Physics 211B : Assignment #2

[**1**] *Rectangular Barrier* – Consider a symmetric planar barrier consisting of a layer of $\mathrm{Al}_x\mathrm{Ga}_{1-x}\mathrm{As}$ of width $2a$ imbedded in GaAs. The barrier height V_0 is simply the difference between conduction band minima ΔE_{c} at the Γ point; energies are defined relative to E_Γ^{GaAs}. Derive the $\mathcal{S}$-matrix for this problem. Show that

$$T(E) = \frac{1}{1 + \left[\frac{\sinh\left(b\sqrt{1-\eta}\right)}{2\sqrt{\eta(1-\eta)}}\right]^2} \qquad (\eta \le 1)$$

and

$$T(E) = \frac{1}{1 + \left[\frac{\sin\left(b\sqrt{\eta-1}\right)}{2\sqrt{\eta(\eta-1)}}\right]^2} \qquad (\eta \ge 1) \ ,$$

where $\eta = E/V_0$ and $b = a/\ell$ with $\ell = \hbar/\sqrt{2m^* V_0}$. Sketch $T(E)$ *versus* E/V_0 for various values of the dimensionless thickness b.

[**2**] *Multichannel Scattering* – Consider a multichannel scattering process defined by the Hamiltonian matrix

$$\mathcal{H}_{ij} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \varepsilon_i \right)\delta_{ij} + \Omega_{ij}\,\delta(x) \ ,$$

which describes the scattering among N channels by a δ-function impurity at $x = 0$. The matrix Ω_{ij} allows a particle in channel j passing through $x = 0$ to be scattered into channel i. The $\{\varepsilon_i\}$ are the internal (transverse) energies for the various channels. For $x \neq 0$, we can write the channel j component of the wavefunction as

$$\begin{aligned} \psi_j(x) &= I_j\, e^{ik_j x} + O'_j\, e^{-ik_j x} && (x < 0) \\ &= O_j\, e^{ik_j x} + I'_j\, e^{-ik_j x} && (x > 0) \ , \end{aligned}$$

where the k_j are positive and determined by

$$\varepsilon_{\mathrm{F}} = \frac{\hbar^2 k_j^2}{2m} + \varepsilon_j \ .$$

Show that the incoming and outgoing flux amplitudes are related by a $2N \times 2N$ $\mathcal{S}$-matrix:

$$\begin{pmatrix} \sqrt{v}\, O' \\ \sqrt{v}\, O \end{pmatrix} = \overbrace{\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}}^{\mathcal{S}} \begin{pmatrix} \sqrt{v}\, I \\ \sqrt{v}\, I' \end{pmatrix}$$

where $v = \mathrm{diag}(v_1, \ldots, v_N)$ with $v_i = \hbar k_i / m > 0$. Find explicit expressions for the component $N \times N$ blocks r, t, t', r', and show that $\mathcal{S}$ is unitary, *i.e.* $\mathcal{S}^\dagger \mathcal{S} = \mathcal{S}\mathcal{S}^\dagger = \mathbb{I}$.

[**3**] *Spin Valve* – Consider a barrier between two halves of a ferromagnetic metallic wire. For $x < 0$ the magnetization lies in the $\hat{\boldsymbol{z}}$ direction, while for $x > 0$ the magnetization is

directed along the unit vector $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The Hamiltonian is given by

$$\mathcal{H} = -\frac{\hbar^2}{2m^*}\frac{d^2}{dx^2} + \mu_{\rm B}\boldsymbol{H}_{\rm int}\cdot\boldsymbol{\sigma} \ ,$$

where $\boldsymbol{H}_{\rm int}$ is the (spontaneously generated) internal magnetic field and $\mu_{\rm B} = e\hbar/2m_{\rm e}c$ is the Bohr magneton[1]. The magnetization $\boldsymbol{M}$ points along $\boldsymbol{H}_{\rm int}$[2]. For $x < 0$ we therefore have

$$E_{\rm F} = \frac{\hbar^2 k_\uparrow^2}{2m^*} + \Delta = \frac{\hbar^2 k_\downarrow^2}{2m^*} - \Delta \ ,$$

where $\Delta = \mu_{\rm B} H_{\rm int}$. A similar relation holds for the Fermi wavevectors corresponding to spin states $\big|\, \hat{\boldsymbol{n}}\,\big\rangle$ and $\big|\, -\hat{\boldsymbol{n}}\,\big\rangle$ in the region $x > 0$.

Consider the $\mathcal{S}$-matrix for this problem. The 'in' and 'out' states should be defined as local eigenstates, which means that they have different spin polarization axes for $x < 0$ and $x > 0$. Explicitly, for $x < 0$ we write

$$\begin{pmatrix}\psi_\uparrow(x)\\ \psi_\downarrow(x)\end{pmatrix} = \Big\{A_\uparrow\, e^{ik_\uparrow x} + B_\uparrow\, e^{-ik_\uparrow x}\Big\}\begin{pmatrix}1\\0\end{pmatrix} + \Big\{A_\downarrow\, e^{ik_\downarrow x} + B_\downarrow\, e^{-ik_\downarrow x}\Big\}\begin{pmatrix}0\\1\end{pmatrix} \ ,$$

while for $x > 0$ we write

$$\begin{pmatrix}\psi_\uparrow(x)\\ \psi_\downarrow(x)\end{pmatrix} = \Big\{C_\uparrow\, e^{ik_\uparrow x} + D_\uparrow\, e^{-ik_\uparrow x}\Big\}\begin{pmatrix}u\\v\end{pmatrix} + \Big\{C_\downarrow\, e^{ik_\downarrow x} + D_\downarrow\, e^{-ik_\downarrow x}\Big\}\begin{pmatrix}-v^*\\u\end{pmatrix} \ ,$$

where $u = \cos(\theta/2)$ and $v = \sin(\theta/2)\,\exp(i\phi)$. The $\mathcal{S}$-matrix relates the *flux amplitudes* of the in-states and out-states:

$$\begin{pmatrix}b_\uparrow\\ b_\downarrow\\ c_\uparrow\\ c_\downarrow\end{pmatrix} = \overbrace{\begin{pmatrix}r_{11} & r_{12} & t'_{11} & t'_{12}\\ r_{21} & r_{22} & t'_{21} & t'_{22}\\ t_{11} & t_{12} & r'_{11} & r'_{12}\\ t_{21} & t_{22} & r'_{21} & r'_{22}\end{pmatrix}}^{\mathcal{S}}\begin{pmatrix}a_\uparrow\\ a_\downarrow\\ d_\uparrow\\ d_\downarrow\end{pmatrix} \ .$$

Derive the 2×2 transmission matrix t (you don't have to derive the entire $\mathcal{S}$-matrix) and thereby obtain the dimensionless conductance $g = {\rm Tr}\,(t^\dagger t)$. Define the polarization P by

$$P = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} \ ,$$

where $n_\sigma = k_\sigma/\pi$ is the electronic density. Find $g(P,\theta)$.

[1] Note that it is the bare electron mass $m_{\rm e}$ which appears in the formula for $\mu_{\rm B}$ and *not* the effective mass m^*!).

[2] For weakly magnetized systems, the magnetization is $\boldsymbol{M} = \mu_{\rm B}^2\, g(\varepsilon_{\rm F})\,\boldsymbol{H}_{\rm int}$, where $g(\varepsilon_{\rm F})$ is the total density of states per unit volume at the Fermi energy.

[4] *Distribution of Resistances of a One-Dimensional Wire* – In this problem you are asked to derive an equation governing the probability distribution $P(\mathcal{R}, L)$ for the dimensionless resistance $\mathcal{R}$ of a one-dimensional wire of length L. The equation is called the Fokker-Planck equation. Here's a brief primer on how to derive Fokker-Planck equations.

Suppose $x(t)$ is a stochastic variable. We define the quantity

$$\delta x(t) \equiv x(t+\delta t) - x(t) \ , \tag{1}$$

and we assume

$$\begin{aligned} \big\langle \delta x(t) \big\rangle &= F_1\big(x(t)\big)\,\delta t \\ \big\langle \big[\delta x(t)\big]^2 \big\rangle &= 2\,F_2\big(x(t)\big)\,\delta t \end{aligned}$$

but $\big\langle \big[\delta x(t)\big]^n \big\rangle = \mathcal{O}\big((\delta t)^2\big)$ for $n > 2$. The $n = 1$ term is due to *drift* and the $n = 2$ term is due to *diffusion*. Now consider the conditional probability density, $P(x, t \,|\, x_0, t_0)$, defined to be the probability distribution for $x \equiv x(t)$ given that $x(t_0) = x_0$. The conditional probability density satisfies the composition rule,

$$P(x, t \,|\, x_0, t_0) = \int\limits_{-\infty}^{\infty} \! dx' \, P(x, t \,|\, x', t') \, P(x', t' \,|\, x_0, t_0) \ ,$$

for any value of t'. Therefore, we must have

$$P(x, t+\delta t \,|\, x_0, t_0) = \int\limits_{-\infty}^{\infty} \! dx' \, P(x, t+\delta t \,|\, x', t) \, P(x', t \,|\, x_0, t_0) \ .$$

Now we may write

$$\begin{aligned} P(x, t+\delta t \,|\, x', t) &= \big\langle \delta\big(x - x' - \delta x(t)\big) \big\rangle \\ &= \bigg\{ 1 + \big\langle \delta x(t) \big\rangle \frac{d}{dx'} + \tfrac{1}{2} \big\langle \big[\delta x(t)\big]^2 \big\rangle \frac{d^2}{dx'^2} + \ldots \bigg\} \, \delta(x - x') \ , \end{aligned}$$

where the average is over the random variables. Upon integrating by parts and expanding to $\mathcal{O}(\delta t)$, we obtain the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \big[F_1(x) \, P(x,t) \big] + \frac{\partial^2}{\partial x^2} \big[F_2(x) \, P(x,t) \big] \ .$$

That wasn't so bad, now was it?

For our application, $x(t)$ is replaced by $\mathcal{R}(L)$. We derived the composition rule for series quantum resistors in class:

$$\begin{aligned} \mathcal{R}(L+\delta L) &= \mathcal{R}(L) + \mathcal{R}(\delta L) + 2\,\mathcal{R}(L)\,\mathcal{R}(\delta L) \\ &\qquad - 2\cos\beta \sqrt{\mathcal{R}(L)\,\big[1+\mathcal{R}(L)\big]\,\mathcal{R}(\delta L)\,\big[1+\mathcal{R}(\delta L)\big]} \ , \end{aligned}$$

where β is a random phase. For small values of δL, we needn't worry about quantum interference and we can use our Boltzmann equation result. Show that

$$\mathcal{R}(\delta L) = \frac{e^2}{h}\,\frac{m^*}{ne^2\tau}\,\delta L = \frac{\delta L}{2\ell}\ ,$$

where $\ell = v_{\rm F}\tau$ is the elastic mean free path. (Assume a single spin species throughout.)

Find the drift and diffusion functions $F_1(\mathcal{R})$ and $F_2(\mathcal{R})$. Show that the distribution function $P(\mathcal{R}, L)$ obeys the equation

$$\frac{\partial P}{\partial L} = \frac{1}{2\ell}\,\frac{\partial}{\partial \mathcal{R}}\left\{\mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}}\right\}\ .$$

Show that this equation may be solved in the limits $\mathcal{R} \ll 1$ and $\mathcal{R} \gg 1$, with

$$P(\mathcal{R}, z) = \frac{1}{z}\,e^{-\mathcal{R}/z}$$

for $\mathcal{R} \ll 1$, and

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\,\frac{1}{\mathcal{R}}\,e^{-(\ln \mathcal{R} - z)^2/4z}$$

for $\mathcal{R} \gg 1$, where $z = L/2\ell$ is the dimensionless length of the wire. Compute $\langle \mathcal{R} \rangle$ in the former case, and $\langle \ln \mathcal{R} \rangle$ in the latter case.