Maximally Symmetric Spaces

Spaces with high degree of symmetry are easier to analyze.

What is the highest degree of symmetry?

Consider $\mathbb{R}^n$ – Euclidean space. Then we had

n translations

$\frac{1}{2}n(n-1)$ rotations

$= \frac{1}{2}n(n+1)$ symmetries in total

~~Rota~~ Symmetry under rotations at a point p is called "isotropy" (at p)

Symmetry under translation is called "homogeneity" of the spacetime.

This is as much as we can have, and we define a

"maximally symmetric space" = one with $\frac{1}{2}n(n+1)$ killing vector fields

Let's find them.

At $p \in M$ choose locally inertial coordinates, so that $g_{\mu\nu}$ is given by $\eta_{\mu\nu}$. Obviously (by construction) this is invariant under local Lorentz transformations. But isotropy means, in this coordinates, at this point p, $R_{\mu\nu\sigma\lambda}$ should also be invariant,

$$R_{\mu\nu\rho\sigma} \propto \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}$$

only tensor with proper symmetries and invariant.

NOTE: A local Lorentz transformation acts only on $T_p(M)$, ie, it is a change of basis vectors $\vec{E}^{(a)}$. It is these vectors that are used to define the components of $\mathbf{R}$.

If we write this as

~~$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} \left(g_{\rho\mu} g_{\sigma\nu} \right.$$~~

$$R_{\mu\nu\rho\sigma} = \kappa \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

since this is a tensorial relation it holds at p in any coordinate system. But then use homogeneity ⇒ it holds everywhere on M with same constant κ.

Contracting indices

$$\boxed{R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)}$$

So, in particular, the Ricci scalar is a constant (should be obvious by homogeneity: same R everywhere).

A maximally symmetric space is determined by

- dimension
- signature
- R
- additional topological considerations (global issues).

Warm-up:

$n = 2$, $\eta = (++)$ ($n=2$ almost trivial, since only one component of $R^\mu{}_{\nu\rho\sigma}$).

$R > 0$ the sphere S^2 $\quad ds^2 = a^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$ $\quad R = \frac{2}{a^2}$

$R = 0$ is $\mathbb{R}^2$ $\quad ds^2 = dx^2 + dy^2$

$R < 0$ less familiar, the hyperboloid H^2 $\quad ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2)$ $\quad y > 0$

Exercise: For H^2 show

(i) $R = -\frac{2}{a^2}$ (ii) The distance between y_1 & y_2 along $x = $ constant is $a \ln \frac{y_2}{y_1}$

(iii) Geodesics satisfy $(x - x_0)^2 + y^2 = b^2$ for x_0, b constants.

Now do $n=4$ with $(-+++)$

$R>0$ de Sitter space

$R=0$ (flat) Minkowski space

$R<0$ anti-de Sitter space

Study these. Study casual structure too.

<u>Minkowski Space-time</u>: (trivial, but will help us understand key concepts for other spacetimes)

$$ds^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$
$$= -(dt)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$(r \geq 0)$
$(0 \leq \theta \leq \pi)$
$(0 \leq \phi < 2\pi)$

Null coordinates:

$$v = t + r$$
$$w = t - r$$

$\infty > v \geq w > -\infty$

$$ds^2 = -dv\,dw + \frac{1}{4}(v-w)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$v = \text{const}$ and $w = \text{const}$ are null hypersurfaces.

Can we change coordinates to have only finite ranges? Let

$$W = \arctan w$$
$$V = \arctan v$$

$W < V$
and both in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Now

$$ds^2 = \frac{1}{\omega^2}\left[-4\,dV\,dW + \sin^2(V-W)(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

where $\omega \equiv 2\cos W \cos V$

Finally write $T = V + W$ $\quad R = V - W$

$0 \leq R < \pi$
$|T| + R < \pi$

so

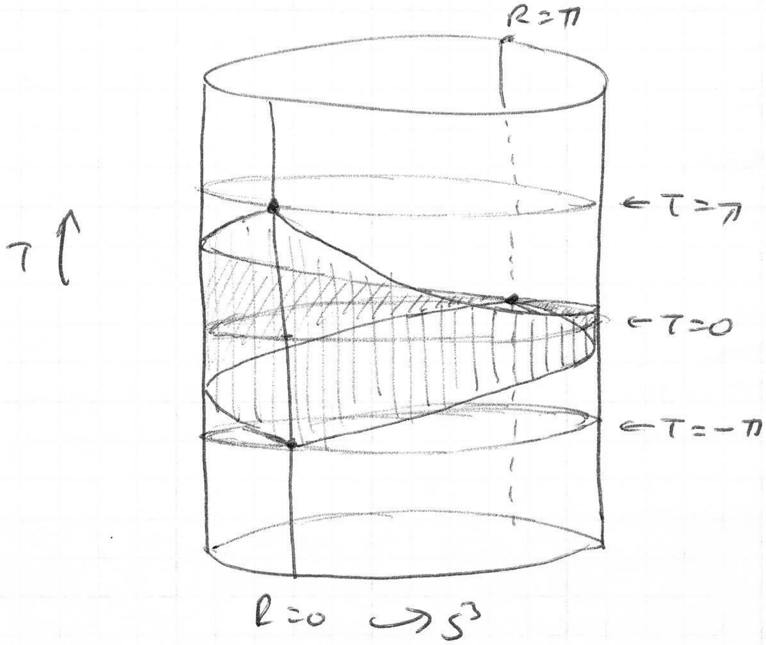
$$ds^2 = \frac{1}{\omega^2}(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2)$$

with $\omega = \cos T + \cos R$ (kind of irrelevant for us).

$$ds^2 = \frac{1}{\omega^2} ds_E^2$$

where $ds_E^2 = -dT^2 + dR^2 + \sin^2 R\, d\Omega^2$ "the metric for Einstein's static universe"

So Minkowski space is conformal to (a part of) the Einstein static universe.
(A conformal transformation is a local change of scale; $\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}$)



Conformal Diagrams (or Penrose Diagrams)

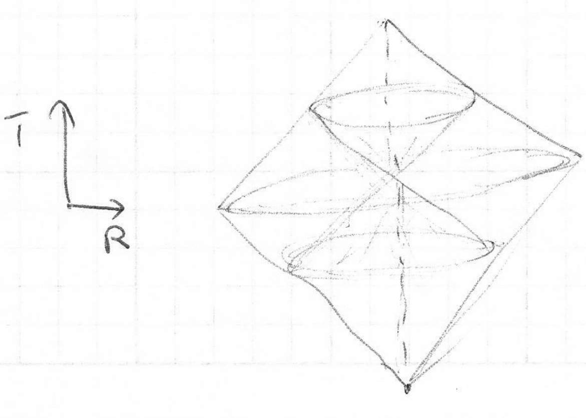
Space-time diagram for space-time (M, g). It has a "time" coordinate and a "radial" coordinate, with light-cones always at 45°. Also infinity is at finite coordinate distance (so we can fit it on a page).

Conformal transformation leave light-cones invariant

(if $d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = 0$ then $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0$)

They are useful in displaying the causal structure of spacetime.

Drawing a circle for each sphere in θ, φ



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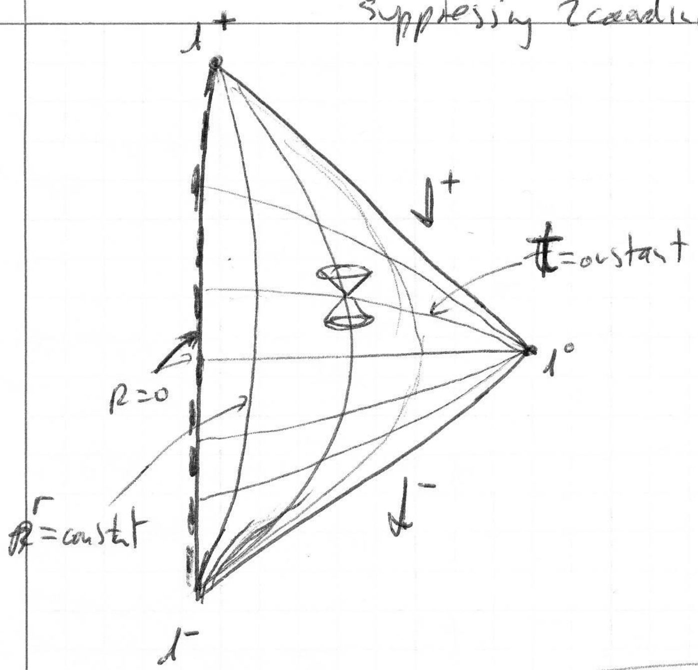
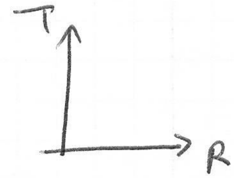
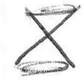
suppressing 2 coordinates (always done when with spherical symmetry):





Note: light cones always  everywhere (i.e. 45°)

(SKIP IN CLASS) The constant t, r surfaces are obtained from

$$T = V + W = tg^{-1}v + tg^{-1}w = tg^{-1}(t+r) + tg^{-1}(t-r)$$

$$R = V - W \qquad\qquad = tg^{-1}(t+r) - tg^{-1}(t-r)$$

More easily: for t = constant, eliminate $r \Rightarrow w + v = 2t$ fixed

$\Rightarrow \quad \Rightarrow \quad \tan V + \tan W = 2t$

$\Rightarrow \quad \tan(\frac{1}{2}(T+R)) + \tan(\frac{1}{2}(T-R)) = 2t$

etc.

---

i^+ = future timelike infinity

i^- = past ✓ ✓

i^0 = spatial infinity

$\mathscr{I}^+$ = ("scri-plus") future null infinity

$\mathscr{I}^-$ = past ✓ ✓

Features:

(i) light cones at 45°

ii) i's are points, $\mathscr{I}$'s are surfaces (null) with topology $R \times S^2$

iii) timelike geodesics: from i^- to i^+; spacelike from i^0 to i^0

null geodesics from $\mathscr{I}^-$ to $\mathscr{I}^+$

Some obvious things are more obvious in the Penrose diagram.

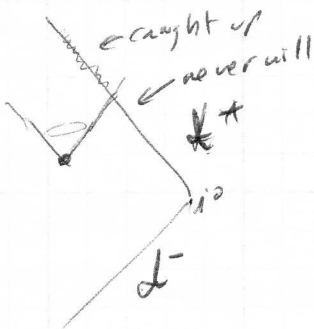
- $\mathscr{I}^+$ is in the future lightcone of any event
- $\mathscr{I}^-$ is in the past lightcone of any event

i.e. you can reach any point, no matter how far, with a signal if you are willing to wait enough

- i^0 is neither in the future nor past light cone of an event

i.e. you can not reach space-like infinity with a signal in finite time.

If you are willing to wait an infinite time, then a signal can reach spatial infinity, but will not catch up with other signals emitted by you earlier:

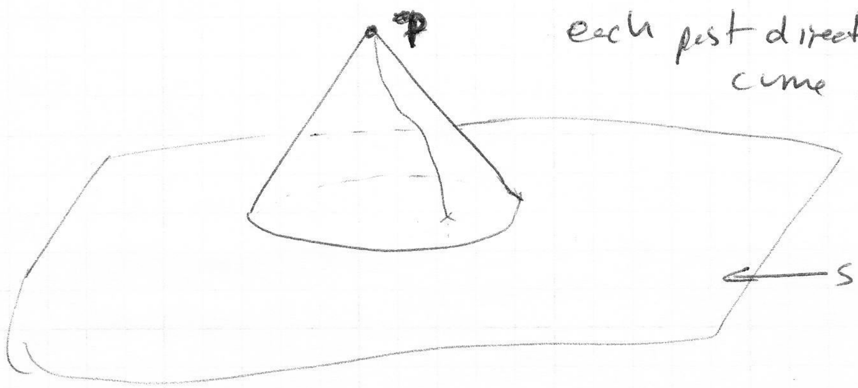


Cauchy Surfaces

$D^+(S)$: "future Cauchy development" of S :

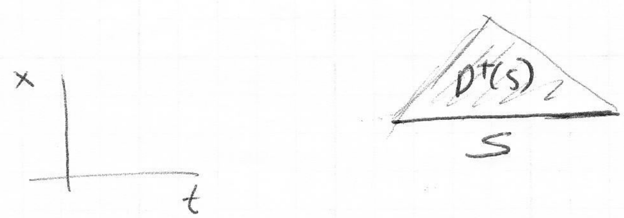
If S is a space-like 3-surface then

$D^+(S) = \{ p \in \mathcal{M} \mid$ each past directed inextendible non-spacelike curve through p intersects $S \}$



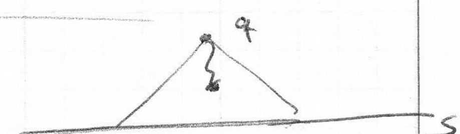
each past directed ~~timelike~~ non-spacelike curve through p intersects S.
$\Rightarrow p$ is in $D^+(S)$

so in Minkowski 1+1 space



Notes:

- inextendible so that we avoid:



- non-spacelike, we want the part of space that can be causaly affected by S.

~~- Similarly define $D^-(S)$, "future directed curves" giving~~

Since signals can ~~not~~ only travel on non-spacelike curves, if $p \in D^+(S)$ then knowing data (value of fields and first derivatives, or particle velocities, etc) on S is enough to predict them at p.

Similarly, if we want to evolve back into the past, but have information only on S, we can only infer the state in $D^-(S)$ (defined by "future" → "past" in def above).

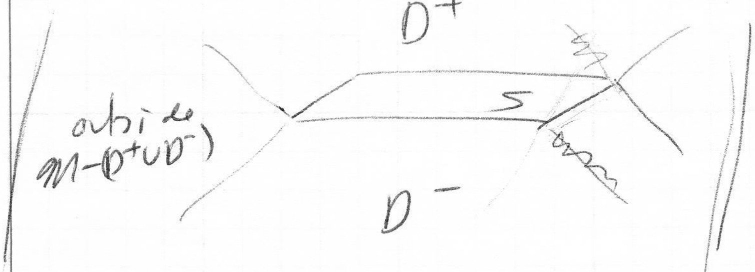
If $D^+(S) \cup D^-(S) = \mathcal{M}$

S is called a <u>Cauchy surface</u>

→ In words, ~~If~~ every inextendible non-spacelike curve in $\mathcal{M}$ intersects S. ~~S is Cauchy~~

In Minkowski space $t=0$ is a Cauchy surface. In fact $t = c$ = a constant is a collection of Cauchy surfaces that cover the whole of $\mathcal{M}$.

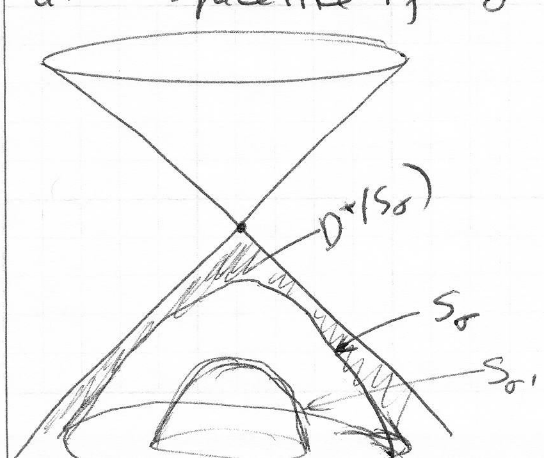
Not every spacelike surface in (Minkowski space) is Cauchy. Clearly extendible surfaces are not



More interestingly, some inextendible surfaces are not Cauchy. For example the surfaces

$$-t^2 + x^2 + y^2 + z^2 = \sigma \in \mathbb{R}$$

are spacelike if $\sigma < 0$. Let S_σ be the surface with $t<0$



These S_σ are not Cauchy, ~~and their~~ but are inextendible spacelike. The collections fills the past lightcone of the origin.

de Sitter space-time

This is $R > 0$. Note that this means (R = const)

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -\tfrac{1}{4} R g_{\mu\nu} \qquad R = \text{const} > 0$$

Comparing with Einstein's equation, (8.8) in Schutz

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = k T^{\mu\nu}$$

we see that either $k T^{\mu\nu} = -\frac{R}{4} g^{\mu\nu}$ (a weird fluid??)

or $\Lambda = \frac{1}{4} R > 0$. That is, de Sitter spacetime is a solution

to Einstein's equations with a ~~cosmological~~ positive cosmological constant, but no matter.

Since we have observed $\Lambda \neq 0$, with $\Lambda \sim \rho_{\text{dark matter}}$, and since Λ is const while ρ is decreasing with the (observed) expansion of the universe, soon ρ will be negligible and the future of the universe will be described by ~~appr~~ (approximately) de Sitter spacetime.

It is defined by embedding the hyperboloid ~~(5 dimensions)~~

$$-U^2 + X^2 + Y^2 + Z^2 + W^2 = \alpha^2$$

defined in 5-dim Minkowski space, $ds_5^2 = -dU^2 + dX^2 + dY^2 + dZ^2 + dW^2$

~~in 4 dimensions~~

Let

$$U = \alpha \sinh(t/\alpha)$$

$$W = \alpha \cosh(t/\alpha) \cos\chi$$

$$X = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi$$

$$Y = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi$$

$$Z = \alpha \cosh(t/\alpha) \sin\chi \cos\theta$$

Then

$$ds^2 = -dt^2 + \alpha^2 \cosh^2\left(\tfrac{t}{\alpha}\right)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

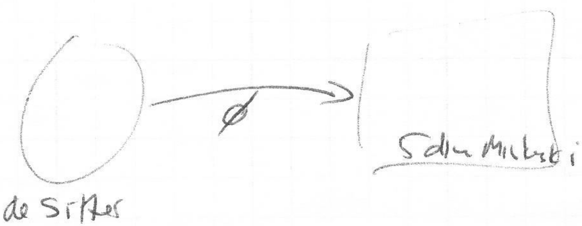
(Exercise: but not for class time!)

(1) $-U^2 + X^2 + Y^2 + Z^2 + W^2 = -\alpha^2 \sinh^2\left(\tfrac{t}{\alpha}\right) + \alpha^2\cosh^2\left(\tfrac{t}{\alpha}\right)\left[\cos^2\chi + \sin^2\chi(\cos^2\theta + \sin^2\theta(\cdots))\right]$

$= \alpha^2$ ✓✓

(2)

$g_{\mu\nu} = \dfrac{\partial y^a \partial y^b}{\partial x^\mu \partial x^\nu} g_{ab}$ is a pull back $\phi^* : g \to \phi^* g$



ϕ given by the above eqs, ie, $U = \alpha \sinh(t/\alpha)$ etc.

So $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \dfrac{\partial y^a}{\partial x^\mu}\dfrac{\partial y^b}{\partial x^\nu} g_{ab}\, dx^\mu dx^\nu$

$= -\dfrac{\partial U}{\partial x^\mu}\dfrac{\partial U}{\partial x^\nu} dx^\mu dx^\nu + \dfrac{\partial W}{\partial x^\mu}\dfrac{\partial W}{\partial x^\nu} dx^\mu dx^\nu + \cdots + \dfrac{\partial Z}{\partial x^\mu}\dfrac{\partial Z}{\partial x^\nu} dx^\mu dx^\nu$

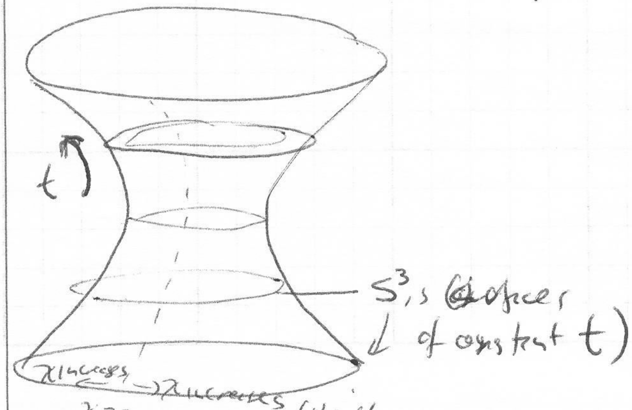
$= -\cosh^2\frac{t}{\alpha}\, dt^2 + \sinh^2\frac{t}{\alpha}(\cos^2\chi + \sin^2\chi(\cos^2\theta + \cdots))\,dt^2$

$+ \alpha^2\cosh^2\frac{t}{\alpha}\left[\sin^2\chi + \cos^2\chi\left(\cos^2\theta + \sin^2\theta(\cos^2\phi + \sin^2\phi)\right)\right] d\chi^2$

$+ \alpha^2\cosh^2\frac{t}{\alpha}\sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$ ✓

Note $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ metric on a 2sphere

Similarly $d\Omega_3^2 = d\chi^2 + \sin^2\chi\, d\Omega_2^2 \to$ metric on a 3sphere



de Sitter space: spatial 3-sphere that shrinks to a minimum radius α, then re-expands.

topology: $\mathbb{R}^1 \times S^3$

Another coordinate system that is common is

$$\hat{t} = \alpha \log\left(\frac{w+u}{\alpha}\right) \qquad \hat{x} = \frac{\alpha x}{w+u} \qquad \hat{y} = \frac{\alpha y}{w+u} \qquad \hat{z} = \frac{\alpha z}{w+u}$$

restricted to the hyperboloid (you can simply write ~~a~~ $\hat{t}, \ldots, \hat{z}$ as a function of t, x, θ, ϕ). In terms of these

$$ds^2 = -d\hat{t}^2 + e^{2\hat{t}/\alpha}\left(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2\right)$$

NOT INCLUDED [To see this, write

$$w+u = \alpha e^{\hat{t}/\alpha}$$

$$x = \hat{x} e^{\hat{t}/\alpha}$$

$$y = \hat{y} e^{\hat{t}/\alpha}$$

$$z = \hat{z} e^{\hat{t}/\alpha}$$

and insist on the hyperboloid, $(w-u)(w+u) + r^2 = \alpha^2 \Rightarrow w-u = \dfrac{\alpha^2 - \hat{r}^2 e^{2\hat{t}/\alpha}}{\alpha e^{\hat{t}/\alpha}}$ or

$$w - u = \alpha e^{-\hat{t}/\alpha} - \frac{1}{\alpha}(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) e^{\hat{t}/\alpha}$$

Now proceed with the pull-back of η^{ab}:

$$ds^2 = d(w+u)d(w-u) + dx^2 + dy^2 + dz^2 = (e^{\hat{t}/\alpha} d\hat{t})\left[\left(-e^{-\hat{t}/\alpha} - \frac{\hat{r}^2}{\alpha^2} e^{\hat{t}/\alpha}\right) d\hat{t} - \frac{\hat{\vec{r}} \cdot d\hat{\vec{r}}}{\alpha} e^{\hat{t}/\alpha}\right]$$

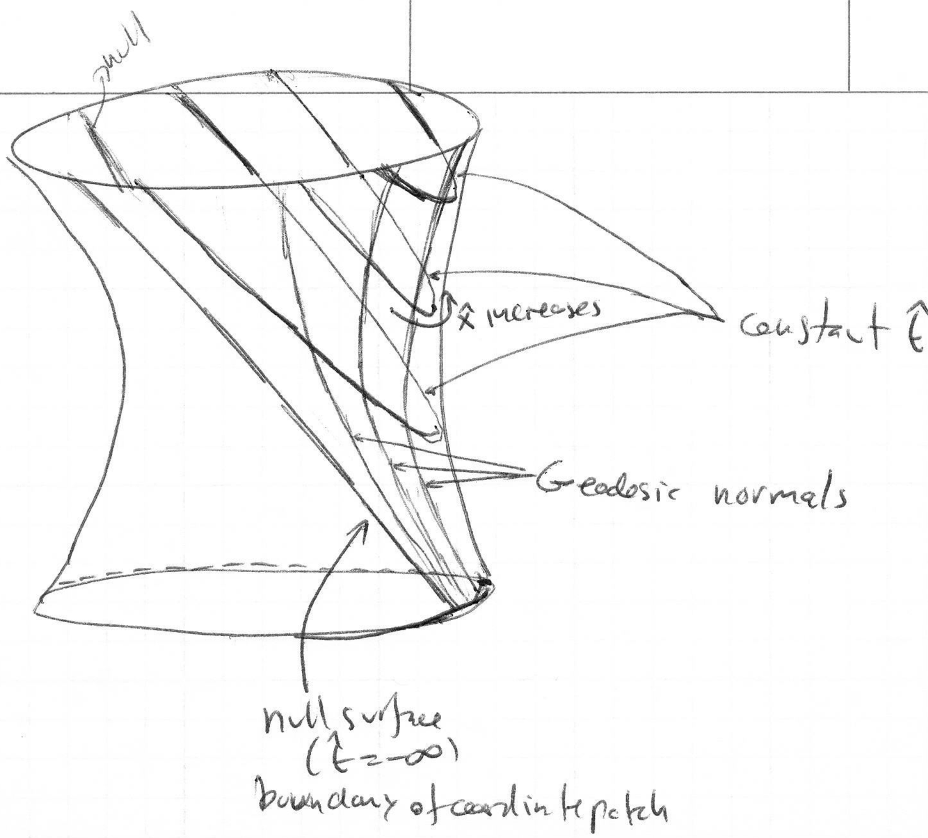
$$+ \left(d\hat{\vec{r}} - \frac{\hat{\vec{r}}}{\alpha} d\hat{t}\right)^2 e^{2\hat{t}/\alpha}$$

cross terms cancel!]

This coordinates cover only the region

$$w + u \geq 0$$

of the hyperboloid



[Check $w + U = 0 \iff \alpha \sinh\left(\frac{t}{\alpha}\right) + \alpha \cosh\left(\frac{t}{\alpha}\right) \cos\chi = 0$

$$\Rightarrow \quad \cos\chi = -\tanh\left(\frac{t}{\alpha}\right)$$

As $t \to \mp\infty$, $\tanh\left(\frac{t}{\alpha}\right) \to \mp 1$ so $\cos\chi \to \pm 1$ or $\chi \to 0$ or π]

Penrose diagram for de Sitter:

~~Let~~ ~~t~~ Change coordi[nate] t to t' by

$$\tan\left(\tfrac{1}{2}t' + \tfrac{\pi}{4}\right) = e^{t/\alpha}$$

with $t' \in (-\pi/2, \pi/2)$

Not needed [then

$$dt^2 \frac{e^{2t/\alpha}}{\alpha^2} = \left(\frac{1/2}{\cos^2(\frac{1}{2}t' + \frac{\pi}{4})}\right)^2 dt'^2$$

or

$$dt^2 = \frac{\alpha^2}{4} \frac{1}{\cos^4(\frac{t'}{2}+\frac{\pi}{4})} \frac{\cos^2}{\sin^2} (dt')^2 = \frac{\alpha^2}{4\cos^2\sin^2} dt'^2$$

$$= \frac{\alpha^2}{\sin^2(t'+\frac{\pi}{2})} dt'^2$$

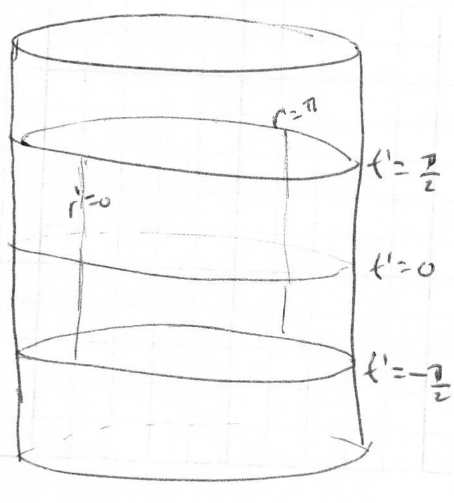
and

$$\cosh\frac{t}{\alpha} = \frac{1}{2}\left(\tan + \frac{1}{\tan}\right) = \frac{1}{2}\frac{\sin^2 + \cos^2}{\sin\cos} = \frac{1}{\sin(t'+\frac{\pi}{2})}$$ so]

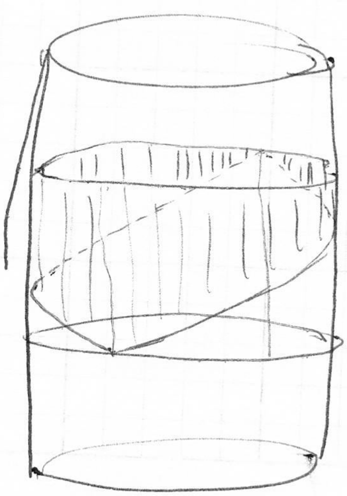
$$ds^2 = \frac{\alpha^2}{\sin^2(t'+\frac{\pi}{2})} d\bar{s}^2 \qquad (d\bar{s}^2 = ds_E^2 \text{ in previous notation})$$

where $d\bar{s}^2 = -dt'^2 + d\chi^2 + d\Omega_2^2 = -dt'^2 + d\Omega_3^2$

So de-Sitter is conformal to the metric $d\bar{s}^2$ = Einstein Static ~~E~~ familiar from Minkowski. Now

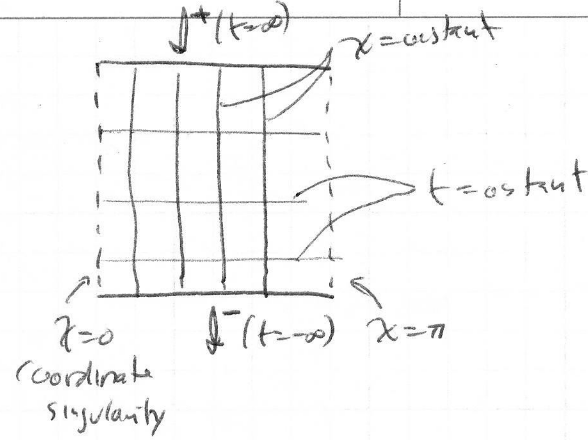


and

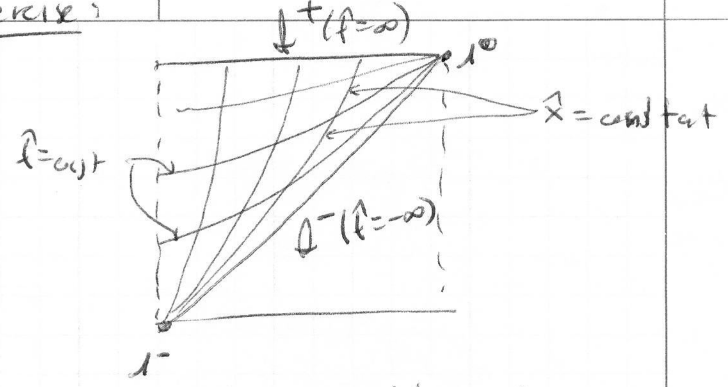


steady state universe (Lemaitre?)

Exercise:



de-Sitter



Steady-state universe of Bondi & Gold, and Hoyle (circa 1948)

$\mathcal{I}^\pm$ spacelike future/past infinity.

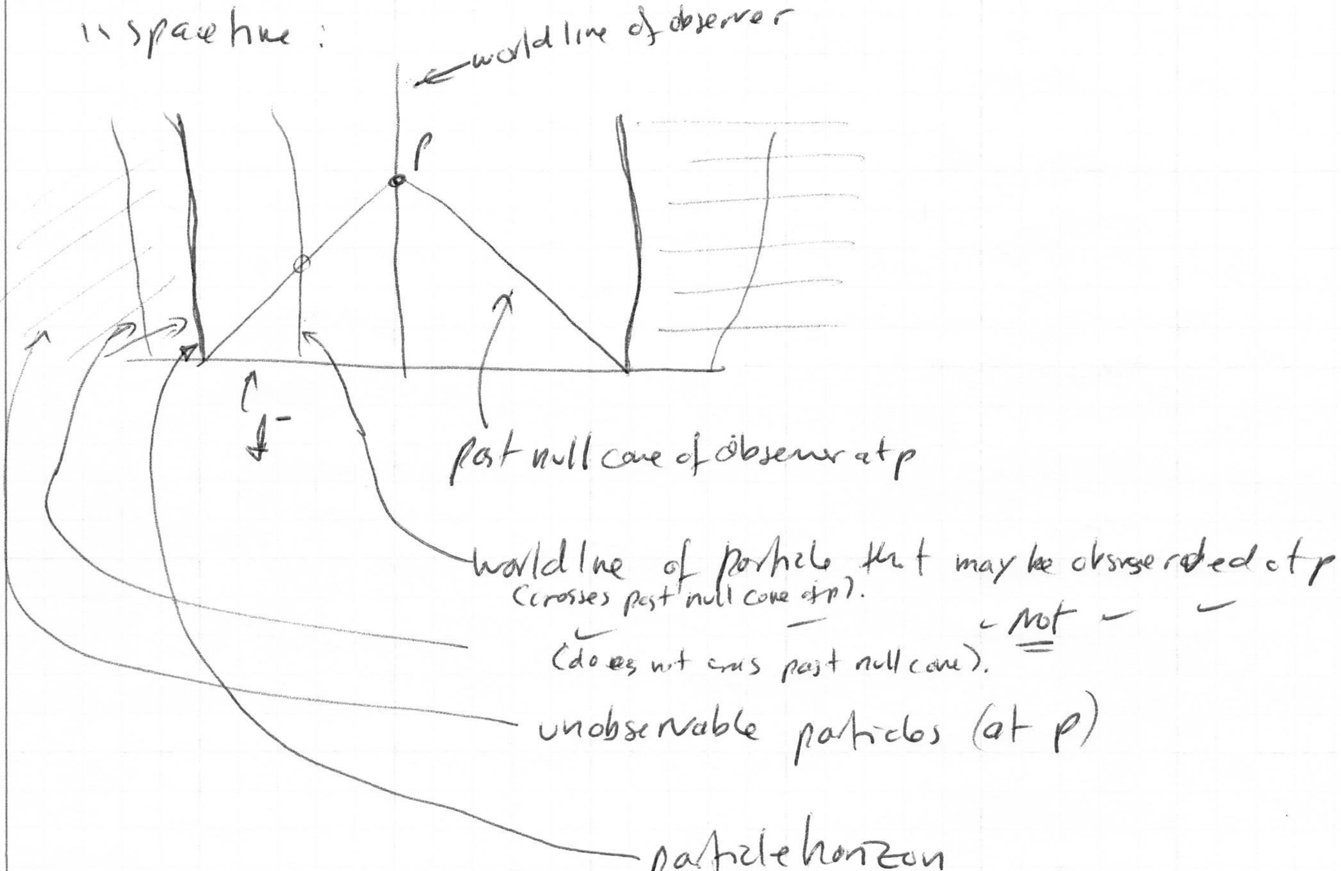
Horizons: (NEXT PAGE)

Horizons

~~Beer~~ deSitter future & past infinities are spacelike (Contrast with Minkowski's timelike).

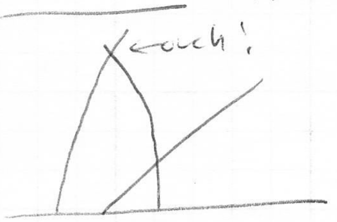
This gives rise to both particle & event horizons

Particle Horizon: defined for an observer at some event p in spacetime:



~~So~~ The particle horizon separates the region of spacetime occupied by particles that may have been seen at p from those that can not be seen at p.

Particle Horizons are defined with respect to a* congruence of world-lines. Problem is

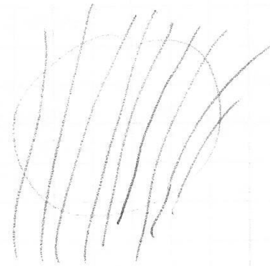


→ so we wouldn't be able to separate space into two pieces → no "horizon".

~~So we~~

Congruence is a set of ~~lines~~ curves such that each point p (in some ~~open~~ open set $U \subset \mathcal{M}$) is in exactly one ~~curve~~ curve.

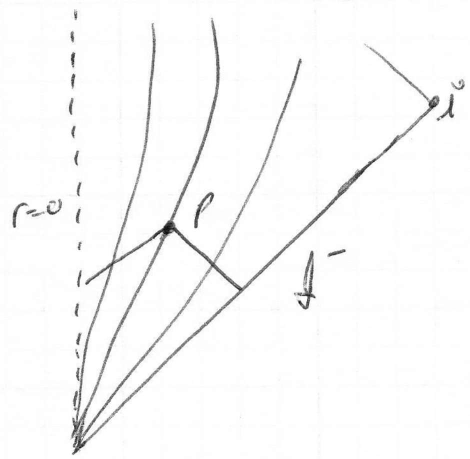
~~image~~



By definition, curves in a congruence do not cross.

Examples:

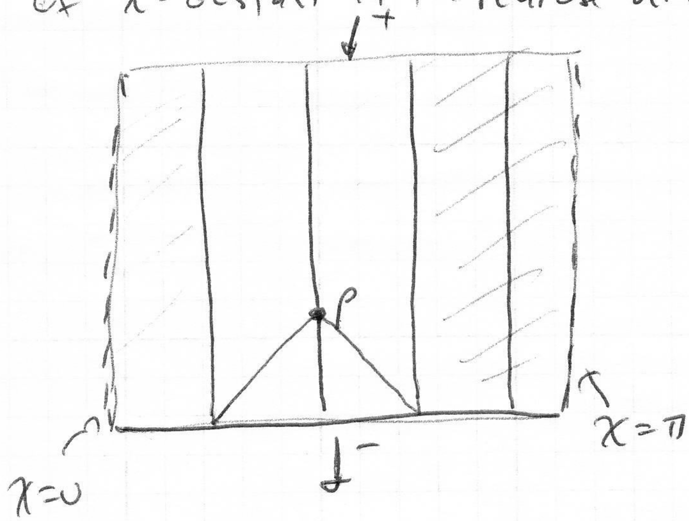
(i) There are no particle horizons in Minkowski space



every timelike geodesic crosses the past light cone of p.

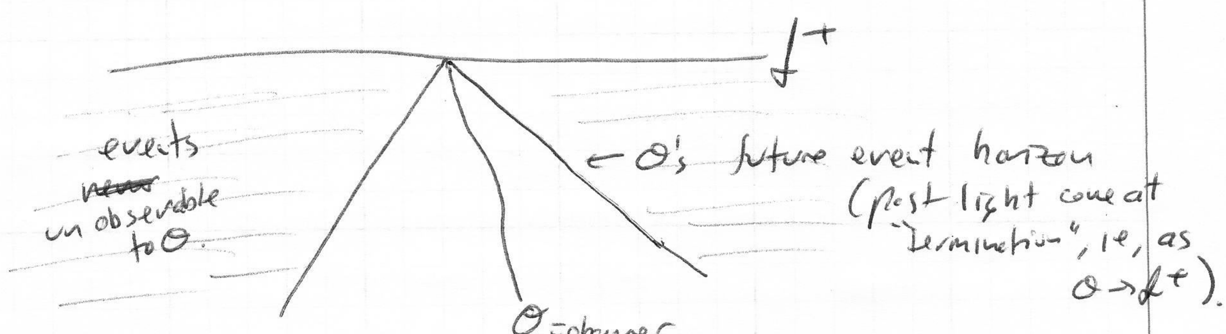
More generally, this is true if $\mathscr{I}^-$ is null.

(ii) de-Sitter does have particle horizons. Consider the congruence of $\chi = \text{constant}$ in the Penrose diagram

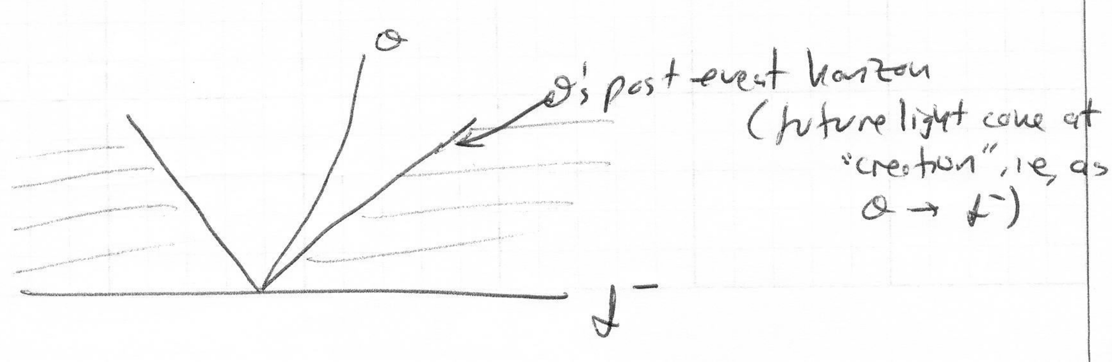


Event Horizons: ~~while~~ Particle horizon tells us which ~~observers~~ 'particles' may have been seen at p_0. We may ask instead ~~of~~ which particles may influence p at all throughout its whole history. That is, if the space time is expanding faster than the speed of light then if some observers far away from us, light sent to us will never reach us. We want to characterize this situation with an "event horizon" separating those events that can never influence us from those that can. ~~Clearl~~

Clearly, at any event p_0 the events (in)side its past light cone are observable, while those outside are not. The ~~future~~ future event horizon is the limiting light cone of an observer as it goes into future infinity, $\mathcal{I}^+$.

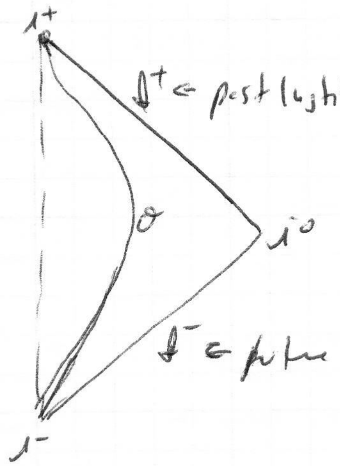


Similarly, past event horizon is defined to separate events that O will be able to influence in its history from those it won't:

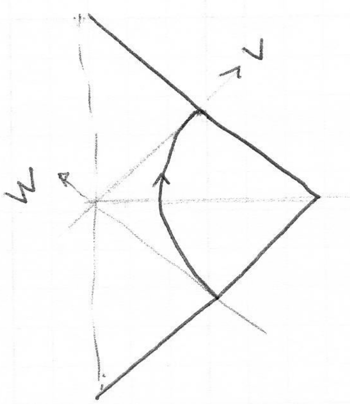


Examples: (i) Minkowski space time.

If $\mathcal{O}$ is a geodesic (free falling) observer $\Rightarrow$ no event horizons



(ii) Uniformly accelerated 'observer' in Minkowski space-time



picture is $r^2 - t^2 = a^2$

his both future and past event horizons.

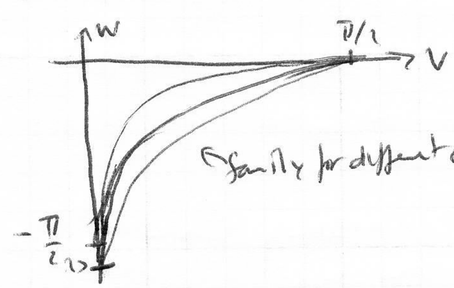
[Work it out: Recall $ds^2 = \frac{1}{\omega^2} ds_E^2$, see above.

and uniformly accelerated $\rightarrow$ $r^2 - t^2 = a^2$ or $vw = -a^2$

$\Rightarrow$ $\operatorname{tg} W \operatorname{tg} V = a^2$ $\Rightarrow$ $\operatorname{tg}\left(\frac{1}{2}(T+R)\right) \operatorname{tg}\left(\frac{1}{2}(T-R)\right) = -a^2$

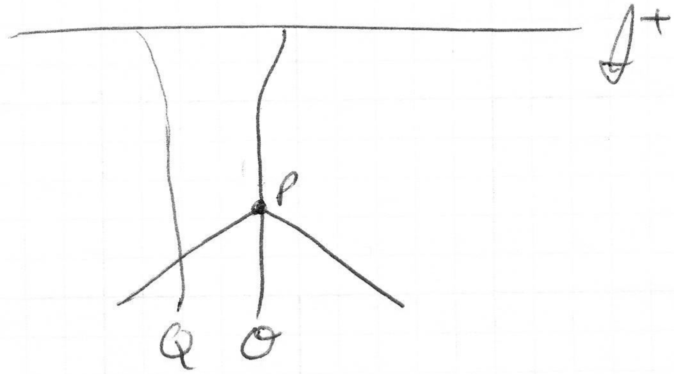
Here $ds_E^2 = -dT^2 + dR^2 + \sin^2 R \, d\Omega^2$ $\quad 0 \leq R \leq \pi \quad |T| + R < \pi$

Now $\operatorname{tg} W \operatorname{tg} V = -a^2$ is easy to draw

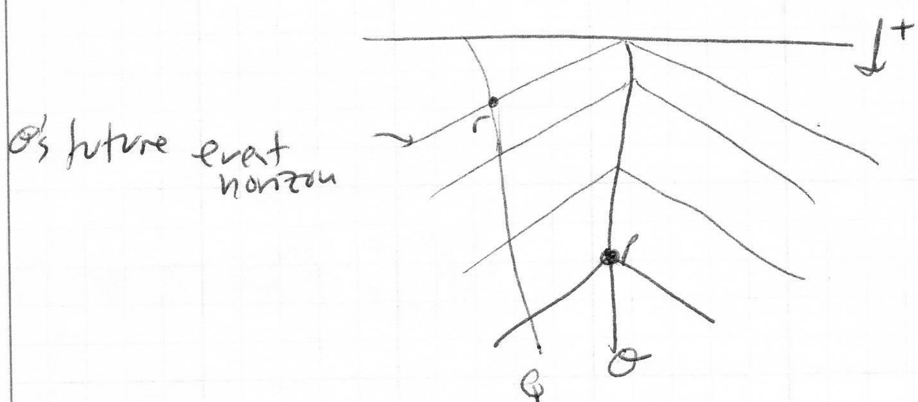


]

Consider (in de-Sitter space, or any space with $\mathscr{I}^+$ spacelike) an observer $\mathcal{O}$ and a particle worldline Q. Suppose Q intersects the past light cone of event p on $\mathcal{O}$:



$\Rightarrow$ Q is observable to $\mathcal{O}$ at any time after p:



But note, there is a point r on Q that lies on $\mathcal{O}$'s future event horizon $\Rightarrow$ Events on Q after r are NOT observable to $\mathcal{O}$.

Since r is seen at $\mathscr{I}^+$, it takes ∞ proper time from any event on $\mathcal{O}$ until observation of r on $\mathcal{O}$.

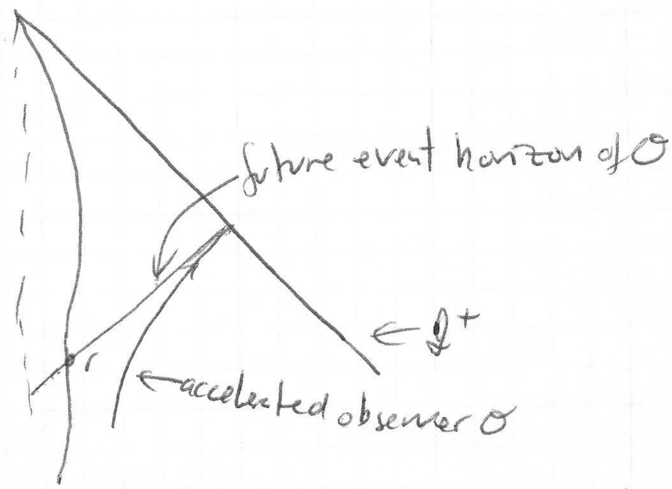
On Q, of course, it takes finite proper time from any past event to r.

It takes an infinite time in $\mathcal{O}$ to see a finite part of Q's history

(eg, $\mathcal{O}$ observes infinite redshift of light from Q as it approaches r).

Likewise, Q will see finite history of $\mathcal{O}$ in infinite time.

Even in Minkowski space if we have non-geodesic observers:



which seems perfectly logical (redshifted light from accelerated light source); light from r appears ∞ redshifted as $\mathcal{O} \to \mathscr{I}^+$).

anti-de Sitter space

($R < 0$ case) we now will have $\Lambda = \frac{1}{4} R < 0$.

Consider hyperboloid

$$-u^2 - w^2 + x^2 + y^2 + z^2 = -\alpha^2$$

in flat R^5 with $--+++$ signature

$$ds^2 = -du^2 - dw^2 + dx^2 + dy^2 + dz^2$$

(compare signs with de-Sitter! both w^2 & a^2 (& dw^2) flipped).

Let

$$u = \alpha \sin t' \cosh \rho$$

$$w = \alpha \cos t' \cosh \rho$$

$$\left.\begin{aligned} x &= \alpha \sinh \rho \sin\theta \cos\phi \\ y &= \alpha \sinh\rho \sin\theta \sin\phi \\ z &= \alpha \sinh\rho \cos\theta \end{aligned}\right\} \text{ spherical coordinates in } R^3 \text{ with radius } \alpha \sinh \rho$$

This defines a map from the Hyperboloid H^4, to R^5

$$\varphi : H^4 \to R^5$$

with induced metric $\varphi^* g$ (pull back of g).

Then $$ds^2 = \alpha^2 \left[-\cosh^2\rho \, dt'^2 + d\rho^2 + \sinh^2\rho \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]$$

Exercise: Check this

$$\Big[\frac{1}{\alpha^2} ds^2 = -dt'^2 \left\{ \cosh^2\rho \left(\cos^2 t' + \sin^2 t' \right) \right. + d\rho^2 \left[-\sinh^2\rho \left(\sin^2 t' + \cos^2 t' \right) + \cosh^2\rho \left(\cos^2\theta + \sin^2\theta \left(s_\varphi^2 + c_\varphi^2 \right) \right) \right]$$

$$+ \sinh^2\rho \, d\theta^2 \left[\sin^2\theta + \cos^2\theta \left(s_\varphi^2 + c_\varphi^2 \right) \right] + \sin^2\theta \, d\phi^2 \Big] \checkmark$$

Note that with $\rho \geq 0$ a radius-like coordinate, the $t' =$ constant sections are R^3 (topologically).

But for ρ, θ, ϕ fixed, t' lines are periodic $t' \to t' + 2\pi$

$\to$ Space has closed timelike curves (a no-no). (maybe... see later, causality).

Another coordinate system:

$$U = \alpha \sin t$$

$$V = \alpha \cos t \cosh r$$

$$x = \alpha \cos t \, \sinh r \sin\theta \cos\varphi$$
$$y = \alpha \cos t \, \sinh r \sin\theta \sin\varphi$$
$$z = \alpha \cos t \, \sinh r \cos\theta$$

Now $\varphi^* g$ is

$$\Big[\frac{1}{\alpha^2} ds^2 = \left(-\cos^2 t - \sin^2 t\left(\cosh^2 r - \sinh^2 r(\cos^2\theta + \sin^2\theta)\right)\right) dt^2$$
$$+ \left(+\cos^2 t(-\sinh^2 r + \cosh^2 r(-\cdot))\right) dr^2 + \cos^2 t \sinh^2 r \, (\sin^2\theta + \cos^2) d\theta^2 - $$

$$\frac{1}{\alpha^2} ds^2 = -dt^2 + \cos^2 t \left[dr^2 + \sinh^2 r \, d\Omega_2^2 \right]$$

As we'll see this system has simple geodesics: $(r, \theta, \varphi) =$ constant. So these lines are orthogonal to $t =$ constant surface.

But note that at $t = \pm \frac{1}{2}\pi$ there are singularities. Clearly these are only coordinate singularities, but this frame can only be used for one piece of the space.

So the space described so far is one with topology $S^1 \times R^3$.

We take de-Sitter space to be the universal covering space of this, meaning, take $t' \in [-\infty, \infty]$ and keep the metric as above (the embedding no longer makes sense).

Structure at infinity and Penrose diagram: Let's define (similar to the de-Sitter case)

$$\cos\chi = \frac{1}{\cosh\rho}$$

[so $\sinh\rho \, d\rho = \frac{\sin\chi}{\cos^2\chi} d\chi$

$$\Rightarrow (-1+\cosh^2\rho)\,d\rho^2 = \frac{\sin^2\chi}{\cos^4\chi} d\chi^2 \qquad -1+\frac{1}{\cos^2\chi} = \frac{1-\cos^2\chi}{\cos^2\chi} = \tan^2\chi$$

$$\Rightarrow d\rho^2 = \frac{\cos^2\chi}{\sin^2\chi}\frac{\sin^2\chi}{\cos^4\chi} d\chi^2 = \frac{1}{\cos^2\chi} d\chi^2$$

and $ds^2 = \alpha^2 \left[-\frac{1}{\cos^2\chi} dt'^2 + \frac{1}{\cos^2\chi} d\chi^2 + \tan^2\chi \, d\Omega_2^2 \right]$]

which has $\chi \in [0, \frac{\pi}{2})$ and

$$ds^2 = \frac{\alpha^2}{\cos^2\chi}\left[-dt'^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2 \right] = \frac{\alpha^2}{\cos^2\chi} d\bar{s}^2$$

recognizing again the metric of Einstein-static universe.

Note that with $t' \in (-\infty, \infty)$ but $\chi \in [0, \frac{\pi}{2}]$ anti-de Sitter is conformally related to half of the Einstein-static universe (the $\chi \in [\frac{\pi}{2}, \pi]$ is missing).

Geodesics in anti de Sitter (not for class)

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Find geodesics? Start $\Gamma_{\mu\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu})$

$\Gamma_{\rho\rho\rho} = 0$

$\Gamma_{tt\rho} = \Gamma_{t\rho t} = -\frac{1}{2}(\cosh^2\rho)_{,\rho} = -\cosh\rho\sinh\rho \quad \Rightarrow \Gamma^t_{t\rho} = \Gamma^t_{\rho t} = \frac{\sinh\rho}{\cosh\rho}$

$\Gamma_{\rho tt} = \cosh\rho\sinh\rho \quad \Rightarrow \Gamma^\rho_{tt} = \cosh\rho\sinh\rho$

$\Gamma_{\theta\theta\rho} = \Gamma_{\theta\rho\theta} = \frac{1}{2}(\sinh^2\rho)_{,\rho} = \cosh\rho\sinh\rho \quad \Rightarrow \Gamma^\theta_{\theta\rho} = \Gamma^\theta_{\rho\theta} = \frac{\cosh\rho}{\sinh\rho}$

$\Gamma_{\rho\theta\theta} = -\cosh\rho\sinh\rho \quad \Rightarrow \Gamma^\rho_{\theta\theta} = -\cosh\rho\sinh\rho$

Ignore ϕ: always look at $\phi = $ const plane (could have done that w/ θ?).
Then

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

To be sure, let's keep ϕ:

$\Gamma_{\phi\phi\rho} = \Gamma_{\phi\rho\phi} = \frac{1}{2}\sin^2\theta\; 2\sinh\rho\cosh\rho = \sin^2\theta\sinh\rho\cosh\rho \quad \Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = \frac{\cosh\rho}{\sinh\rho}$

$\Gamma_{\rho\phi\phi} = -\sin^2\theta\sinh\rho\cosh\rho \quad \Gamma^\rho_{\phi\phi} = -\sin^2\theta\sinh\rho\cosh\rho$

$\Gamma_{\phi\phi\theta} = \sin\theta\cos\theta\sinh^2\rho \quad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

$\Gamma_{\theta\phi\phi} = -\sin\theta\cos\theta\sinh^2\rho \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

Conserved quantities $\quad g_{t\mu}\frac{dx^\mu}{d\tau} = -\cosh^2\rho\frac{dt}{d\tau} = T$

$\frac{d^2t}{d\tau^2}+$

but it works if $\Phi = 0$, see below ← not this, since g does depend on θ → $\boxed{g_{\theta\mu}\frac{dx^\mu}{d\tau} = \sinh^2\rho\frac{d\theta}{d\tau} = \Theta}$?? $\quad g_{\phi\mu}\frac{dx^\mu}{d\tau} = \sin^2\theta\sinh^2\rho\frac{d\phi}{d\tau} = \Phi$

So

$$\frac{d^2\rho}{d\tau^2} + \cosh\rho\sinh\rho\left[\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{d\theta}{d\tau}\right)^2 - \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] = 0$$

$$\frac{d^2\rho}{d\tau^2} + \frac{\sinh\rho}{\cosh^3\rho}T^2 - \frac{\cosh\rho}{\sinh^3\rho}\Theta^2 - \frac{\cosh\rho}{\sin^2\theta\sinh^3\rho}\Phi^2 = 0$$

This equation has a 1^{st} integral that is easy to find. But, even easier, use τ = proper time so

$$g_{\mu\nu} U^\mu U^\nu = -1$$

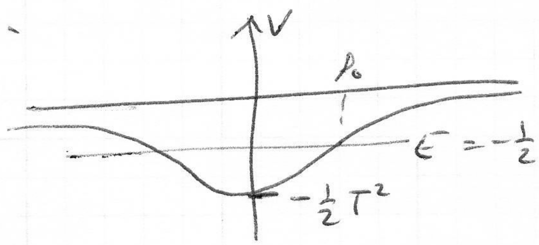
or

$$\left(\frac{d\rho}{d\tau}\right)^2 - \frac{T^2}{\cosh^2\rho} + \frac{\Theta^2}{\sinh^2\rho} + \frac{\Phi^2}{\sinh^2\rho \sin^2\theta} = -1$$

Look for solutions with $\Theta = \Phi = 0$. Then

$$\frac{d\rho}{d\tau} = \sqrt{\frac{T^2}{\cosh^2\rho} - 1} \qquad (\star)$$

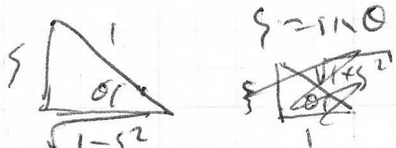
This is like motion in a potential $-\frac{T^2}{2\cosh^2\rho}$ with total energy $-\frac{1}{2}$.



And clearly there are "bound state" solutions, with turning points at $\cosh^2\rho_0 = T^2$ or $\rho_0 = \operatorname{arccosh} T$. Now, it is easy to integrate $(\star)$

$$\int \frac{d\rho}{\sqrt{\frac{T^2}{\cosh^2\rho} - 1}} = \int \frac{\cosh\rho \, d\rho}{\sqrt{T^2 - \cosh^2\rho}} = \int \frac{d \sinh\rho}{\sqrt{T^2 - 1 - \sinh^2\rho}}$$

Let $\sinh\rho = \sqrt{T^2-1}\,\xi \Rightarrow = \int \frac{d\xi}{\sqrt{1-\xi^2}}$



$\xi = \sin\theta$

$$\Rightarrow \int \frac{\cos\theta \, d\theta}{\cos\theta} = \theta = \arcsin\xi = \operatorname{arctg}\frac{\xi}{\sqrt{1-\xi^2}}$$

$$= \arcsin\left(\frac{\sinh\rho}{\sqrt{T^2-1}}\right)$$

or $\operatorname{arctg}\left(\frac{\sinh\rho}{\sqrt{T^2-1-\sinh^2\rho}}\right) = \operatorname{arctg}\left(\frac{\tanh\rho}{\sqrt{\frac{T^2}{\cosh^2\rho} - 1}}\right)$

Then $t(\tau)$ is obtained from

$$\frac{dt}{d\tau} = -\frac{T}{\cosh^2\rho}$$

For this we need

$$\boxed{\sin\tau = \frac{\sinh\rho}{\sqrt{T^2-1}}}$$

or $(T^2-1)\sin^2\tau = \sinh^2\rho = \cosh^2\rho - 1$

so

$$\frac{dt}{d\tau} = -\frac{T}{1+(T^2-1)\sin^2\tau}$$

We need

$$\int \frac{d\tau}{1+k^2\sin^2\tau} = \frac{tg^{-1}\left[\sqrt{1+k^2}\, tg\,\tau\right]}{\sqrt{1+k^2}}$$ (mathematica)

so

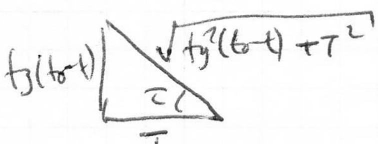
$$-\frac{(t-t_0)}{T} = \frac{1}{\sqrt{1+(T^2-1)}}\, \mathrm{arctg}\left[T\, tg\,\tau\right]$$

or

$$\boxed{-tg(t-t_0) = T\, tg\,\tau}$$

(The sign is because ~~T(o) is proper distance, not proper time~~).

We can also obtain the trajectory. Since $tg\,\tau = \frac{1}{T}\, tg(t_0-t)$



$$\Rightarrow \sin\tau = \frac{tg(t_0-t)}{\sqrt{tg^2(t_0-t)+T^2}} = \frac{1}{\sqrt{1+T^2 c\,tg^2(t_0-t)}}$$

so

$$\boxed{\frac{1}{\sqrt{1+T^2 c\,tg^2(t_0-t)}} = \frac{\sinh\rho}{\sqrt{T^2-1}}}$$

In all these it's worth remembering $T = -\cosh\rho_0$

Check the θ piece (recall $g = g(\rho)$ so we were justified in using $g_{\theta\mu}\frac{dx^\mu}{d\tau}$ = constant)

Now

$$\frac{d^2\theta}{d\tau^2} + 2\frac{\cosh\rho}{\sinh\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

But if ϕ = constant ($\Phi = 0$) we have

$$\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) + 2\frac{\cosh\rho}{\sinh\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} = 0$$

Now, check: $\frac{d\theta}{d\tau} = \frac{\Theta}{\sinh^2\rho}$ gives $\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) = -2\frac{\cosh\rho}{\sinh^3\rho}\Theta\frac{d\rho}{d\tau}$

while the 2nd term is $2\frac{\cosh}{\sinh\rho}\frac{d\rho}{d\tau}\frac{\Theta}{\sinh^2\rho}$

so they cancel ✓

Connecting both coordinates systems:

in (t, r, θ, ϕ) system geodesics are $r, \theta, \phi = $ constant with $r = \rho_0$

Comparing both systems:

u: $\sin t' \cosh\rho = \sin t$

v: $\cos t' \cosh\rho = \cos t \cosh r$

z: $\sinh\rho = \cos t \sinh r$

u/v: $\operatorname{tg} t' = \frac{1}{\cosh r} \operatorname{tg} t$

(θ, φ remain the same)

Geodesics
$$\begin{cases} \sinh\rho = \sinh\rho_0 \sin\tau \\ \operatorname{tg}(t' - t_0) = \cosh\rho_0 \operatorname{tg}\tau \end{cases}$$

Going to other system:

$$\sinh\rho_0 \sin\tau = \cos t \sinh r$$

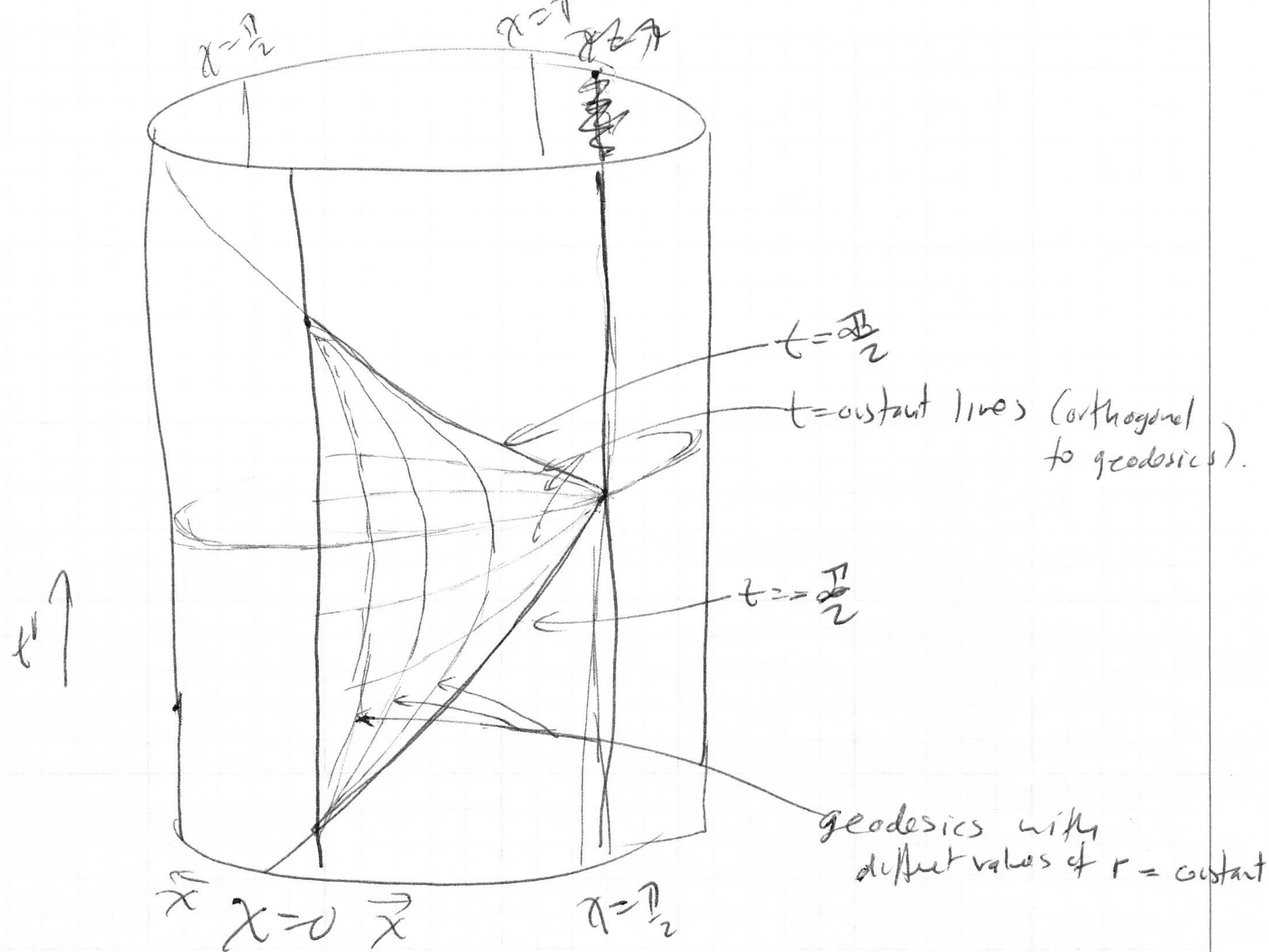
$$\tau \to \tau + \frac{\pi}{2} \qquad \rho_0 = r$$

and then

$$\operatorname{tg}(t' - t_0) = \cosh\rho_0 \operatorname{tg}\left(\tau + \frac{\pi}{2}\right) = \cosh\rho_0 \frac{\cos\tau}{-\sin\tau}$$

$$\operatorname{ctg}(t' - t_0) = -\frac{1}{\cosh\rho_0} \operatorname{tg}\tau$$

$\Rightarrow t_0 = \frac{\pi}{2}$ $\Rightarrow$ it works



(The lines $t=\pm\frac{\pi}{2}$ are easy to understand. Since

$$\sin t = \sin t' \cosh\rho$$

we have

$$\pm 1 = \sin t' \cosh\rho = \sin t' \frac{1}{\cos\chi}$$

where we introduced the variable χ for the conformal mapping

$\Rightarrow$

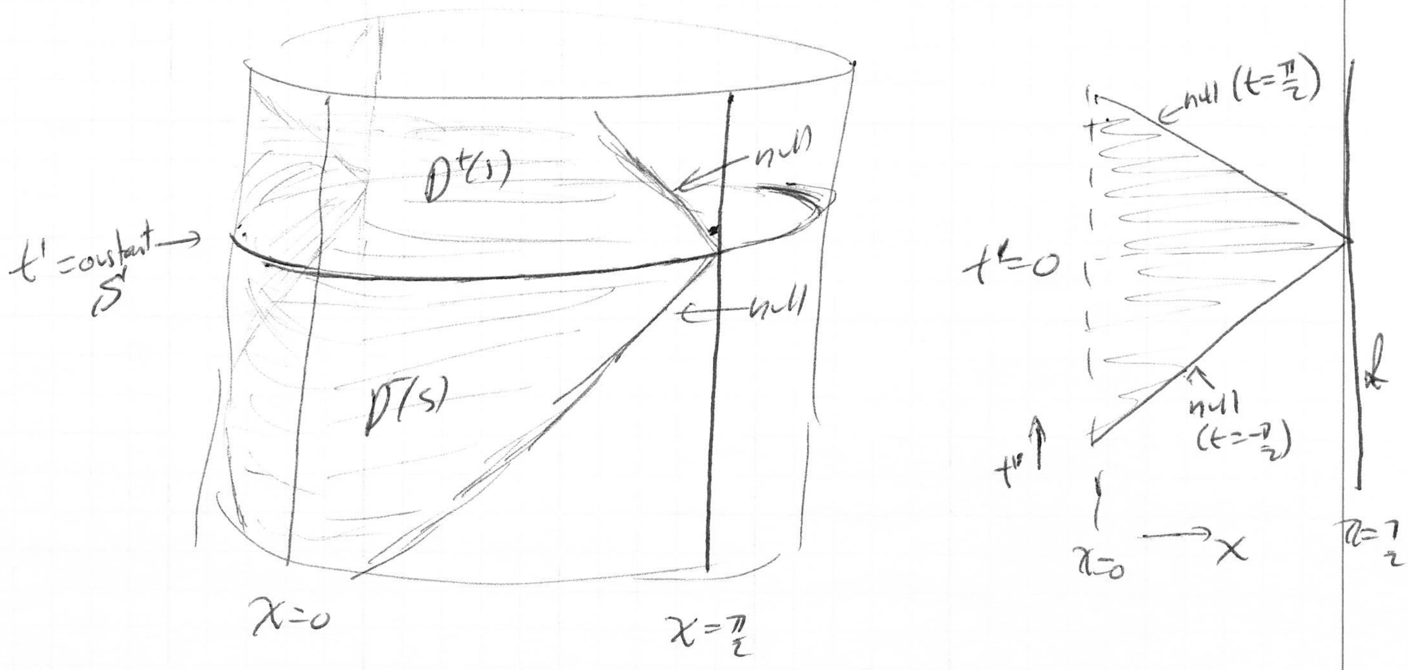
$$\cos\chi = \pm\sin t'$$

or $\chi = \frac{\pi}{2} \pm t'$).

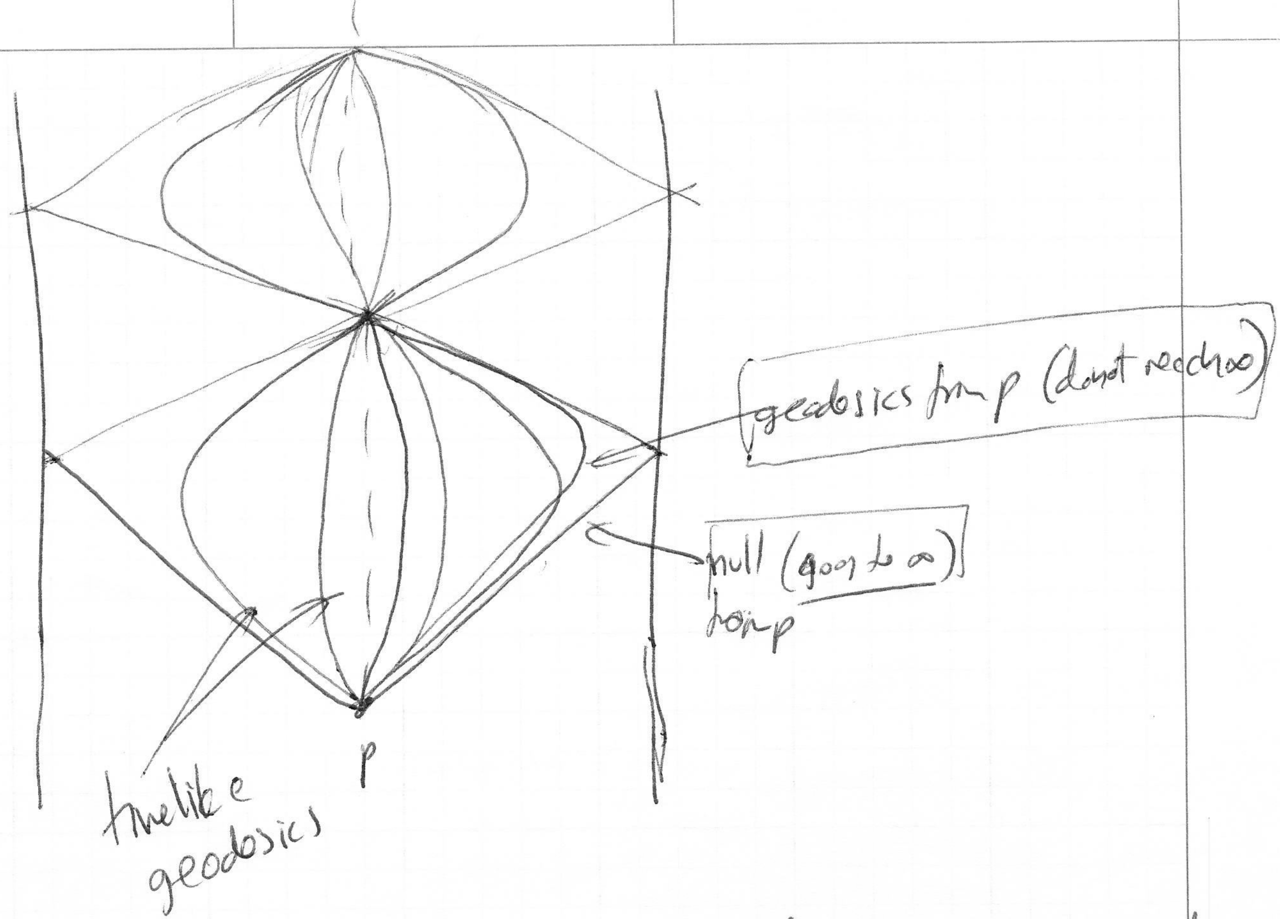
Note that the apparent singularity in t, r, θ, φ coord's is related to convergence of geodesics.

Causal structure of anti-de Sitter space:

NO CAUCHY SURFACE !

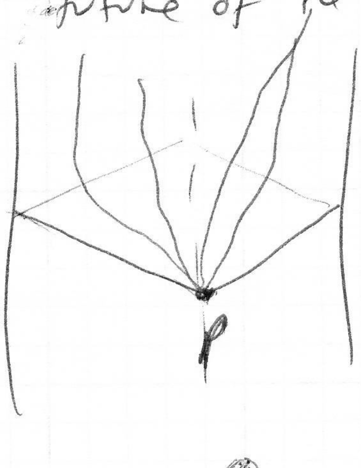


Evident ⇒ information flows in/out from boundary at ∞.

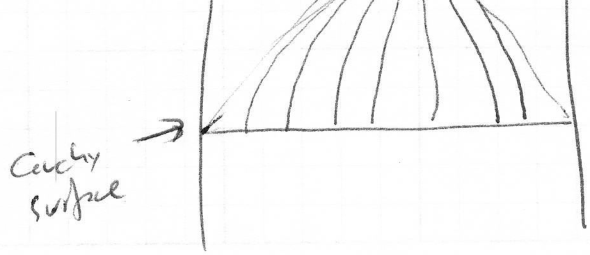


timelike geodesics from p are confined to infinite sequence of diamonds

But there are timelike curves (non-geodesic) from p that can reach any point future of the null-cone from p.



Also



every point in $D^+(S)$ can be reached by a unique geodesic from S, normal to S.