

Experiment 3, Physics 2BL

Construct and test a critically damped shock absorber.

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Preparation

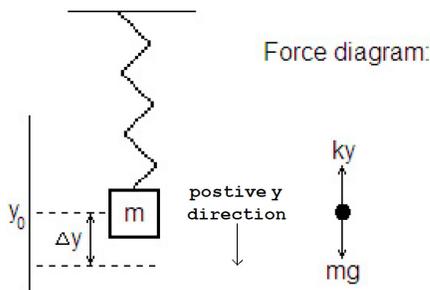
Before this experiment, we recommend you review or familiarize yourself with the following:

- Chapters 7 in Taylor
- Simple Harmonic Motion

1. PHYSICS

For this experiment you will need to be familiar with 3 physical systems and the equations of motion that govern them. Our method for discussing these systems will be as follows. First we will set up a force body diagram and specify initial conditions for the position and velocity. Then we will determine the equilibrium condition that occurs when the forces cancel out. When we sum up the forces and set them equal to mass times acceleration (Newton's Second Law) we get a differential equation. After we write the general solution to this differential equation, we can solve for the constants using the initial conditions and plot the resulting motion over time.

1.1. Spring Harmonic Oscillator



The first system is very simple but is a good warm up problem. We have a mass hanging from a spring as shown in the figure. The equilibrium position, y_0 is determined when the forces cancel out:

$$mg = ky_0 \rightarrow y_0 = \frac{mg}{k}$$

We also set the initial conditions so that the initial position is given a slight displacement, Δy , from equilibrium. This is similar to the small angle displacement we used for the simple pendulum, but in this case we are assuming that the spring constant is not going to change

for a small amount of stretching. Also we are releasing the mass at rest, so the initial velocity is zero. We will be using the notation that a dot is a derivative in time.

$$y(t=0) = y_0 + \Delta y; \quad \dot{y}(t=0) = 0$$

We set the forces equal to the mass times acceleration.

$$\Sigma F = -ky + mg = m\ddot{y}$$

And rearrange to form a differential equation.

$$\ddot{y} + \frac{k}{m}y = g$$

Define $\omega \equiv \sqrt{\frac{k}{m}}$. Then the roots to the characteristic equation $r^2 + \omega^2 = 0$ are $r_1 = +i\omega$ and $r_2 = -i\omega$. Given these roots, the general solution to the homogeneous differential equation, $\ddot{y} + \frac{k}{m}y = 0$, is:

$$y_h(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

While the particular solution is:

$$y_p(t) = \frac{mg}{k} = y_0$$

Then the general solution to our differential equation is:

$$y(t) = y_h(t) + y_p(t) = c_1 \cos \omega t + c_2 \sin \omega t + y_0$$

Now we substitute our initial conditions to determine the constants c_1 and c_2 .

$$y(0) = c_1 + y_0 = y_0 + \Delta y \quad \rightarrow c_1 = \Delta y$$

$$\dot{y}(t) = -c_1\omega \sin \omega t + c_2\omega \cos \omega t$$

$$\dot{y}(0) = c_2\omega = 0 \quad \rightarrow c_2 = 0$$

Plugging in the constants we have the following solution for the position, which we have plotted below.

$$y(t) = \Delta y \cos \omega t + y_0$$

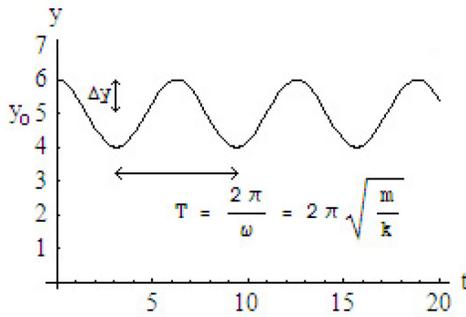
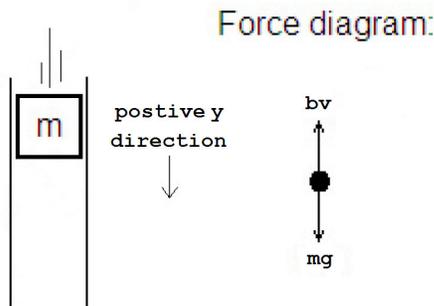


FIG. 1: Motion of a Spring Harmonic Oscillator

Questions

1. What are the differences between this ideal system and a spring oscillator in the presence of real life conditions? Hint: What are we assuming about the spring? Why would we never see motion exactly like the plot in FIG. 1?
2. What are the physical units of the spring constant, k ?
3. Say you have two oscillating ideal spring systems. First, suppose both have the same spring constant k , but have different masses hanging from the springs, m_1 and m_2 , where $m_1 < m_2$. How do the periods of these two systems compare? Next, suppose they have different spring constants k_1 and k_2 , where $k_1 < k_2$, but they both have the same mass m . You displace each mass by the same amount, Δy , from their respective equilibrium and release them. How do the amplitudes compare?

1.2. Mass falling in a drag force



The next system describes a mass in free fall with a drag force that is linearly proportional to the velocity of the mass (i.e. $\mathbf{F}_{drag} = -b\mathbf{v}$). In order to vary the

drag force in the lab, we use a tube that has a variable amount of air pressure under the mass. Now, instead of an *equilibrium length*, we have an *equilibrium velocity*. We will call this the terminal velocity, v_t , because during free fall the mass will speed up to the terminal velocity and then remain fixed at that speed.

Note that in general the damping force may not depend linearly on velocity, but can have quadratic or other functional dependence on it. The purpose of this lab is to test the validity of this assumption.

In order to solve for v_t we set the forces equal to each other.

$$bv_t = mg \rightarrow v_t = \frac{mg}{b}$$

We'll set the initial position and velocity at zero because we are dropping the mass from rest from the top of the tube.

$$y(t=0) = 0; \quad \dot{y}(t=0) = 0$$

We set the sum of the forces equal to the mass times acceleration.

$$\Sigma F = -by + mg = m\ddot{y}$$

And rearrange to form a differential equation.

$$\ddot{y} + \frac{b}{m}\dot{y} = g$$

In this case, the roots to the characteristic equation $r^2 + \frac{b}{m}r = 0$ are $r_1 = 0$ and $r_2 = -\frac{b}{m}$. The homogeneous and particular solutions to the differential equation end up as

$$y_h(t) = c_1 + c_2 e^{-\frac{b}{m}t}$$

$$y_p(t) = \frac{mg}{b}t = v_t t$$

Then the general solution is:

$$y(t) = y_h(t) + y_p(t) = c_1 + c_2 e^{-\frac{b}{m}t} + v_t t$$

Now we substitute in our initial conditions to determine the constants c_1 and c_2 . We arrive at the following solution for $y(t)$:

$$y(t) = v_t \left[\frac{m}{b} (e^{-\frac{b}{m}t} - 1) + t \right]$$

This function is not terribly enlightening because it's just the equation of a mass falling. What is more interesting is if we take a derivative and look at the velocity:

$$\dot{y}(t) = v_t[1 - e^{-\frac{b}{m}t}]$$

The prefactor v_t , which equals $\frac{mg}{b}$, determines the final value that the velocity will reach. The constant in the exponent, $\frac{b}{m}$ determines how fast the velocity will reach its final value. This function is plotted in the figure below. These graphs are a plot of the velocity function for four different values of the drag constant, b , while keeping the mass constant.

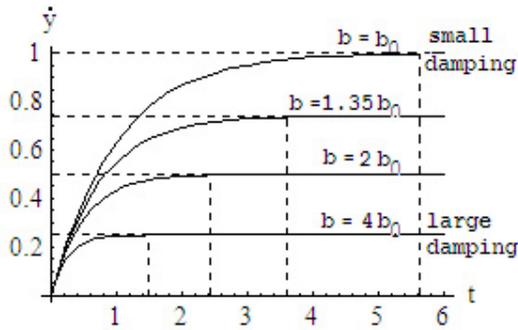
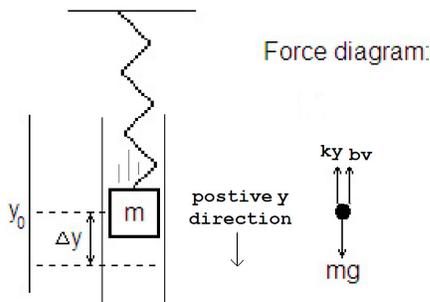


FIG. 2: Velocity of a mass falling in a drag force for varying drag constant.

Questions

4. What are the physical units of the drag constant, b ?
5. According to the FIG. 2, what happens to the terminal velocity as the damping is increased? (horizontal lines)
What happens to the time it takes for the velocity to reach terminal velocity as the damping is increased? (vertical lines)

1.3. Damped Harmonic Motion



Now we will combine the spring force and the drag force to show what happens when the harmonic motion of a spring is damped. We will use the same initial conditions as the first system. Also the equilibrium position,

$y_0 = \frac{mg}{k}$, will be the same because when the mass is stationary, it feels no drag force.

We set the sum of the forces equal to the mass times acceleration.

$$\Sigma F = -ky - b\dot{y} + mg = m\ddot{y}$$

And rearrange to form the differential equation.

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = g$$

In this case, the roots to the characteristic equation $r^2 + \frac{b}{m}r + \frac{k}{m} = 0$ are found using the quadratic formula.

This gives us $r = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$. Let's introduce some variables to make this notation look nicer. If we define $\gamma = \frac{b}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$ then we get $r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$. The damping coefficient b , the mass m , and the spring constant k are parameters of the system that we can control. Since we can give b , m , and k any values we want, in the equation for r , the value under the square root sign can be positive, negative, or zero. These three options give us the three types of solutions to the differential equation. A list of the three cases and the conditions under which they occur is given here.

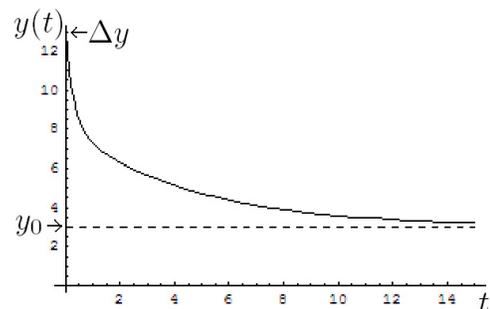
- Overdamped: When $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m} > 0$
- Underdamped: When $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m} < 0$
- Critically damped: When $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m} = 0$

Note: In this experiment, m and k are fixed and we only adjust the value of b .

Overdamped

This is the condition where the value under the radical, $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m}$ is positive. In this case the solution to the differential equation is:

$$y(t) = \frac{\Delta y}{2} e^{-\gamma t} [e^{-(\sqrt{\gamma^2 - \omega_0^2})t} + e^{+(\sqrt{\gamma^2 - \omega_0^2})t}] + y_0$$

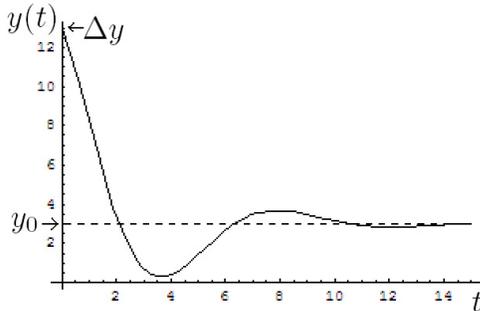


How do we interpret this graph physically? This solution/graph tells us, for a system that is overdamped, if we drop the mass from a position Δy above the equilibrium position y_0 , then it fall toward the equilibrium position exponentially in time. In other words, it will initially fall toward y_0 quickly, then it will fall more slowly toward y_0 as time passes.

Underdamped

This is the condition where the value under the radical, $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m}$ is negative. In order to deal with this imaginary root, we can factor out a $\sqrt{-1}$. This switches the values under the radical. As in the simple harmonic case, imaginary roots to the characteristic equation give oscillatory solutions to the differential equation. In this case the solution to the differential equation is:

$$y(t) = \Delta y e^{-\gamma t} \cos[(\sqrt{\omega_0^2 - \gamma^2})t] + y_0$$



How do we interpret this graph physically? The solution/graph tells us, for a system that is underdamped, if we drop a mass from a position Δy above y_0 then the mass will oscillate about y_0 but the amplitude of this oscillation will decrease exponentially in time. In a non-ideal case the mass will eventually stop at the equilibrium point y_0 .

Critically Damped

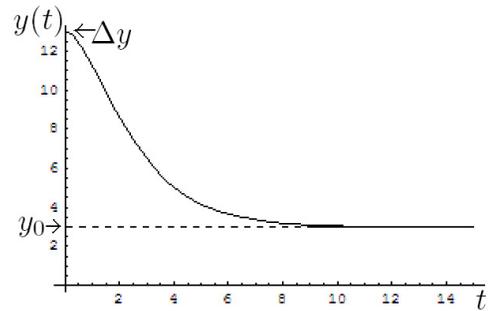
This is the condition where the value under the radical, $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m}$ is zero. Solving for b gives us

$$b_{crit} = 2\sqrt{mk}$$

In this case the solution to the differential equation is simply :

$$y(t) = \Delta y(1 + \gamma t)e^{-\gamma t} + y_0$$

How do we interpret this graph physically? The solution/graph here tell us, for a critically damped system, a mass dropped from Δy above y_0 will behave much like the



overdamped system. It will initially fall quickly toward y_0 and then fall more slowly toward y_0 as time passes. The difference is that in this critically damped system the mass will fall to y_0 in the shortest time possible without oscillating about y_0 .

2. METHODS FOR STATISTICAL ANALYSIS

2.1. Plotting graphs

Plotting data makes it easy to visualize trends and in some cases it can end up saving you a lot of work. You will be required to create graphs in your report for this experiment, so here are some guidelines to keep in mind.

1. Give each graph a title
2. Decide which variable is the independent variable (values that you have chosen, i.e. height to drop the mass down the tube) and which is the dependent variable (the output of your experiment or a calculated value from that output). Always plot the independent variables on the horizontal axis and the dependent variables on the vertical axis.
3. Figure out what boundaries will be appropriate for each axis, mark off the divisions, and label each axis with its units. Each graph should be about half a page in size.
4. Include error bars for the data points when appropriate.

2.2. Short-cut method for error propagation

The following method is very useful for saving time in the error propagation process, however, you should only use it when it is appropriate. ***It can only be used when the function you are propagating error to is***

a product of measured variables to given powers.
For example:

$$f(x, y) = Ax^n y^m$$

Starting with the general formula for error propagation we use the rule for taking derivatives of powers, in this case n and m .

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x} \sigma_x\right)^2 + \left(\frac{\partial f}{\partial y} \sigma_y\right)^2}$$

$$\frac{\partial f}{\partial x} = Anx^{n-1}y^m; \quad \frac{\partial f}{\partial y} = Amx^n y^{m-1}$$

Now we divide both sides by f . Notice that on the right we have replaced f with its functional form.

$$\frac{\sigma_f}{f} = \sqrt{\left(\frac{Anx^{n-1}y^m}{Ax^n y^m} \sigma_x\right)^2 + \left(\frac{Amx^n y^{m-1}}{Ax^n y^m} \sigma_y\right)^2}$$

This reduces to a very simple formula.

$$\boxed{\frac{\sigma_f}{f} = \sqrt{\left(n \frac{\sigma_x}{x}\right)^2 + \left(m \frac{\sigma_y}{y}\right)^2}} \quad (1)$$

Notice that this works just as well for cases where n and m are negative numbers. This formula can also be useful because you can compare fractional errors and in some cases rule out errors that are negligible. If you are interested in calculating σ_f then simply multiply your result by f .

Questions

6. Find an expression for σ_k where $k = \frac{4\pi^2 m}{T^2}$.

First write out the error propagation the long way, using partial derivatives. then divide the left hand side by k and the right hand side by $\frac{4\pi^2 m}{T^2}$ and reduce it to get an expression that only has fractional errors for m and T . You should get the same thing that you would have gotten using the short cut method.

3. EXPERIMENTAL PROCEDURE

In this experiment you will be testing a model using an "engineering-approach" in which you will need to answer: does the data support the assumed model? The procedure below guides you to determine the spring constant k of a spring by two different methods. The first method is based on the model discussed in section 1.1. In this

case, to determine k , you only need to measure the mass and period of a mass on a spring. The second method is based on the model discussed in section 1.3 where we assume that the damping force is linearly dependent on velocity. Then you can test the validity of this assumption made in the model of section 1.3 by comparing the value of k obtained by both methods using the quantitative analysis tools you've learned so far in this course. A brief outline of the procedure is:

- first calculate the spring constant, k , of a spring using model of section 1.1.
- from this you compute the desired value of b to critically damp a mass m .
- setup an experiment to determine b for a variety of damping setups, until you are able to adjust it to the desired value, b_{crit} .
- Check whether critical damping has been achieved by observing the motion of the spring/mass system within the damping mechanism.
- Drop the mass through the critically damped damping mechanism without the spring attached.
- Use collected data to calculate the spring constant, k .

3.1. Spring Harmonic Oscillator

Step 1 Go to the back of the lab and get a spring, silver colored piston (with eye-hook), and a damping tube (this will be used in the procedure described in the next section). Note: The pistons and damping tubes are numbered. The number on the piston indicates which tube the piston belongs too. Be sure to pick the piston and damping tube with the same number.

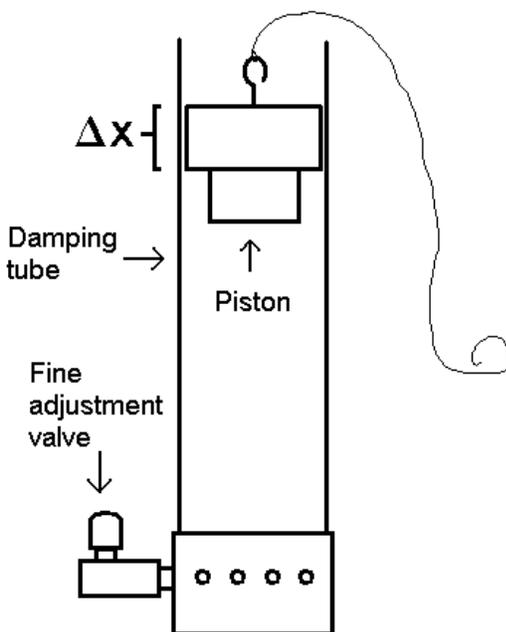
Measure the mass m of your piston. Hang your piston from the spring and measure the period T of small oscillations. In order to increase accuracy, measure the time for N oscillations and divide the total time by N (It is recommended N be *at least* 10). You will, however, have to determine the uncertainty on the period. This can be done by making p measurements of N periods and calculating the mean and standard deviation, which you then divide by N in order to obtain the mean period and the standard deviation. Then you can calculate the standard deviation of the mean (i.e. the uncertainty on the mean). This procedure is the same as was done in experiment 1 for the pendulum. Refer to experiment 1 guidelines for more detail if this is unclear. Now calculate $k = \frac{4\pi^2 m}{T^2}$, its uncertainty, and $b_{crit} = 2\sqrt{mk}$

3.2. Damped Free Fall

NOTE: Steps 2 and 3 that follow may take a considerable amount of time. In order to complete the

experiment in time, please come well prepared for this experiment and work as quickly and diligently as possible.

Step 2 Measure the thickness of the piston Δx (Your piston will look like two stacked cylinders of differing diameters. IMPORTANT: make sure that the Δx you measure corresponds to the one the gate timer "sees"!!! See for example the diagram below.). Set up the photogate at the bottom of the tube and set the photogate to "Gate" mode and the little black switch to 0.1ms. This set up will allow you to measure the time Δt it takes for Δx of the piston to fall through the photogate. From this data you can then calculate the velocity of the mass at the bottom of the tube (i.e. $v = \frac{\Delta x}{\Delta t}$). The orientation of your piston is important. Orient your piston so the lower half is the cylinder with the smaller diameter (See the diagram below.) Attach an eye-hook to the top of the piston and tie a string to the eye-hook so you can pull the piston out of the damping tube easily. Position the photogate so that it triggers at the beginning and end of the Δx of your piston. Be sure it does not trigger from the bottom to the top of the piston nor on the eye-hook. Clamp the tube and photogate down to the table.



Now that the setup is complete, *close the valve* on the tube if it isn't closed already and place a piece of masking tape over 1 hole at the base of the tube so that you have 5 holes OPEN. You will next be dropping the piston from various heights in the tube. Using the length scale attached to the tube, choose a length by which you would like to increment the drop height. Then, using the string attached to the piston, drop your piston from each chosen height at least 10 times and record the times Δt that correspond to each of these release heights. DO

NOT WRITE ON THE TUBES! Now, place tape over an additional hole so 4 holes remain OPEN and repeat the procedure just used.

Calculate the average times $\bar{\Delta t}$ for each height from the data you just collected.

Step 3 Convert your time data to velocities using $v = \frac{\Delta x}{\Delta t}$ and make a plot of h vs. v for 5 and 4 holes open. These graphs should look very similar to the graph on page 3 of this lab guide. Because it would be very difficult to measure velocity of something falling as a function of time, we can measure it as a function of release height. The purpose of this step is to see whether the piston is reaching terminal velocity by the time it is dropped from the top of the tube.

The following questions, Q, are not quiz questions. They are intended to be answered as part of your lab report.

Q: Will the piston reach v_t for 6 holes? Do we need to check for 0,1,2,3 holes open?

Questions

7. Suppose your damping tube has 4 holes open and you find that your piston does reach terminal velocity when released from a certain height below the maximum release height. Sketch what you would expect a h vs. v plot to look like for this experiment. (h is the release height.) Briefly describe what your plot means.

Now, using an "engineering approach" to test the model of section 1.3 implies that you assume the model is correct from the beginning and that you use the theoretical results from this model while conducting the experiment. The remainder of this procedure does just this. It is important that you keep in mind that all quantities obtained are a result of assuming this model is correct from the beginning.

Step 4 Determine the terminal velocity v_t for 0-5 holes open. This can be done by simply measuring Δt of the piston dropped from the top of the tube for each configuration of holes open. It is recommended that you measure Δt many times and determine the average $\bar{\Delta t}$ for each hole setting.

Step 5 Convert your terminal velocity data for 0-5 holes open to b values using $b = \frac{mg}{v_t}$ and make a plot of b for 0-5 holes open (i.e. *number of holes open vs. b*). Plot b_{crit} as a horizontal line across your graph.

Q: Where do the two lines intersect? What is the rough estimate for how many holes need to be open for the damping tube to be critically damped according to b_{crit} ?

Step 6 Solve for the Δt_1^* which gives b_{crit} . The result is

$$\Delta t_1^* = \frac{2\Delta x}{mg} \sqrt{mk}$$

Calculate Δt_1^* .

Step 7 Next, you want to open the number of holes according to what you answered in the previous **Q** question. Open up the lower number in your range and then you will use the valve on the tube to open a hole partially. You need to determine how many revolutions of the fine adjustment valve correspond to the valve being fully opened. Also you can use the 10 tick marks on the fine adjustment valve to get an extra digit of significance in your fraction of a hole open.

3.3. Damped Harmonic Motion

Step 8 Now that you have achieved Δt_1^* , your system is damped according to the value you calculated for b_{crit} . So, if b_{crit} is correct, then your system should be critically damped. Next, attach the spring and observe the motion in the tube.

Q: Does this system look underdamped or overdamped? How can you tell?

Step 9 Next, adjust the valve until the system is underdamped, then adjust the valve until you have critically damped the system by eye.

Step 10 Take the piston off the spring, release the piston from the top of the tube, and measure Δt_2^* . Drop the piston down the tube N more times and record Δt_2^* . Calculate $\Delta t_2^* \pm \sigma_{\Delta t_2^*}$. Δt_2^* is the time it takes the piston to fall through the photogate when the tube has been critically damped by eye and the piston is falling at its terminal velocity.

Questions

8. In this experiment, how can you tell the difference between overdamped and critically damped motion? Do not give a mathematical definition. Describe what you would do in this experiment to distinguish between an overdamped and critically damped system.

3.4. Cautions

It is HIGHLY recommended that you do the following.

- Clamp down setup to reduce wobbling.

- Don't overstretch the spring
- Attach a string to the piston so you can pull it out of the tube.
- Do NOT use clear tape for any part of this experiment.
- Make sure you are triggering on the correct Δx by pulling the mass slowly through the photogate.
- Make sure the spring is colinear with the tube in steps 8 and 9.

4. ANALYSIS

As previously mentioned, in order to test the model assumed for damping (linear dependence on velocity), you will need to compare the value of the spring constant obtained by the two methods, which can be calculated using the following formula.

$$k_{spring} = \frac{4\pi^2 m}{T^2} \pm \sigma_{k_{spring}}$$

$$k_{by-eye} = m \left(\frac{g\Delta t_2^*}{2\Delta x} \right)^2 \pm \sigma_{k_{be}}$$

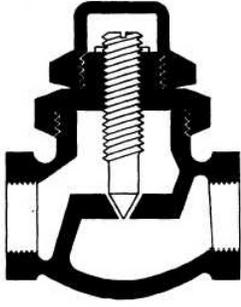
Find these two values and propagate the errors from m , T , Δt , and Δx . Also, determine the discrepancy and its uncertainty. Then, using t-score, determine the level at which these two values are discrepant and explain what this means about the model you are testing.

Questions

9. Three students measure the period of their piston-spring system with the results (in units of seconds):
 1.5 ± 0.5
 1.17 ± 0.03
 1.82 ± 0.19
 Determine the best estimate and its uncertainty for the period.

Appendix 1: Lab Equipment, The Damping Tube

In order to damp our spring motion we will be using a long vertical tube that has holes drilled in the bottom. You will be able to cover up the holes in order to increase the pressure underneath the piston that you are damping. The more holes that are open, the less damping, and the piston will be able to move more freely. To fine adjust between the levels of holes open, there is a valve that can go anywhere between an open hole and a closed hole. The insides of the valve look something like this. A screw on the side of the tube allows air to flow through.



Cross section of the fine adjustment valve.