Physics 211B : Solution Set #2

[**1**] *Rectangular Barrier* – Consider a symmetric planar barrier consisting of a layer of $\mathrm{Al}_x\mathrm{Ga}_{1-x}\mathrm{As}$ of width $2a$ imbedded in GaAs. The barrier height V_0 is simply the difference between conduction band minima ΔE_{c} at the Γ point; energies are defined relative to E_Γ^{GaAs}. Derive the $\mathcal{S}$-matrix for this problem. Show that

$$T(E) = \frac{1}{1 + \left[\frac{\sinh\left(b\sqrt{1-\eta}\right)}{2\sqrt{\eta(1-\eta)}}\right]^2} \qquad\qquad (\eta \le 1)$$

and

$$T(E) = \frac{1}{1 + \left[\frac{\sin\left(b\sqrt{\eta-1}\right)}{2\sqrt{\eta(\eta-1)}}\right]^2} \qquad\qquad (\eta \ge 1)\ ,$$

where $\eta = E/V_0$ and $b = a/\ell$ with $\ell = \hbar/\sqrt{2m^* V_0}$. Sketch $T(E)$ *versus* E/V_0 for various values of the dimensionless thickness b.

<u>Solution:</u> Let the barrier extend from $x = 0$ to $x = d \equiv 2a$. The energy is

$$E = \frac{\hbar^2 k^2}{2m^*} = \frac{\hbar^2 q^2}{2m^*} + V_0\ .$$

Thus, with $\eta = E/V_0$, and $\ell = \hbar/\sqrt{2m^* V_0}$, the wavevectors k and q outside and inside the barrier region are given by $k = \ell^{-1}\sqrt{\eta}$ and $q = \ell^{-1}\sqrt{\eta - 1}$, respectively.

The wavefunction in the three regions is written

$$\begin{aligned}
\psi(x) &= A\, e^{ikx} + B\, e^{-ikx} && (x \le 0)\\
&= C\, e^{iqx} + D\, e^{-iqx} && (0 \le x \le d)\\
&= E\, e^{ikx} + F\, e^{-ikx} && (d \le x)\ .
\end{aligned}$$

Matching the wavefunction and its derivative at the points $x = 0$ and $x = d$ gives four equations in the six unknowns A, B, C, D, E, and F:

$$\begin{aligned}
A + B &= C + D\\
k(A - B) &= q(C - D)\\
C\, e^{iqd} + D\, e^{-iqd} &= E\, e^{ikd} + F\, e^{-ikd}\\
q\big(C\, e^{iqd} + D\, e^{-iqd}\big) &= k\big(E\, e^{ikd} - F\, e^{-ikd}\big).
\end{aligned}$$

Solving the first two equations for C and D yields

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ q & -q \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ k & -k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

The bottom pair says

$$\begin{pmatrix} E \\ F \end{pmatrix} = \begin{pmatrix} e^{ikd} & e^{-ikd} \\ k\, e^{ikd} & -k\, e^{ikd} \end{pmatrix}^{-1} \begin{pmatrix} e^{iqd} & e^{-iqd} \\ q\, e^{iqd} & -q\, e^{-iqd} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}\ .$$

Thus, the transfer matrix for this problem is

$$\begin{aligned}
\mathcal{M} &= \frac{1}{4kq}\begin{pmatrix} k\,e^{-ikd} & e^{-ikd} \\ k\,e^{ikd} & -e^{ikd}\end{pmatrix}\begin{pmatrix} e^{iqd} & e^{-iqd} \\ q\,e^{iqd} & -q\,e^{-iqd}\end{pmatrix}\begin{pmatrix} q & 1 \\ q & -1\end{pmatrix}\begin{pmatrix} 1 & 1 \\ k & -k\end{pmatrix} \\
&= \frac{1}{4kq}\begin{pmatrix} (k+q)^2\,e^{-i(k-q)d} - (k-q)^2\,e^{-i(k+q)d} & -2i\,(k^2-q^2)\,e^{-ikd}\,\sin(qd) \\ 2i\,(k^2-q^2)\,e^{ikd}\,\sin(qd) & (k+q)^2\,e^{i(k-q)d} - (k-q)^2\,e^{i(k+q)d}\end{pmatrix} \\
&= \begin{pmatrix} 1/t^* & -r^*/t^* \\ -r/t' & 1/t'\end{pmatrix} \ .
\end{aligned}$$

Thus,

$$t^* = \frac{4kq\,e^{ikd}}{(k+q)^2\,e^{iqd} - (k-q)^2\,e^{-iqd}}$$

and (see sketch in figure 1):

$$\begin{aligned}
T(E) = |t|^2 &= \frac{1}{1 + \left(\frac{k^2-q^2}{2kq}\right)^2 \sin^2(qd)} \\
&= \frac{1}{1 + \left[\frac{\sin\left(2b\sqrt{\eta-1}\right)}{2\sqrt{\eta(\eta-1)}}\right]^2} \qquad (\eta \ge 1) \\
&= \frac{1}{1 + \left[\frac{\sinh\left(2b\sqrt{1-\eta}\right)}{2\sqrt{\eta(1-\eta)}}\right]^2} \qquad (\eta \le 1) \ .
\end{aligned}$$

[2] *Multichannel Scattering* – Consider a multichannel scattering process defined by the Hamiltonian matrix

$$\mathcal{H}_{ij} = \bigg(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \varepsilon_i\bigg)\delta_{ij} + \Omega_{ij}\,\delta(x) \ ,$$

which describes the scattering among N channels by a δ-function impurity at $x = 0$. The matrix Ω_{ij} allows a particle in channel j passing through $x = 0$ to be scattered into channel i. The $\{\varepsilon_i\}$ are the internal (transverse) energies for the various channels. For $x \ne 0$, we can write the channel j component of the wavefunction as

$$\begin{aligned}
\psi_j(x) &= I_j\,e^{ik_jx} + O'_j\,e^{-ik_jx} \qquad && (x < 0) \\
&= O_j\,e^{ik_jx} + I'_j\,e^{-ik_jx} \qquad && (x > 0) \ ,
\end{aligned}$$

where the k_j are positive and determined by

$$\varepsilon_{\rm F} = \frac{\hbar^2 k_j^2}{2m} + \varepsilon_j \ .$$

Show that the incoming and outgoing flux amplitudes are related by a $2N \times 2N$ $\mathcal{S}$-matrix:

$$\begin{pmatrix} \sqrt{v}\,O' \\ \sqrt{v}\,O\end{pmatrix} = \overbrace{\begin{pmatrix} r & t' \\ t & r'\end{pmatrix}}^{\mathcal{S}}\begin{pmatrix} \sqrt{v}\,I \\ \sqrt{v}\,I'\end{pmatrix}$$

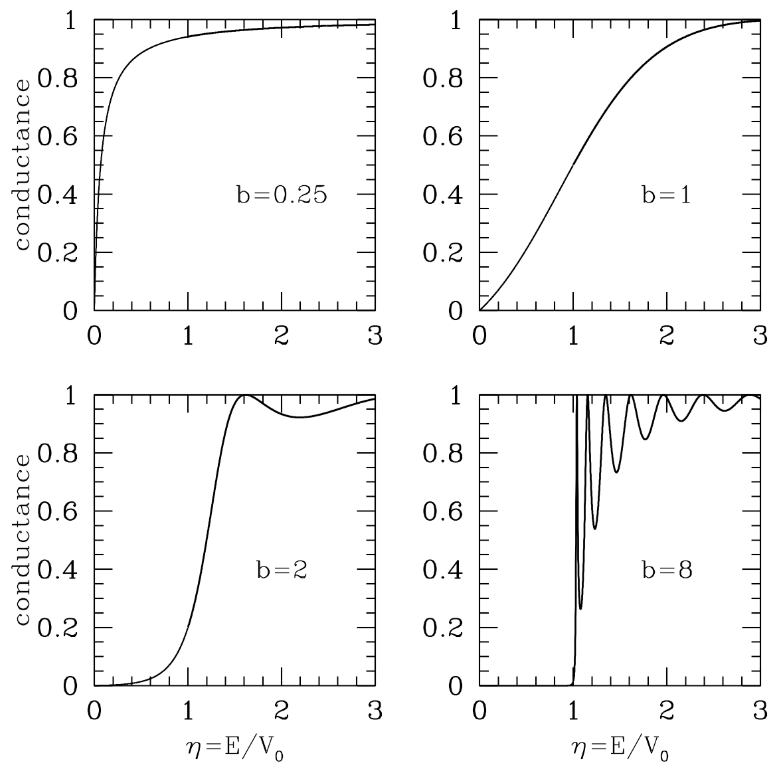

Figure 1: Dimensionless barrier conductance *versus* incident energy for a set of thickness parameters.

where $v = \mathrm{diag}(v_1, \ldots, v_N)$ with $v_i = \hbar k_i/m > 0$. Find explicit expressions for the component $N \times N$ blocks r, t, t', r', and show that $\mathcal{S}$ is unitary, *i.e.* $\mathcal{S}^\dagger \mathcal{S} = \mathcal{S}\mathcal{S}^\dagger = \mathbb{I}$.

Solution: Continuity of the wavefunction at $x = 0$ requires

$$I_j + O'_j = O_j + I'_j \ .$$

Integrating the Schrödinger equation from $x = 0^-$to $x = 0^+$ yields

$$-\frac{\hbar^2}{2m}\Big[\psi'_i(0^+) - \psi'_i(0^-)\Big] + \Omega_{ij}\,\psi_j(0) = 0 \ ,$$

which is equivalent to

$$(i\hbar V + \Omega)_{ij}\,(I_j + I'_j) = (i\hbar V - \Omega)_{ij}\,(O_j + O'_j) \ ,$$

with $V_{ij} = v_i\,\delta_{ij}$. Thus,

$$\begin{pmatrix} 1 & -1 \\ i\hbar V - \Omega & i\hbar V - \Omega \end{pmatrix} \begin{pmatrix} O' \\ O \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ i\hbar V + \Omega & i\hbar V + \Omega \end{pmatrix} \begin{pmatrix} I \\ I' \end{pmatrix} \ .$$

If A is any $N \times N$ matrix, then

$$\begin{pmatrix} 1 & -1 \\ A & A \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 1 & A^{-1} \\ -1 & A^{-1} \end{pmatrix} \ .$$

Consequently,

$$\begin{pmatrix} O' \\ O \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} Q-1 & Q+1 \\ Q+1 & Q-1 \end{pmatrix} \begin{pmatrix} I \\ I' \end{pmatrix}$$

with $Q = (i\hbar V - \Omega)^{-1}(i\hbar V + \Omega)$. This immediately gives the $\mathcal{S}$-matrix as

$$\mathcal{S} = \begin{pmatrix} O' \\ O \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} \widetilde{Q}-1 & \widetilde{Q}+1 \\ \widetilde{Q}+1 & \widetilde{Q}-1 \end{pmatrix}$$

where

$$\widetilde{Q} = V^{1/2}\, Q\, V^{-1/2} = \big(1 + i\hbar^{-1}\,\widetilde{\Omega}\big)^{-1}\big(1 - i\hbar^{-1}\,\widetilde{\Omega}\big) \ ,$$

with $\widetilde{\Omega} = V^{-1/2}\,\Omega\, V^{-1/2}$. Note that the product in the above equation may be taken in either order, as the two factors commute. Since $\widetilde{\Omega} = \widetilde{\Omega}^\dagger$ is Hermitian, $\widetilde{Q}$ is unitary, which in turn guarantees the unitarity of $\mathcal{S}$:

$$\mathcal{S}^\dagger \mathcal{S} = \tfrac{1}{2} \begin{pmatrix} \widetilde{Q}^\dagger\widetilde{Q}+1 & \widetilde{Q}^\dagger\widetilde{Q}-1 \\ \widetilde{Q}^\dagger\widetilde{Q}-1 & \widetilde{Q}^\dagger\widetilde{Q}+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

[3] *Spin Valve* – Consider a barrier between two halves of a ferromagnetic metallic wire. For $x < 0$ the magnetization lies in the $\hat{\boldsymbol{z}}$ direction, while for $x > 0$ the magnetization is directed along the unit vector $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The Hamiltonian is given by

$$\mathcal{H} = -\frac{\hbar^2}{2m^*}\frac{d^2}{dx^2} + \mu_{\rm B} \boldsymbol{H}_{\rm int} \cdot \boldsymbol{\sigma} \ ,$$

where $\boldsymbol{H}_{\rm int}$ is the (spontaneously generated) internal magnetic field and $\mu_{\rm B} = e\hbar/2m_{\rm e}c$ is the Bohr magneton[1]. The magnetization $\boldsymbol{M}$ points along $\boldsymbol{H}_{\rm int}$[2]. For $x < 0$ we therefore have

$$E_{\rm F} = \frac{\hbar^2 k_\uparrow^2}{2m^*} + \Delta = \frac{\hbar^2 k_\downarrow^2}{2m^*} - \Delta \ ,$$

where $\Delta = \mu_{\rm B} H_{\rm int}$. A similar relation holds for the Fermi wavevectors corresponding to spin states $\big|\, \hat{\boldsymbol{n}}\, \big\rangle$ and $\big|\, -\hat{\boldsymbol{n}}\, \big\rangle$ in the region $x > 0$.

Consider the $\mathcal{S}$-matrix for this problem. The 'in' and 'out' states should be defined as local eigenstates, which means that they have different spin polarization axes for $x < 0$ and $x > 0$. Explicitly, for $x < 0$ we write

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} = \Big\{ A_\uparrow\, e^{ik_\uparrow x} + B_\uparrow\, e^{-ik_\uparrow x} \Big\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Big\{ A_\downarrow\, e^{ik_\downarrow x} + B_\downarrow\, e^{-ik_\downarrow x} \Big\} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ ,$$

[1] Note that it is the bare electron mass $m_{\rm e}$ which appears in the formula for $\mu_{\rm B}$ and *not* the effective mass m^*!).

[2] For weakly magnetized systems, the magnetization is $\boldsymbol{M} = \mu_{\rm B}^2\, g(\varepsilon_{\rm F})\, \boldsymbol{H}_{\rm int}$, where $g(\varepsilon_{\rm F})$ is the total density of states per unit volume at the Fermi energy.

while for $x > 0$ we write

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} = \Big\{ C_\uparrow \, e^{ik_\uparrow x} + D_\uparrow \, e^{-ik_\uparrow x} \Big\} \begin{pmatrix} u \\ v \end{pmatrix} + \Big\{ C_\downarrow \, e^{ik_\downarrow x} + D_\downarrow \, e^{-ik_\downarrow x} \Big\} \begin{pmatrix} -v^* \\ u \end{pmatrix} \ ,$$

where $u = \cos(\theta/2)$ and $v = \sin(\theta/2)\, \exp(i\phi)$. The $\mathcal{S}$-matrix relates the *flux amplitudes* of the in-states and out-states:

$$\begin{pmatrix} b_\uparrow \\ b_\downarrow \\ c_\uparrow \\ c_\downarrow \end{pmatrix} = \overbrace{\begin{pmatrix} r_{11} & r_{12} & t'_{11} & t'_{12} \\ r_{21} & r_{22} & t'_{21} & t'_{22} \\ t_{11} & t_{12} & r'_{11} & r'_{12} \\ t_{21} & t_{22} & r'_{21} & r'_{22} \end{pmatrix}}^{\mathcal{S}} \begin{pmatrix} a_\uparrow \\ a_\downarrow \\ d_\uparrow \\ d_\downarrow \end{pmatrix} \ .$$

Derive the 2×2 transmission matrix t (you don't have to derive the entire $\mathcal{S}$-matrix) and thereby obtain the dimensionless conductance $g = \mathrm{Tr}\,(t^\dagger t)$. Define the polarization P by

$$P = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} \ ,$$

where $n_\sigma = k_\sigma/\pi$ is the electronic density. Find $g(P, \theta)$.

<u>**Solution:**</u> Continuity of the wavefunction and its derivatives at $x = 0$ yields four equations, conveniently written in matrix form:

$$\begin{pmatrix} 1 & 0 & -u & v^* \\ 0 & 1 & -v & -u \\ k_\uparrow & 0 & k_\uparrow u & -k_\downarrow v \\ 0 & k_\downarrow & k_\uparrow v & k_\downarrow u \end{pmatrix} \begin{pmatrix} B_\uparrow \\ B_\downarrow \\ C_\uparrow \\ C_\downarrow \end{pmatrix} = \begin{pmatrix} -1 & 0 & u & -v^* \\ 0 & -1 & v & u \\ k_\uparrow & 0 & k_\uparrow u & -k_\downarrow v \\ 0 & k_\downarrow & k_\uparrow v & k_\downarrow u \end{pmatrix} \begin{pmatrix} A_\uparrow \\ A_\downarrow \\ D_\uparrow \\ D_\downarrow \end{pmatrix} \ .$$

Defining the 2×2 blocks,

$$\Sigma \equiv \begin{pmatrix} u & -v^* \\ v & u \end{pmatrix} \qquad , \qquad K \equiv \begin{pmatrix} k_\uparrow & 0 \\ 0 & k_\downarrow \end{pmatrix} \ ,$$

we have

$$\begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 1 & -\Sigma \\ K & \Sigma K \end{pmatrix}^{-1} \begin{pmatrix} -1 & \Sigma \\ K & \Sigma K \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix} \ .$$

Converting to flux amplitudes, we have

$$\mathcal{S} = \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \sqrt{K} \end{pmatrix} \begin{pmatrix} 1 & -\Sigma \\ K & \Sigma K \end{pmatrix}^{-1} \begin{pmatrix} -1 & \Sigma \\ K & \Sigma K \end{pmatrix} \begin{pmatrix} \sqrt{K^{-1}} & 0 \\ 0 & \sqrt{K^{-1}} \end{pmatrix} \ .$$

We now invoke the general result

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & (C - DB^{-1}A)^{-1} \\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

to obtain the blocks of $\mathcal{S}$:

$$\begin{aligned}
r &= K^{1/2}\Big\{\big(1+K^{-1}\Sigma K\Sigma^{-1}\big)^{-1} - \big(1+\Sigma K^{-1}\Sigma^{-1}K\big)^{-1}\Big\}K^{-1/2}\\
t' &= 2K^{1/2}\big(\Sigma^{-1}+K^{-1}\Sigma^{-1}K\big)^{-1}K^{-1/2}\\
t &= 2K^{1/2}\big(\Sigma+K^{-1}\Sigma K\big)^{-1}K^{-1/2}\\
r' &= K^{1/2}\Big\{\big(1+K^{-1}\Sigma^{-1}K\Sigma\big)^{-1} - \big(1+\Sigma^{-1}K^{-1}\Sigma K\big)^{-1}\Big\}K^{-1/2} \ .
\end{aligned}$$

We find

$$t = \frac{1}{u^2+|v|^2\,\cosh^2\!y}\begin{pmatrix} u & v^*\,\cosh y \\ -v\cosh y & u\end{pmatrix}$$

with $y=\frac{1}{2}\ln(k_\uparrow/k_\downarrow)$. The dimensionless conductance is

$$g(P,\theta) = \mathrm{Tr}\,(t^\dagger t) = \frac{2}{u^2+|v|^2\,\cosh^2\!y} = \frac{2\,(1-P^2)}{(1-P^2)\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta} \ ,$$

where P is the polarization. Note that $g(P=\pm 1,\theta)=0$, since it is impossible to match boundary conditions on the lower components. One can also compute the reflection matrix,

$$r = \frac{\sinh y\,\sin\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta\,\cosh^2\!y}\begin{pmatrix} \cos\frac{1}{2}\theta & \cosh y\,\sin\frac{1}{2}\theta\,e^{-i\phi} \\ -\cosh y\,\sin\frac{1}{2}\theta\,e^{i\phi} & \cos\frac{1}{2}\theta\end{pmatrix} \ .$$

[4] *Distribution of Resistances of a One-Dimensional Wire* – In this problem you are asked to derive an equation governing the probability distribution $P(\mathcal{R},L)$ for the dimensionless resistance $\mathcal{R}$ of a one-dimensional wire of length L. The equation is called the Fokker-Planck equation. Here's a brief primer on how to derive Fokker-Planck equations.

Suppose $x(t)$ is a stochastic variable. We define the quantity

$$\delta x(t) \equiv x(t+\delta t) - x(t) \ , \tag{1}$$

and we assume

$$\begin{aligned}
\big\langle \delta x(t)\big\rangle &= F_1\big(x(t)\big)\,\delta t\\
\big\langle \big[\delta x(t)\big]^2\big\rangle &= 2\,F_2\big(x(t)\big)\,\delta t
\end{aligned}$$

but $\big\langle \big[\delta x(t)\big]^n\big\rangle = \mathcal{O}\big((\delta t)^2\big)$ for $n>2$. The $n=1$ term is due to *drift* and the $n=2$ term is due to *diffusion*. Now consider the conditional probability density, $P(x,t\,|\,x_0,t_0)$, defined to be the probability distribution for $x\equiv x(t)$ given that $x(t_0)=x_0$. The conditional probability density satisfies the composition rule,

$$P(x,t\,|\,x_0,t_0) = \int\limits_{-\infty}^{\infty}\! dx'\,P(x,t\,|\,x',t')\,P(x',t'\,|\,x_0,t_0) \ ,$$

for any value of t'. Therefore, we must have

$$P(x,t+\delta t\,|\,x_0,t_0)=\int\limits_{-\infty}^{\infty}\!dx'\,P(x,t+\delta t\,|\,x',t)\,P(x',t\,|\,x_0,t_0)\ .$$

Now we may write

$$\begin{aligned}P(x,t+\delta t\,|\,x',t)&=\big\langle\delta\big(x-x'-\delta x(t)\big)\big\rangle\\&=\bigg\{1+\big\langle\delta x(t)\big\rangle\,\frac{d}{dx'}+\tfrac{1}{2}\big\langle\big[\delta x(t)\big]^2\big\rangle\,\frac{d^2}{dx'^2}+\ldots\bigg\}\,\delta(x-x')\ ,\end{aligned}$$

where the average is over the random variables. Upon integrating by parts and expanding to $\mathcal{O}(\delta t)$, we obtain the Fokker-Planck equation,

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\big[F_1(x)\,P(x,t)\big]+\frac{\partial^2}{\partial x^2}\big[F_2(x)\,P(x,t)\big]\ .$$

That wasn't so bad, now was it?

For our application, $x(t)$ is replaced by $\mathcal{R}(L)$. We derived the composition rule for series quantum resistors in class:

$$\begin{aligned}\mathcal{R}(L+\delta L)=\mathcal{R}(L)&+\mathcal{R}(\delta L)+2\,\mathcal{R}(L)\,\mathcal{R}(\delta L)\\&-2\cos\beta\,\sqrt{\mathcal{R}(L)\,\big[1+\mathcal{R}(L)\big]\,\mathcal{R}(\delta L)\,\big[1+\mathcal{R}(\delta L)\big]}\ ,\end{aligned}$$

where β is a random phase. For small values of δL, we needn't worry about quantum interference and we can use our Boltzmann equation result. Show that

$$\mathcal{R}(\delta L)=\frac{e^2}{h}\,\frac{m^*}{ne^2\tau}\,\delta L=\frac{\delta L}{2\ell}\ ,$$

where $\ell=v_{\rm F}\tau$ is the elastic mean free path. (Assume a single spin species throughout.)

Find the drift and diffusion functions $F_1(\mathcal{R})$ and $F_2(\mathcal{R})$. Show that the distribution function $P(\mathcal{R},L)$ obeys the equation

$$\frac{\partial P}{\partial L}=\frac{1}{2\ell}\,\frac{\partial}{\partial\mathcal{R}}\bigg\{\mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial\mathcal{R}}\bigg\}\ .$$

Show that this equation may be solved in the limits $\mathcal{R}\ll 1$ and $\mathcal{R}\gg 1$, with

$$P(\mathcal{R},z)=\frac{1}{z}\,e^{-\mathcal{R}/z}$$

for $\mathcal{R}\ll 1$, and

$$P(\mathcal{R},z)=(4\pi z)^{-1/2}\,\frac{1}{\mathcal{R}}\,e^{-(\ln\mathcal{R}-z)^2/4z}$$

for $\mathcal{R}\gg 1$, where $z=L/2\ell$ is the dimensionless length of the wire. Compute $\langle\mathcal{R}\rangle$ in the former case, and $\langle\ln\mathcal{R}\rangle$ in the latter case.

Solution: We have

$$\begin{aligned}\mathcal{R}(\delta L) &= \frac{e^2}{h}\,\rho\,\delta L = \frac{e^2}{h}\,\frac{m^*}{ne^2\tau}\,\delta L = \frac{e^2}{h}\,\frac{m^* v_{\rm F}}{ne^2\ell}\,\delta L \\ &= \frac{k_{\rm F}}{2\pi n}\,\frac{\delta L}{\ell} = \frac{\delta L}{2\ell}\ .\end{aligned}$$

From the composition rule for series quantum resistances, we derive the phase averages

$$\begin{aligned}\big\langle \delta\mathcal{R}\big\rangle &= \Big(1+2\,\mathcal{R}(L)\Big)\frac{\delta L}{2\ell} \\ \big\langle (\delta\mathcal{R})^2\big\rangle &= \Big(1+2\,\mathcal{R}(L)\Big)^2\left(\frac{\delta L}{2\ell}\right)^2 + 2\,\mathcal{R}(L)\,\Big(1+\mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell}\left(1+\frac{\delta L}{2\ell}\right) \\ &= 2\,\mathcal{R}(L)\,\Big(1+\mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell} + \mathcal{O}\big((\delta L)^2\big)\ ,\end{aligned}$$

whence we obtain the drift and diffusion terms

$$F_1(\mathcal{R}) = \frac{2\,\mathcal{R}+1}{2\ell} \qquad , \qquad F_2(\mathcal{R}) = \frac{\mathcal{R}(1+\mathcal{R})}{2\ell}\ .$$

Note that $F_1(\mathcal{R}) = dF_2/d\mathcal{R}$, which allows us to write the Fokker-Planck equation as

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial \mathcal{R}}\left\{\frac{\mathcal{R}\,(1+\mathcal{R})}{2\ell}\,\frac{\partial P}{\partial \mathcal{R}}\right\}\ .$$

Defining the dimensionless length $z = L/2\ell$, we have

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}}\left\{\mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}}\right\}\ .$$

In the limit $\mathcal{R} \ll 1$, this reduces to

$$\frac{\partial P}{\partial z} = \mathcal{R}\,\frac{\partial^2 P}{\partial \mathcal{R}^2} + \frac{\partial P}{\partial \mathcal{R}}\ ,$$

which is satisfied by $P(\mathcal{R},z) = z^{-1}\,\exp(-\mathcal{R}/z)$. In the opposite limit, $\mathcal{R} \gg 1$, we have

$$\begin{aligned}\frac{\partial P}{\partial z} &= \mathcal{R}^2\,\frac{\partial^2 P}{\partial \mathcal{R}^2} + 2\,\mathcal{R}\,\frac{\partial P}{\partial \mathcal{R}} \\ &= \frac{\partial^2 P}{\partial \nu^2} + \frac{\partial P}{\partial \nu}\ ,\end{aligned}$$

where $\nu \equiv \ln\mathcal{R}$. This is solved by the log-normal distribution,

$$P(\mathcal{R},z) = (4\pi z)^{-1/2}\,e^{-(\nu+z)^2/4z}\ .$$

Note that

$$P(\mathcal{R},z)\,d\mathcal{R} = (4\pi z)^{-1/2}\,\exp\left\{-\frac{(\ln\mathcal{R}-z)^2}{4z}\right\}d\ln\mathcal{R}\ .$$