Gaussians Distributions, Simple Harmonic Motion & Uncertainty Analysis Review

Lecture # 5

Physics 2BL

Summer 2015

Outline

- Significant figures
- Gaussian distribution and probabilities
- Experiment 2 review
- Experiment 3 intro
- Physics of damping and SHM

Clicker Question 6

What is the correct way to report 653 ± 55.4 m

(a) 653.0 ± 55.4 m
(b) 653 ± 55 m
(c) 650 ± 55 m
(d) 650 ± 60 m

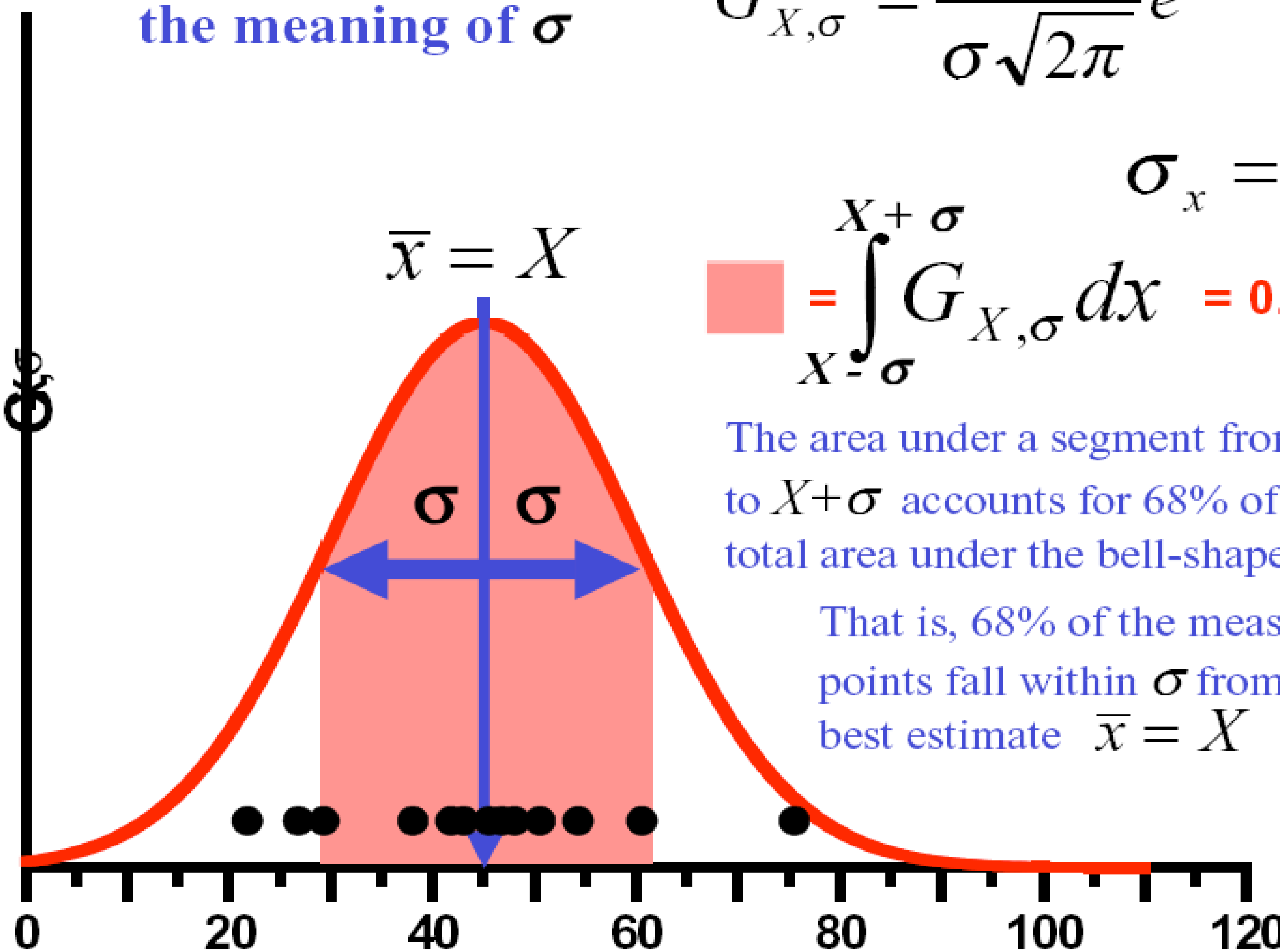
Keep one significant figure

Last sig fig of answer should be same order of magnitude as error

Gauss distribution: the meaning of σ

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$

$$\sigma_x = \sigma$$

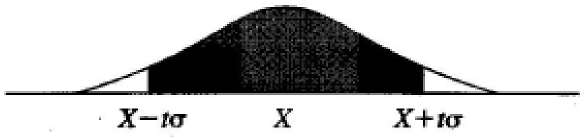


$$= \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}\, dx = 0.68$$

The area under a segment from $X - \sigma$ to $X + \sigma$ accounts for 68% of the total area under the bell-shaped curve.

That is, 68% of the measured points fall within σ from the best estimate $\bar{x} = X$

Table A. The percentage probability, $Prob(\text{within } t\sigma) = \int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)\,dx$, as a function of t.



t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.2	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.8	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.0	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.1	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.2	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
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1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.1	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.2	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80

$t = 1$

p. 287 Taylor

Compatibility of a measured result(s): t-score

- Best estimate of x: $x_{best} \pm \sigma_{\overline{X}}$

- Compare with expected answer x_{exp} and compute t-score:

$$t \equiv \frac{\left| x_{best} - x_{\exp ected} \right|}{\sigma_X}$$

- This is the number of standard deviations that x_{best} differs from x_{exp}.
- Therefore,the probability of obtaining an answer that differs from x_{exp} by t or more standard deviations is:

Prob(outside tσ) = 1-Prob(within tσ))

Example problem

Measure wavelength λ four times:

479 ± 10 nm

485 ± 8 nm

466 ± 20 nm

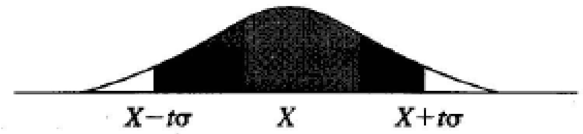
570 ± 20 nm

Should we reject the last data point?

$$t_{sus} = \Delta\lambda = \frac{|570 - 500| \text{ nm}}{\sqrt{47^2 + 40^2} \text{ nm}} = 1.37\ \sigma$$

Prob of λ outside Δλ =

Table A. The percentage probability, $Prob(\text{within } t\sigma) = \int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)\,dx$, as a function of t.



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Example problem

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Should we reject the last data point?

$$t_{sus} = \Delta\lambda = \frac{\left|570 - 500\right| \text{ nm}}{\sqrt{47^2 + 40^2} \text{ nm}} = 1.37\,\sigma$$

Prob of λ outside Δλ = 100 % - 82.9 % = 17.1 %

Total Prob = N x Prob = 4 * 17.1 % = 68.4 %

Is Total Prob < 50 % ?

NO, therefore CANNOT reject data point

Clicker Question 5

Suppose you roll the ball down the ramp 5 times and measure the rolling times to be [3.092 s, 3.101 s, 3.098 s, 3.095 s, 4.056 s]. For this set, the average is 3.288 s and the standard deviation is 0.4291 s. According to Chauvenet's criterion, would you be justified in rejecting the time measurement t = 4.056 s?

(A) Yes
(B) No
(C) Give your partner a time-out

(A) t-score = (4.056 s – 3.288 s) / 0.4291 s = 1.78 σ
(B) Prob within t-score = 92.5
(C) Prob outside t-score = 7.5
(D) Total prob = 5*7.5 = 37.5 %
(E) < 50%, reject

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1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34

The Four Experiments

- **Determine the average density of the earth**

Weigh the Earth, Measure its volume

– Measure simple things like lengths and times

– Learn to estimate and propagate errors

- **Non-Destructive measurements of densities, inner structure of objects**

– Absolute measurements *vs.* Measurements of variability

– Measure moments of inertia

– Use repeated measurements to reduce random errors

- **Construct and tune a shock absorber**

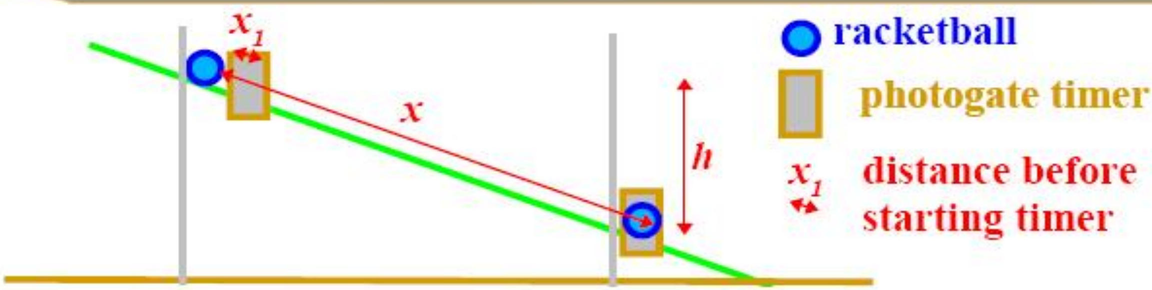
– Adjust performance of a mechanical system

– Demonstrate critical damping of your shock absorber

- **Measure coulomb force and calibrate a voltmeter.**

– Reduce systematic errors in a precise measurement.

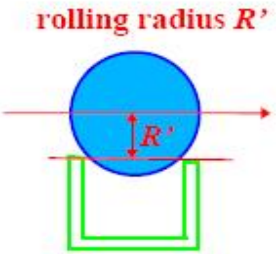
Measuring I by Rolling Objects



1. Measure M and R
2. Using photo gate timer measure the time, t, to travel distance x

$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ energy conservation

$v = R'\omega$ rolling radius



$v = \frac{2x}{t}$ for uniform acceleration

$$Mgh = \frac{1}{2}v^2\left(M + \frac{I}{R'^2}\right)$$

$$gh = \frac{2x^2}{t^2}\left(1 + \frac{I}{MR'^2}\right)$$

$$\frac{I}{MR'^2} = \left(\frac{ght^2}{2x^2} - 1\right)$$

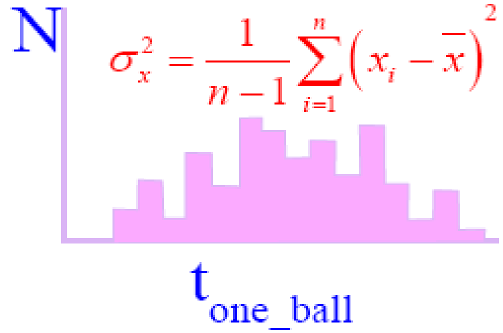
$$\tilde{I} \equiv \frac{I}{MR^2} = \frac{R'^2}{R^2}\left(\frac{ght^2}{2x^2} - 1\right)$$

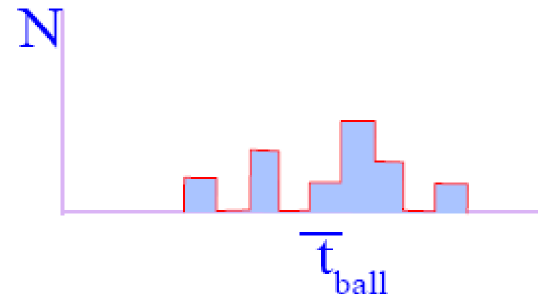
Measuring the Variation in Thickness of the Shell

- 1. Measure rolling time of one ball many times to determine the measurement error in *t*, $\sigma_{measurement}$
- 2. Measure rolling time of many balls to determine the total spread in *t*, σ_{total}
- 3. Calculate the spread in time due to ball manufacture, $\sigma_{manufacture}$, by subtracting the measurement error
- 4. Propagate error on *t* into error on *I* and then into error on thickness *d*

$$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

statistical analysis





$$\sigma_{total} = \sigma_{manufacture} \oplus \sigma_{measruement}$$

$\sigma_t \longrightarrow \sigma_I \longrightarrow \sigma_d$

variation in *t* → variation in *I* → variation in *d*

Propagate Error from *I* to *d*

$$I = \frac{2}{5} M \frac{R^5 - r^5}{R^3 - r^3}$$

$$z \equiv \frac{r}{R} \approx \frac{28.25 - 4.5\text{ mm}}{28.25\text{ mm}} \approx 0.841$$

← measured thickness and radius for one ball

d=4.5 mm R=28.25 mm

d=R-r

$$\tilde{I}(0.841) \equiv \frac{I}{MR^2} = \frac{2}{5}\frac{1-z^5}{1-z^3} \approx 0.571892$$

$$\tilde{I}(0.840) \equiv \frac{I}{MR^2} = \frac{2}{5}\frac{1-z^5}{1-z^3} \approx 0.571366$$

$$\frac{\partial z}{\partial \tilde{I}} = \frac{0.841 - 0.840}{0.571892 - 0.571366} = \frac{0.001}{0.00526} = 1.901$$

← $\delta z \leftrightarrow \delta I$ numerically

$$\frac{\sigma_d}{d} = \frac{\sigma_r}{d} = \frac{R\sigma_z}{d} = \frac{R\tilde{I}}{d}\frac{\partial z}{\partial \tilde{I}}\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx \frac{(28.25\text{ mm})(0.572)}{4.5\text{ mm}}(1.901)\frac{\sigma_{\tilde{I}}}{\tilde{I}} = 6.826\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}}$$

$$\frac{\sigma_d}{d} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}}$$

Useful concept for complicated formula

- Often the quickest method is to calculate with the extreme values
  - $q = q(x)$
  - $q_{max} = q(\overline{x} + \delta x)$
  - $q_{min} = q(\overline{x} - \delta x)$
  - $\delta q = (q_{max} - q_{min})/2$ (3.39)

Propagate Error from t to I

$$\tilde{I} = \frac{I}{MR^2} = \frac{R'^2}{R^2}\left(\frac{ght^2}{2x^2} - 1\right) \approx 0.572$$ from previous page

$$\frac{\partial \tilde{I}}{\partial t} = \frac{R'^2}{R^2}\left(\frac{ght}{x^2}\right)$$ compute derivative

$$\sigma_{\tilde{I}} = \frac{R'^2}{R^2}\left(\frac{ght}{x^2}\right)\sigma_t$$ propagate error

$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} = \frac{\left(\frac{ght}{x^2}\right)}{\left(\frac{ght^2}{2x^2} - 1\right)}\sigma_t \approx \frac{\left(\frac{ght}{x^2}\right)}{\frac{R^2}{R'^2}(0.572)}\sigma_t$$

work out fractional error numerically

$$\left(\frac{ght}{x^2}\right) = \frac{2}{t}\left(\frac{R^2}{R'^2}\tilde{I} + 1\right)$$

$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx \frac{\frac{2}{t}\left(\frac{R^2}{R'^2}\tilde{I} + 1\right)}{\frac{R^2}{R'^2}(0.572)}\sigma_t = \frac{2\left(0.572 + \frac{R'^2}{R^2}\right)}{(0.572)}\frac{\sigma_t}{t} \approx 4\frac{\sigma_t}{t}$$

$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 4\frac{\sigma_t}{t}$$

$$\frac{\sigma_d}{d} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 27\frac{\sigma_t}{t}$$

to get a 10% error on the thickness we need 0.37% error on the rolling time

accuracy can be improved by rolling each ball many times

The Four Experiments

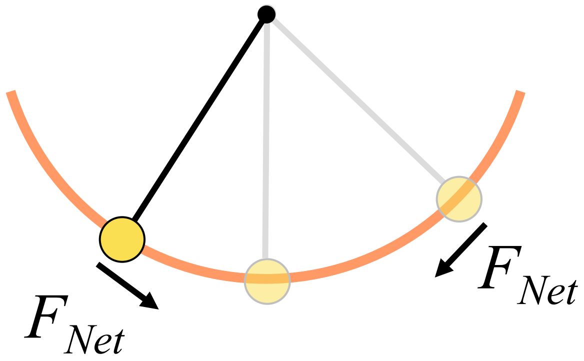
- **Determine the average density of the earth**
  – Measure simple things like lengths and times
  – Learn to estimate and propagate errors
- **Non-Destructive measurements of densities, structure**
  – Measure moments of inertia
  – Use repeated measurements to reduce random errors
- **Test model for damping; Construct and tune a shock absorber**
  – Damping model based on simple assumption
  – Adjust performance of a mechanical system
  – Demonstrate critical damping of your shock absorber
  – Does model work? Under what conditions? If needed, what more needs to be considered?
- **Measure coulomb force and calibrate a voltmeter.**
  – Reduce systematic errors in a precise measurement.

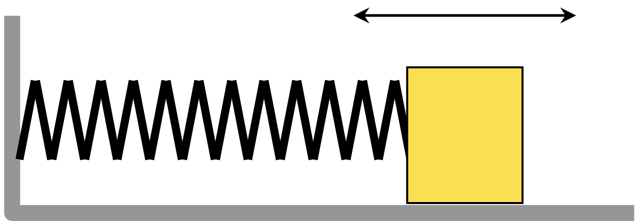
Experiment 3

- Goals: Test model for damping
- Model of a shock absorber in car
- Procedure: develop and demonstrate critically damped system
- check out setup, take data, do data make sense?
- Write up results - Does model work under all conditions, some conditions? Need modification?

Simple Harmonic Motion

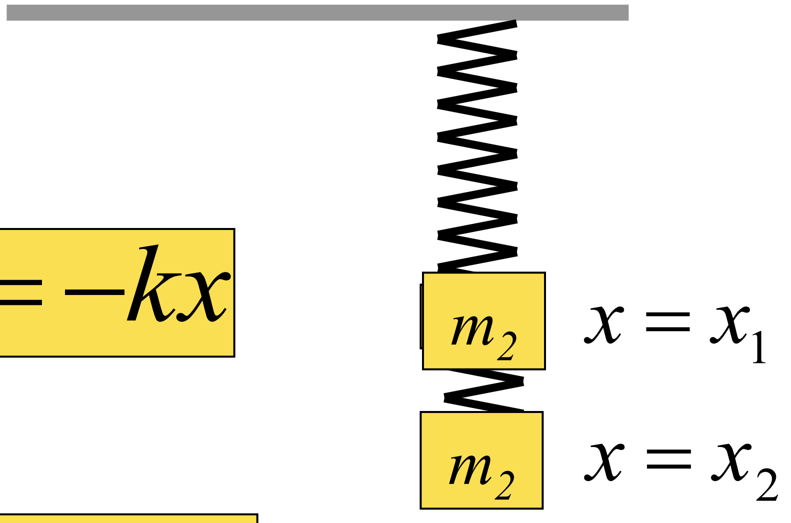
- Position oscillates if force is always directed towards equilibrium position (restoring force).
- If restoring force is ~ position, motion is easy to analyze.





Springs

- Mag. of force from spring ~ extension (compression) of spring
- Mass hanging on spring: forces due to gravity, spring
- Stationary when forces balance



$$F_S = -kx$$

$$F_G = -mg$$

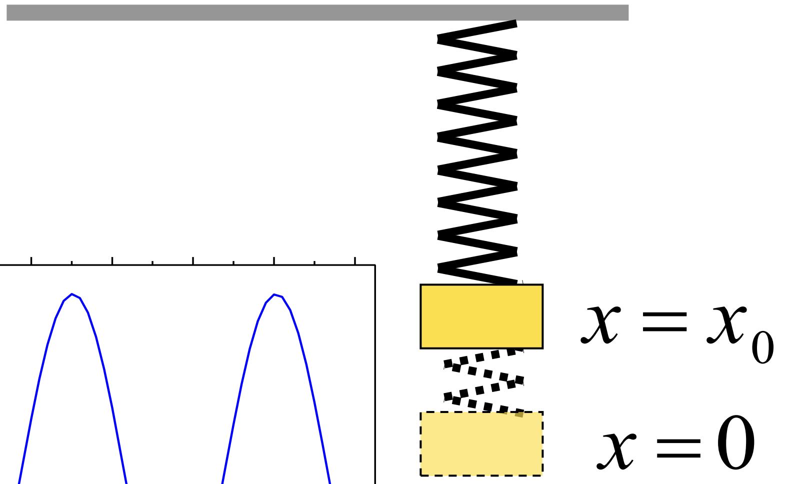
$$F_G = F_S$$
$$mg = kx$$

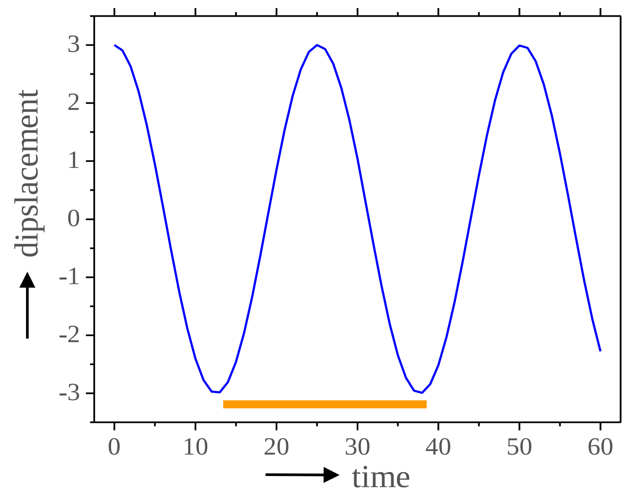
Action Figure

Simple Harmonic Motion

- Spring provides linear restoring force

$\Rightarrow$ Mass on a spring is a harmonic oscillator





$$F = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$x(t) = x_0 \cos \omega t$$

$$T = \frac{2\pi}{\omega}$$

$$\omega = \sqrt{\frac{k}{m}}$$

Damping

- Damping force opposes motion, magnitude depends on speed
- For falling object, constant gravitational force
- Damping force increases as velocity increases until damping force equals gravitational force
- Then no net force so no acceleration (constant velocity)

$$\vec{F}_{damping} = -b\vec{v}$$

$$F_{gravity} = -mg$$

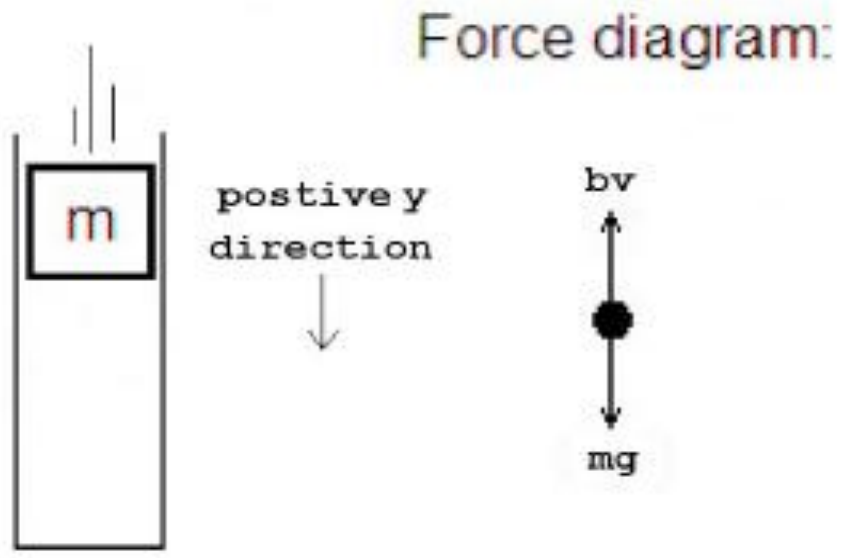
$$bv = mg$$

$$v_{terminal} = (mg)/b$$

Terminal velocity

- What is terminal velocity?
- How can it be calculated?

Falling Mass and Drag



At steady state: $F_{drag} = F_{gravity}$

$$bv_t = mg$$

From rest: $y(t) = v_t[(m/b)(e^{-(b/m)t} - 1) + t]$

Clicker Question 7

What is the uncertainty formula for *P* if

$P = q/t^{1/2}$

(a) $\delta P = [(\delta q)^2 + (\delta t)^2]^{1/2}$
(b) $\delta P = [(\delta q)^2 + (2\delta t)^2]^{1/2}$
(c) $\varepsilon P = [(\varepsilon q)^2 + (\varepsilon t)^2]^{1/2}$
(d) $\varepsilon P = [(\varepsilon q)^2 + (2\varepsilon t)^2]^{1/2}$
(e) $\varepsilon P = [(\varepsilon q)^2 + (0.5\varepsilon t)^2]^{1/2}$

Error propagation

(1) $k_{spring} = 4\pi^2 m/T^2$

$\sigma_{kspring} = \varepsilon_{kspring} * k_{spring}$

$\varepsilon_{kspring} = \sqrt{\varepsilon_m{}^2 + (2\varepsilon_T)^2}$

(2) $k_{by\text{-}eye} = m(g\Delta t^*/2\Delta x)^2$

$\sigma_{by\text{-}eye} = \varepsilon_{by\text{-}eye} * k_{by\text{-}eye}$

$\varepsilon_{by\text{-}eye} = \sqrt{(2\varepsilon_{\Delta t^*)}{}^2 + (2\varepsilon_{\Delta x})^2 + \varepsilon_m{}^2}$

Remember

- Finish Exp. 2 write-up
- Prepare for Exp. 3
- Read Taylor through Chapter 8