $I^-(p)$ & $I^-(S)$ (replace "future" by "past")

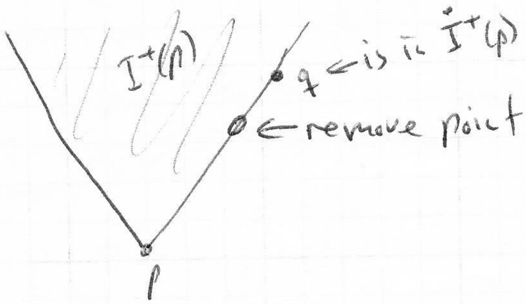
and $J^\pm(p)$ & $J^\pm(S)$ (replace "timelike curve" by "causal curve").
(ie, non-spacelike

$\dot{I}^+(p) \equiv$ boundary of $I^+(p)$

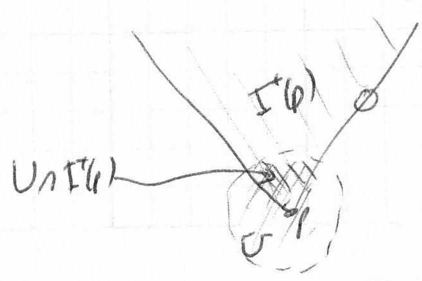
In Minkowski spacetime $I^+(p)$ is the set of points that can be reached by future directed timelike <u>geodesics</u> starting at p.

and $\dot{I}^+(p)$ is ~~geo~~ future light cone (generated by future null geodesics).

This is locally true in any spacetime, but not necessarily globally. An artificial example



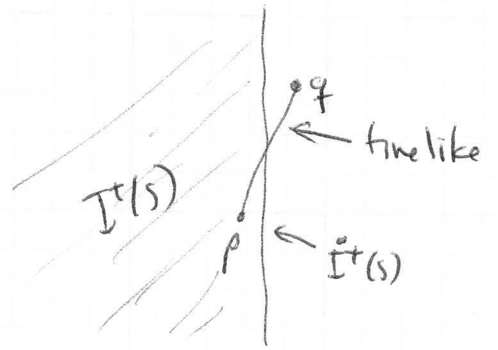
Still for $U \subset M$ small enough, $p \in U$



still holds there.

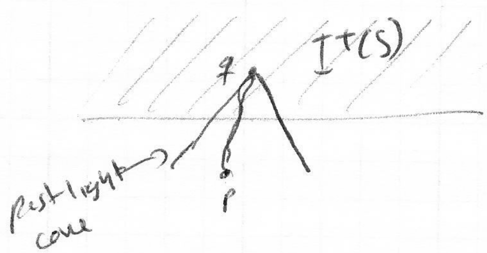
Some fun "theorems"

* $\dot{I}^+(S)$ cannot be timelike:



so $q \in I^+(S)$
but $q \notin I^+(p)$, a contradiction

* $\dot{I}^+(S)$ cannot be spacelike, except for the set S itself



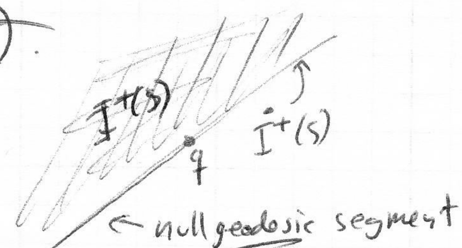
$\dot{I}^+(S) \leftarrow$ part of $\dot{I}^+(S)$ not in S

no point p in q's past light cone is in S
$\Rightarrow$ a contradiction.

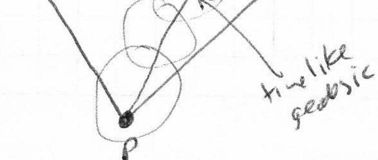
* Therefore $\dot{I}^+(S)$ is null, apart from S itself.

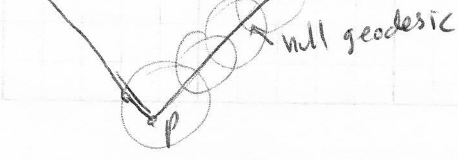
~~Moreover, to be precise~~
~~if $q \in \dot{I}^+(S)$ but $q \notin \bar{S} \Rightarrow \exists$ past directed null geodesic through q lying on $\dot{I}^+(S)$.~~

picture



Extra information: $\dot{I}^+(S)$ is generated by null geodesic segments (we did not get this above, we just got that $\dot{I}^+(S)$ is null (up to S)). The proof uses the fact that it is locally true (see previous page) and that one can show one can find a finite cover with p, q





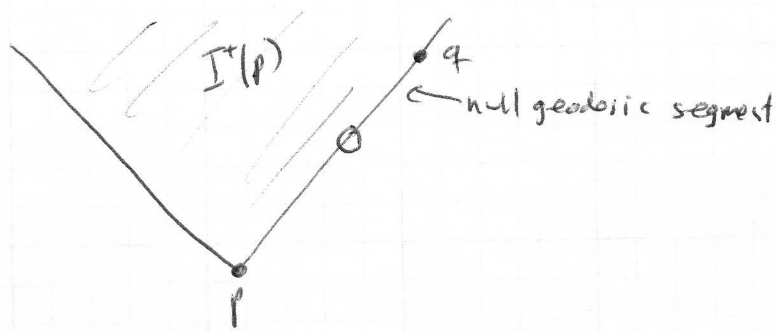
This does not conflict with taking points out of space if one phrases it carefully, for example:

- If $q \in J^+(p) - I^+(p) \Rightarrow$ any causal curve from p to q is a null geodesic

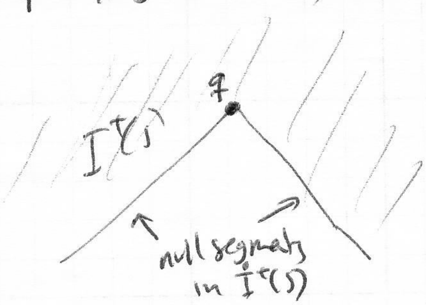
or

- If $q \in \dot{I}^+(S)$ but $q \notin S \Rightarrow$ there is a past directed null geodesic segment through q lying on $\dot{I}^+(S)$.

So in our example



If there is more than one past directed null geodesic segment through q lying on $\dot{I}^+(S) \Rightarrow q$ is the endpoint of the segments



So the structure of $\dot{I}^+(S)$ is ~~as fo~~llows:

- it is generated by null geodesic segments that
  - have past endpoints only on S
  - have future endpoints in the boundary (and would then pass into the interior of $I^+(S)$) if they intersect another generator.
  - may have no endpoints

Example: Minkowski space with a horizontal line segment (in 1+1 dim) removed