Physics 4E | Quiz 5 Solutions | Spring 2015

Problem 1

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right) = \frac{\hbar\omega}{2} + n\hbar\omega \quad \text{for harmonic oscillator}$$

$$\langle E\rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}$$

$\hbar\omega = 0.1\,eV$

(i) $kT = 1\,eV$

$$\langle E\rangle = 0.05\,eV + \frac{0.1\,eV}{e^{0.1}-1} = 1.0008\,eV$$

(ii) $kT = 0.1\,eV$

$$\langle E\rangle = 0.05\,eV + \frac{0.1\,eV}{e^{1}-1} = 0.108\,eV$$

(iii) $kT = 0.01\,eV$

$$\langle E\rangle = 0.05\,eV + \frac{0.1\,eV}{e^{10}-1} = 0.050005\,eV$$

(b) For $kT = 1.1\,eV$

$$\langle E\rangle = 0.05\,eV + \frac{0.1\,eV}{e^{0.1/1.1}-1} = 1.1008\,eV$$

$$C_v = \frac{d\langle E\rangle}{dT} = \frac{d\langle E\rangle}{d\,kT}\cdot k = \frac{1.1008\,eV - 1.0008\,eV}{0.1\,eV}\,k = k$$

$\Rightarrow$ for N_A oscillators, $C_v = N_A k = R$

This is in agreement with equipartition theorem, because $kT \gg \hbar\omega$

(c) For $kT = 10\,eV$ equipartition will hold too $\Rightarrow C_v = R$

(d) For this T we can ignore the -1 $\Rightarrow \langle E\rangle = \frac{\hbar\omega}{2} + \hbar\omega e^{-\hbar\omega/kT}$

$$C_v = \frac{d\langle E\rangle}{dT} = \frac{\hbar\omega}{kT^2}e^{-\hbar\omega/kT} = k\left(\frac{\hbar\omega}{kT}\right)^2 e^{-\hbar\omega/kT} = k\cdot 10^2 e^{-10} = 0.005\,k \text{ per atom}$$

$\Rightarrow \boxed{0.005R \text{ per mol}}$

Problem 2

Electron in $n=3$ can have $l=0$, $l=1$, $l=2$

Energy in magnetic field:

$E_B = -\vec{\mu}\cdot\vec{B} = -\mu_z B$; $\mu_z = -m_l \mu_B$

$E_B = \mu_B B m_l$; $\mu_B = 5.79\times10^{-5}\,eV/T$

Possible values of m_l are $-2, -1, 0, 1, 2$

$\Rightarrow$ 5 different energies. Difference of lowest and highest states:

$E_B(m_l=2) - E_B(m_l=-2) = 4\mu_B B = 0.023\,eV$

(b) Including spin

$E_B = \mu_B B(m_l + 2m_s)$ with $m_s = \pm\frac{1}{2}$

Possible values of $m_l + 2m_s = -3, -2, -1, 0, 1, 2, 3$

$\Rightarrow$ 7 different energies. Difference of lowest and highest

$E_B(m_l=2, m_s=\frac{1}{2}) - E_B(m_l=-2, m_s=-\frac{1}{2}) = 6\mu_B B = 0.035\,eV$

(c)

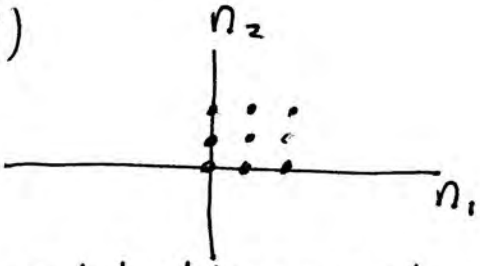
$$\frac{n(m_l=2, m_s=\frac{1}{2})}{n(m_l=-2, m_s=-\frac{1}{2})} = \frac{g(2,\frac{1}{2})}{g(-2,-\frac{1}{2})}\, e^{-0.035eV/kT}$$

The degeneracies are the same, by symmetry, so:

$$\boxed{\frac{n(2,\frac{1}{2})}{n(-2,-\frac{1}{2})} = e^{-0.035\times 11,600/300} = 0.258}$$

Problem 3

$$E_{n_1 n_2} = \frac{\hbar^2 \pi^2}{2 m L^2}\,(n_1^2 + n_2^2) = E_0 (n_1^2 + n_2^2)$$



The points that give energy $< E$ are in a circle of radius $R = \left(\frac{E}{E_0}\right)^{1/2}$. Since $n_1, n_2 > 0$ we need $\frac{1}{4}$ of the area of the circle of radius R

$$N = \frac{1}{4}\cdot \pi R^2 = \frac{1}{4}\pi \frac{E}{E_0}$$ and the density of states is

$$g(E) = \frac{dN}{dE} = \frac{\pi}{4E_0} = \frac{\cancel{\pi}}{\underset{2}{\cancel{4}}\hbar^2 \cancel{\pi^2}}\cdot \cancel{2} m L^2 = \frac{m L^2}{2\pi \hbar^2} \Rightarrow$$

$$\boxed{g = \frac{m \cdot A}{2\pi \hbar^2}}$$ note that it is independent of E, unlike in 3D.

(b) $$N = \int_0^\infty dE\, g(E)\, f_B(E) = g\cdot e^{-\alpha}\cdot \int_0^\infty dE\, e^{-E/kT} = g e^{-\alpha}\cdot kT$$

$$\Rightarrow \boxed{e^{\alpha} = g\cdot \frac{kT}{N}} \qquad \boxed{f_B(E) = \frac{1}{e^{\alpha} e^{E/kT}} = \frac{N}{g kT}\, e^{-E/kT}}$$

(c) $$\boxed{n(E) = g(E)\, f_B(E) = \frac{N}{kT}\, e^{-E/kT}}$$

Verify normalization

$$\boxed{N = \int_0^\infty dE\, n(E) = \frac{N}{kT}\int_0^\infty dE\, e^{-E/kT} = \frac{N}{kT}\, kT = N}$$