Power of Probability
Principle of Maximum Likelihood
Weighted Averages
Linear Least Squares Fitting

Lecture # 6

Physics 2BL

Summer 2015

Principle of Maximum Likelihood

- Best estimates of X and σ from N measurements (x_1 - x_N) are those for which $Prob_{X,\sigma}$ (x_i) is a maximum

Clicker Question 8

Upon flipping a coin three times, what are the chances of three heads in a row?

(a) 1
(b) 0.5
(c) 0.25
(d) 0.125
(e) 0.0625

Clicker Question 8.5

What are the chances that two people in this room have a Birthday within one day of someone else?

(a) > 80%
(b) 60 - 80%
(c) 40 - 60%
(d) 20 - 40%
(e) < 20%

The Principle of Maximum Likelihood

Recall the probability density for measurements of some quantity x

(distributed as a Gaussian with mean X and standard deviation σ)

$$P_{X,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

Normal distribution is one example of *P(x)*.

Now, lets make repeated measurements of x to help reduce our errors.

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n$$

We define the Likelihood as the product of the probabilities. The larger L, the more likely a set of measurements is.

$$L = P(x_1)P(x_2)P(x_3)\ldots P(x_n)$$

Is L a Probability?

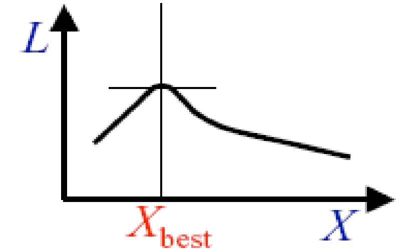
Why does max L give the best estimate?

The best estimate for the parameters of *P(x)* are those that maximize *L*.

Using the Principle of Maximum Likelihood:

Prove the mean is best estimate of X

Assume X is a parameter of $P(x)$.
When L is maximum, we must have: $\dfrac{\partial L}{\partial X} = 0$



Lets assume a Normal error distribution and find the formula for the best value for X.

$$L = P(x_1)P(x_2)\ldots P(x_n) = \prod_{i=1}^{n} P(x_i)$$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - X)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\sum_{i=1}^{n}\frac{(x_i - X)^2}{2\sigma^2}}$$

$$L = Ce^{-\chi^2/2}$$

$$\chi^2 = \sum_{i=1}^{n} \frac{(x_i - X)^2}{\sigma^2}$$ Defininition

$$\frac{\partial L}{\partial X} = 0 = Ce^{-\chi^2/2}\,\frac{-1}{2}\frac{\partial \chi^2}{\partial X}$$

$$\Rightarrow \quad \frac{\partial \chi^2}{\partial X} = 0$$

$$\frac{\partial \chi^2}{\partial X} = \frac{1}{\sigma^2}\sum_{i=1}^{n} 2(x_i - X) = 0$$

$$\sum_{i=1}^{n} (x_i - X) = 0$$

$$\sum_{i=1}^{n} x_i - nX = 0$$

$$X = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Q.E.D.

the mean

What is the Error on the Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Formula for mean of measurements. (We just proved that this is the best estimate of the true x.)

Now, use propagation of errors to get the error on the mean.

$$\sigma_{\bar{x}} = \frac{\partial \bar{x}}{\partial x_1}\sigma_{x_1} \oplus \frac{\partial \bar{x}}{\partial x_2}\sigma_{x_2} \oplus \ldots \oplus \frac{\partial \bar{x}}{\partial x_n}\sigma_{x_n}$$

$$\frac{\partial \bar{x}}{\partial x_i} = \frac{1}{n}$$

$$\sigma_{\bar{x}} = \sqrt{\sum_{i=1}^{n}\left(\frac{\sigma_{x_i}}{n}\right)^2} = \sqrt{n\left(\frac{\sigma}{n}\right)^2} = \frac{\sigma}{\sqrt{n}}$$

What would you do if the x_i had different errors?

We got the error on the mean (SDOM) by propagating errors.

Weighted averages (Chapter 7)

We can use maximum Likelihood (χ^2) to average measurements with different errors.

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - X}{\sigma_i} \right)^2$$

We derived the result that:

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Using error propagation, we can determine the error on the weighted mean:

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{\sum_{i=1}^{n} w_i}}$$

What does this give in the limit where all errors are equal?

$$\frac{\partial \chi^2}{\partial X} = 0 = -2 \sum_{i=1}^{n} \frac{x_i - X}{\sigma_i^2}$$

$$\sum_{i=1}^{n} \frac{x_i}{\sigma_i^2} - X \sum_{i=1}^{n} \frac{1}{\sigma_i^2} = 0$$

$$w_i \equiv \frac{1}{\sigma_i^2}$$

$$\sum_{i=1}^{n} w_i x_i = X \sum_{i=1}^{n} w_i$$

$$X = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Weighted averages

- $X = x = \dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$ where $w_i = \dfrac{1}{\sigma_i^2}$

- $\sigma_{wav} = \dfrac{1}{\sqrt{\sum_{i=1}^{n} w_i}}$

Example: Weighted Average

Suppose 2 students measure the radius of Neptune.

- Student A gets r=80 Mm with an error of 10 Mm and
- Student B gets r=60 Mm with an error of 3 Mm

What is the best estimate of the true radius?

$$\bar{r} = \frac{w_A r_A + w_B r_B}{w_A + w_B} = \frac{\frac{1}{100}80 + \frac{1}{9}60}{\frac{1}{100} + \frac{1}{9}} = 61.65 \text{ Mm}$$

What does this tell us about the importance of error estimates?

Clicker Question 9

Two measurements of the speed of sound give the answers:
$u_A = (332 \pm 1)$ m/s and $u_B = (339 \pm 3)$ m/s.
What is the random chance of getting two results that is ≥ this difference ?

t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.2	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.8	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.0	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.1	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.2	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.5	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.6	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.1	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.2	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80

(A) 2 %
(B) 3 %
(C) 4%
(D) 8 %
(E) 40%

a) To check if the two measurements are consistent, we compute:

$$q = u_A - u_B = 339\text{-}332 = 7 \text{ m/s}$$

and: $$\sigma_q = \sqrt{\sigma_{uA}^2 + \sigma_{uB}^2} = 3.16 \text{ m/s}$$

so that: $$t = \frac{q}{\sigma_q} = \frac{339 - 332}{3.16} = 2.21$$

From Table A we get that 2.21 sigma corresponds to: 97.21%

Therefore the probability to get a worse result is 1-97% ~3%.

Clicker Question 10

Two measurements of the speed of sound give the answers:
$u_A = (332 \pm 1)$ m/s and $u_B = (339 \pm 3)$ m/s.
What is the best estimate (weighted mean)?

(A) 336.5 ± 2 m/s
(B) 336 ± 2 m/s
(C) 336.5 ± 0.9 m/s
(D) 332.7 ± 0.9 m/s
(E) 333 ± 2 m/s

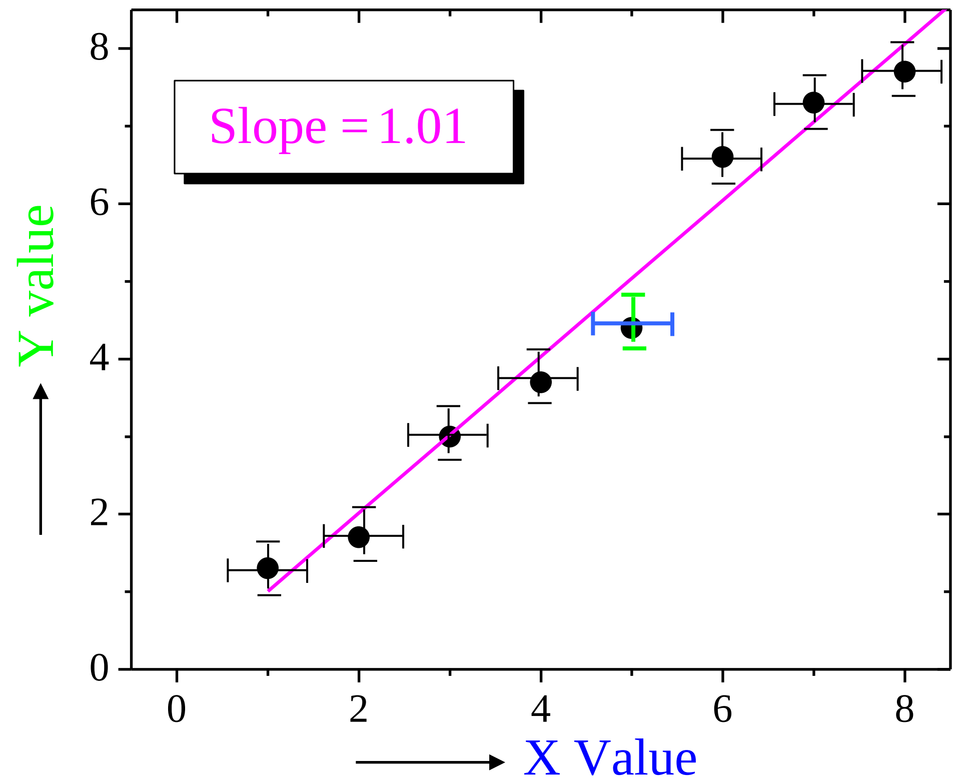
b) Best estimate is the weighted mean:

$$\bar{u} = \frac{w_A u_A + w_B u_B}{w_A + w_B} = \frac{\frac{1}{1}332 + \frac{1}{9}339}{\frac{1}{1} + \frac{1}{9}} = 332.7 \text{ m/s}$$

$$\sigma_{\bar{u}} = \frac{1}{\sqrt{1/w_A + 1/w_B}} = \frac{1}{\sqrt{1/1 + 1/9}} = 0.9 m/s$$

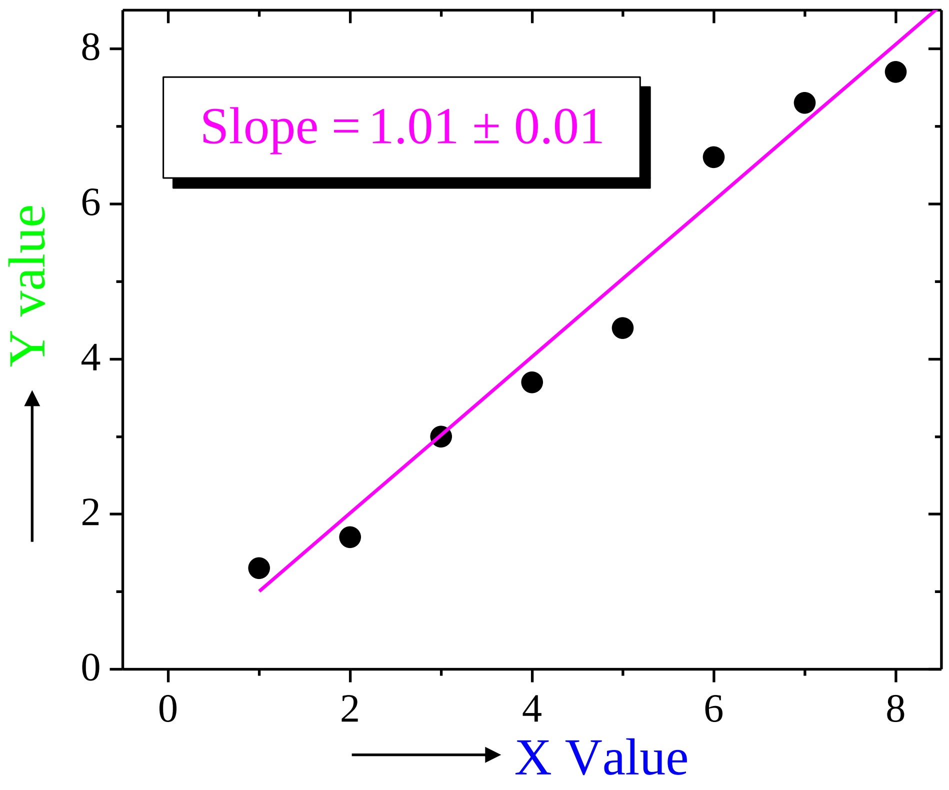
Linear Relationships: $y = A + Bx$ (Chapter 8)

- Data would lie on a straight line, except for errors
- What is 'best' line through the points?
- What is uncertainty in constants?
- How well does the relationship describe the data?



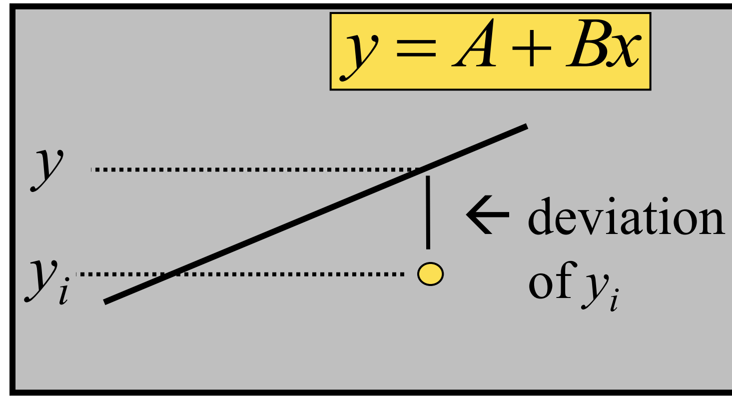
Analytical Fit

- Best means ‘minimize the square of the deviations between line and points’
- Can use error analysis to find constants, error



The Details of How to Do This (Chapter 8)

- Want to find *A, B* that minimize difference between data and line
- Since line above some data, below other, minimize sum of squares of deviations
- Find *A, B* that minimize this sum



$$y_i - y = y_i - A - Bx_i$$

$$\sum_{i=1}^{N}(y_i - A - Bx_i)^2$$

$$\frac{\partial}{\partial A} = \sum y_i - AN - B\sum x_i = 0$$

$$\frac{\partial}{\partial B} = \sum x_i y_i - A\sum x_i + B\sum x_i^{\,2} = 0$$

Finding *A* and *B*

- After minimization, solve equations for *A* and *B*

- Looks nasty, not so bad…
- See Taylor, example 8.1

$$\frac{\partial}{\partial A} = \sum y_i - AN - B\sum x_i = 0$$

$$\frac{\partial}{\partial B} = \sum x_i y_i - A\sum x_i + B\sum x_i^{\ 2} = 0$$

$$A = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{\Delta}$$

$$B = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{\Delta}$$

$$\Delta = N\sum x_i^{\ 2} - \left(\sum x_i\right)^2$$

Uncertainty in Measurements of *y*

- Before, measure several times and take standard deviation as error in *y*
- Can't now, since y_i's are different quantities
- Instead, find standard deviation of deviations

$$\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - A - Bx_i)^2}$$

Uncertainty in *A* and *B*

- *A, B* are calculated from x_i, y_i
- Know error in x_i, y_i ; use error propagation to find error in *A, B*
- A distant extrapolation will be subject to large uncertainty

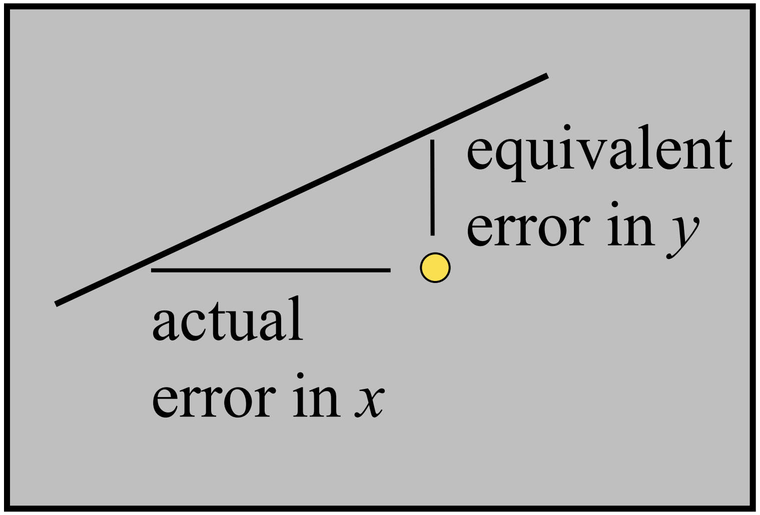
$$\sigma_A = \sigma_y \sqrt{\frac{\sum x_i^{\,2}}{\Delta}}$$

$$\sigma_B = \sigma_y \sqrt{\frac{N}{\Delta}}$$

$$\Delta = N \sum x_i^{\,2} - \left(\sum x_i\right)^2$$

Uncertainty in x

- So far, assumed negligible uncertainty in x
- If uncertainty in x, not y, just switch them
- If uncertainty in both, convert error in x to error in y, then add errors



$$\Delta y = B\Delta x$$

$$\sigma_y(equiv) = B\sigma_x$$

$$\sigma_y(equiv) = \sqrt{\sigma_y{}^2 + (B\sigma_x)^2}$$

Other Functions

- Convert to linear
- Can now use least squares fitting to get ln *A* and *B*

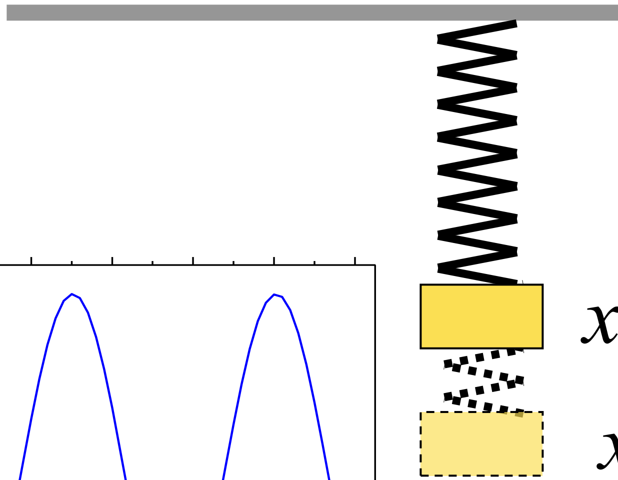
$$y = Ae^{Bx}$$
$$\ln y = \ln A + Bx$$

Experiment 3

- Goals: Test model for damping
- Model of a shock absorber in car
- Procedure: develop and demonstrate critically damped system
- check out setup, take data, do data make sense?
- Write up results - Does model work under all conditions, some conditions? Need modification?

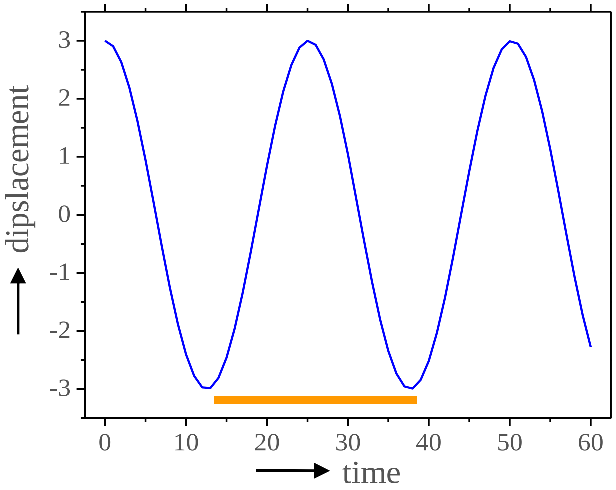
Simple Harmonic Motion

- Spring provides linear restoring force $\Rightarrow$ Mass on a spring is a harmonic oscillator



$x = x_0$

$x = 0$



$$F = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$x(t) = x_0 \cos \omega t$$

$$T = \frac{2\pi}{\omega}$$

$$\omega = \sqrt{\frac{k}{m}}$$

Damped SHM

- Consider both position and velocity dependant forces
- Behavior depends on how much damping occurs during one ‘oscillation’

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$x = x_0 \exp\left(-\frac{b}{2m}t\right)\exp\left(it\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}\right)$$

$$x = x_0 \exp\left(-\frac{b}{2m}t\right)\cos\left(t\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}\right).$$

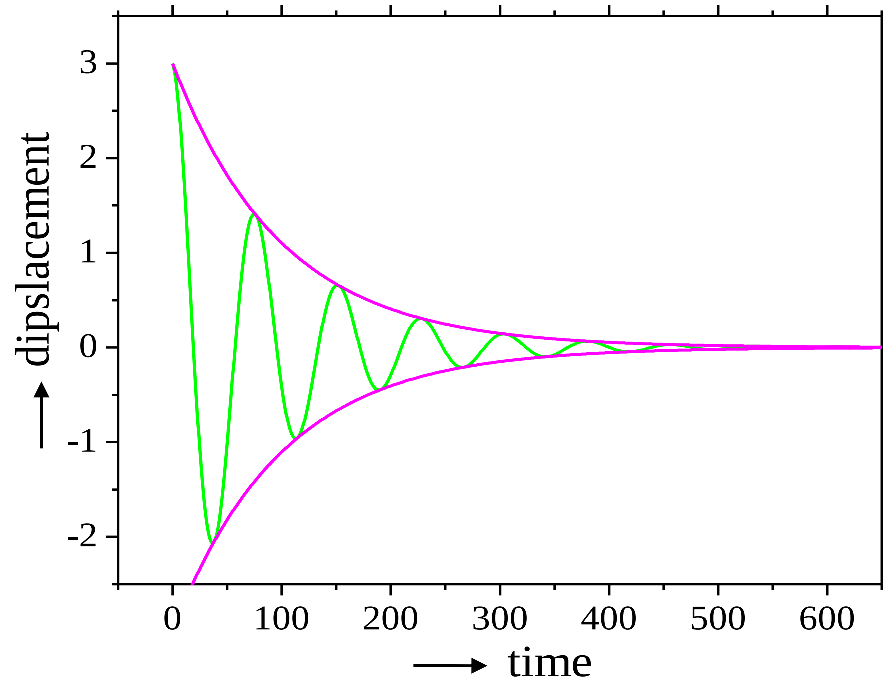
or

$$x = x_0 \exp\left(-\frac{b}{2m}+\sqrt{\frac{b^2}{4m^2}-\frac{k}{m}}\right)t$$

Relative Damping Strength: Weak damping

$$x = x_0 \exp\left(-\frac{b}{2m}t\right)\cos\left(t\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}\right).$$

$$\frac{b^2}{4m^2} << \frac{k}{m}$$



weak damping

(underdamped)

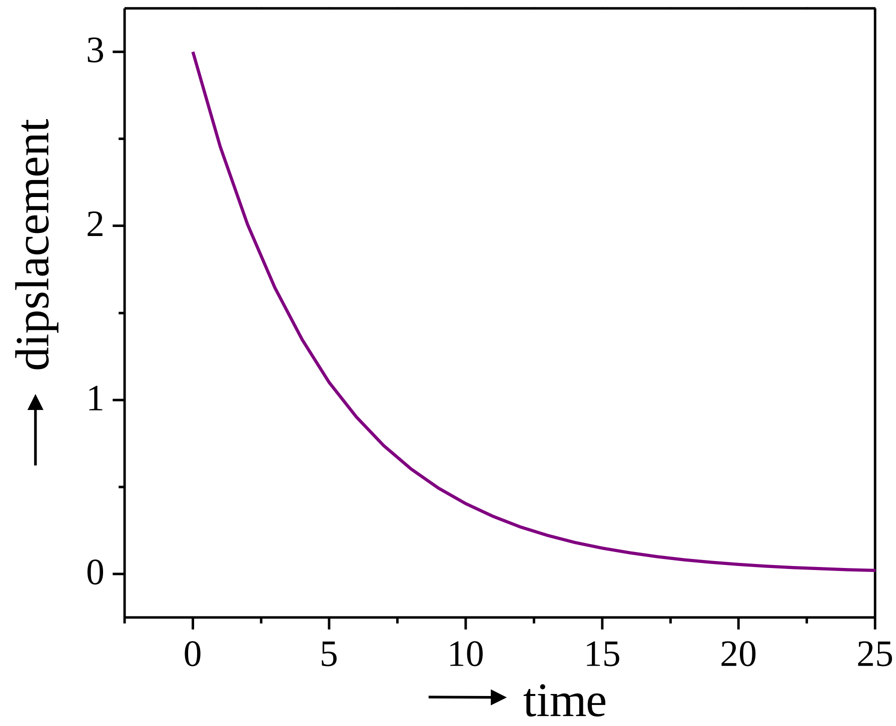
Relative Damping Strength: Strong damping

$$x = x_0 \exp\left(\left(-\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}\right)t\right)$$

$$\frac{b^2}{4m^2} >> \frac{k}{m}$$

strong damping

(overdamped)



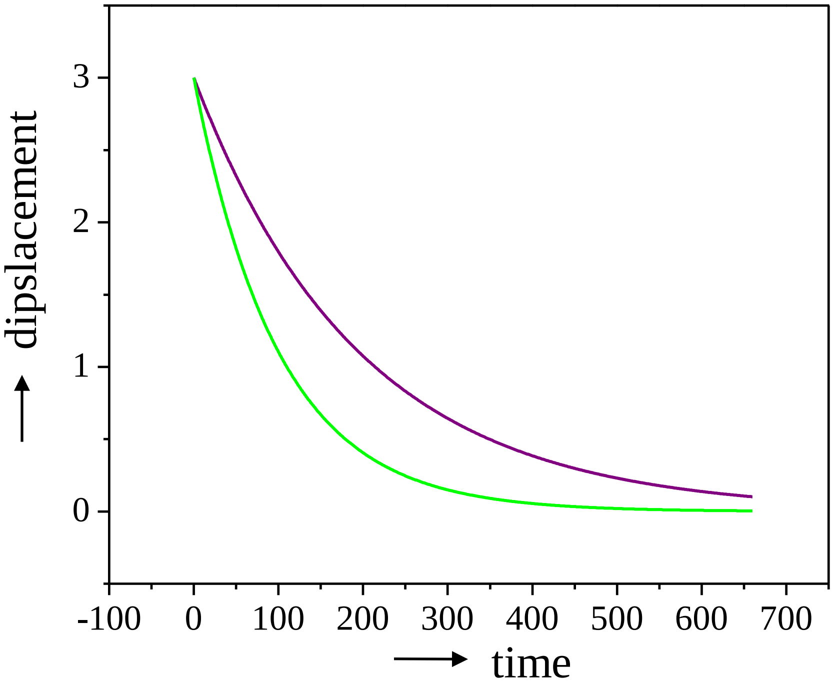
Relative Damping Strength: Critical damping

$$x = x_0 \exp\left(\left(-\frac{b}{2m} + \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right)t\right)$$

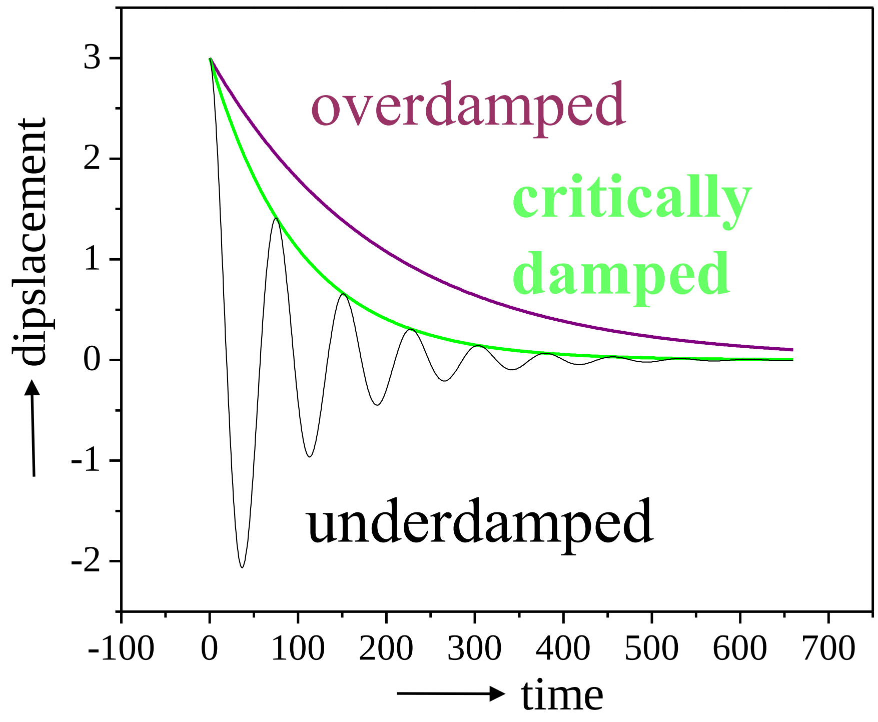
$$\frac{b^2}{4m^2} = \frac{k}{m}$$

critical damping

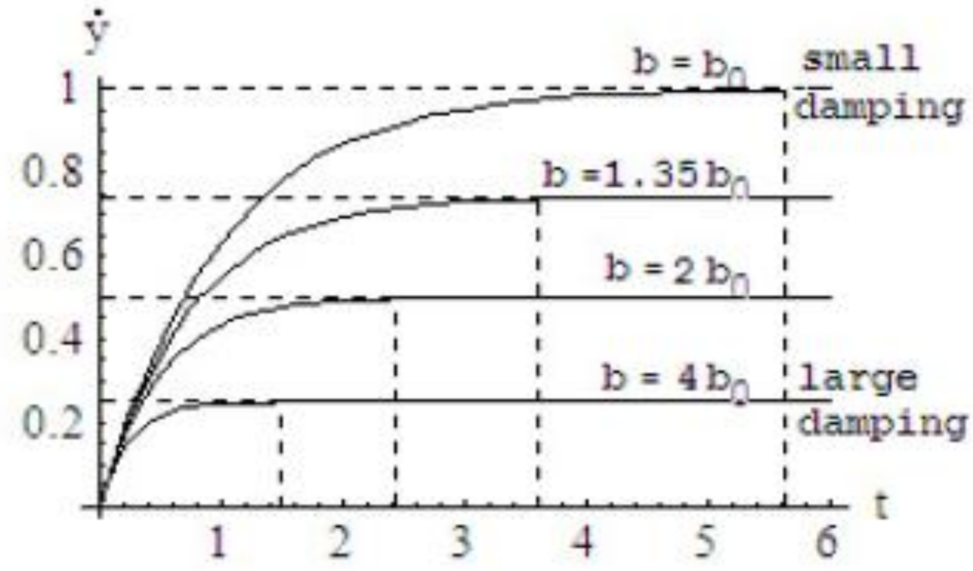
$$b_{crit} = 2\sqrt{\mathrm{mk}}$$



Comparison of the various types of damping

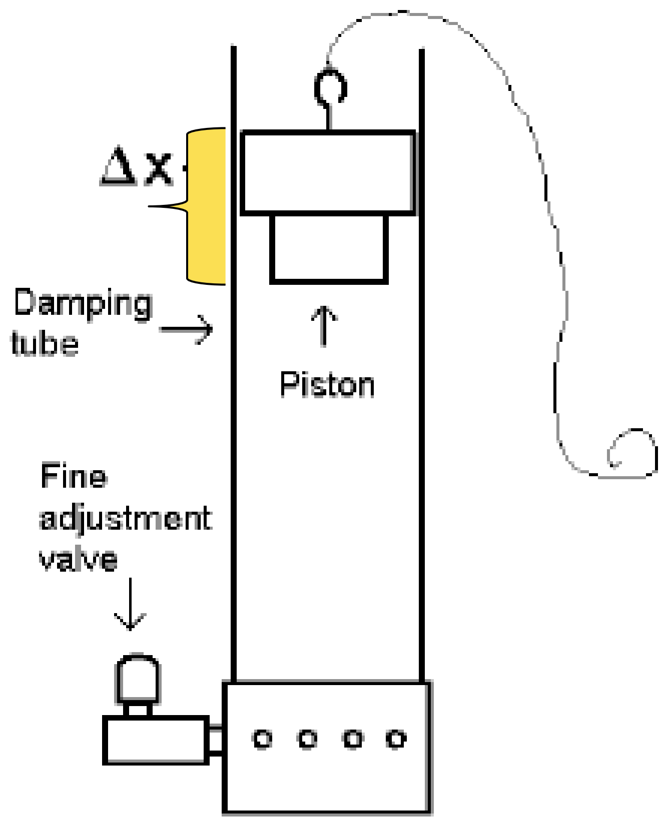


Terminal Velocity



For velocity: $\dot{y}(t) = v_t[1 - e^{-(b/m)t}]$

Experimental Setup for Falling Mass and Drag



How do you measure velocity?

Plotting Graphs

Give each graph a title

Determine independent and dependent variables

Determine boundaries

Include error bars

Demonstrate critical damping: show convincing evidence that critical damping was achieved

- Demonstrate that damping is critical
  - No oscillations (overshoot)
  - Shortest time to return to equilibrium position

Remember

- Write-up for Experiment # 3
- Homework Taylor #8.6, 8.10
  - Last assignment
- Read Taylor Chapter 12