Discussion Session 1

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1 Probability Distributions

Consider a object where possible configurations are labelled by some variable i. The variable i could be a reference to some quantum state, the color of a marble, a score on a quiz, etc. Suppose we have a system, or ensemble, of these objects from which we can choose objects at random and let $P_i \in (0,1)$ denote the probability that the object chosen is in configuration i. The collection of values $\{P_i\}$ forms a discrete probability distribution. We assume that the distribution is normalized

$$\sum_i P_i = 1 \tag{1.1}$$

which is just the statement that if we choose an object, it will be in one of it's possible states. The probability distribution determines the statistical properties of a system. For example, suppose we have some measurable quantity A which takes a value depending on the variable i. Then the average of A is given by

$$\langle A \rangle \equiv \bar{A} = \sum_i A_i P_i \tag{1.2}$$

where A_i is the value of A in the state i. Another useful quantity is the mean square

$$\langle A^2 \rangle = \sum_i A_i^2 P_i \tag{1.3}$$

From the mean and the mean square, we can define the standard deviation, which is the square root of the average square deviation from the mean. It quantifies the variation, or spread, of a set of data. That sounds complicated, so let's unpack this statement. The average square deviation from the mean is:

$$\langle (A_i - \bar{A})^2 \rangle = \sum_i (A_i - \bar{A})^2 P_i \tag{1.4}$$

and then we take the square root at the end. We can do some algebra to simplify this expression: $\bar{A}$ is just a number, so

$$\sum_i (A_i - \bar{A})^2 P_i = \sum_i (A_i^2 - 2\bar{A}A_i + \bar{A}^2)P_i = \langle A^2 \rangle - \langle A \rangle^2 \tag{1.5}$$

so our final results is $\sigma = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$. The analysis we've done here can be extended to systems where the variable i can take on continuous values. In that case, our distribution functions become continuous functions $P(x)$. In that case, the normalization condition becomes

$$\int dx\ P(x) = 1 \tag{1.6}$$

and averages are given by

$$\langle A \rangle = \int dx\ A(x)P(x) \tag{1.7}$$

One subtlety worth mentioning is that in the case of continuous probability distributions, it is no longer well defined to ask about the probability of observing a specific value. Instead, what we can ask is the probability to observe a value within a certain range. The correct way to think about continuous probability distributions is that $P(x)dx$ gives the probability that the configuration lies within the range dx centered at x. To see this, consider a uniform continuous probability distribution normalized on the interval $x \in [0, 1]$. There are infinitely many numbers between 0 and 1, so the probability that some x chosen at random is exactly $\frac{1}{2}$ is essentially zero. However the probability that some x chosen at random will be between 0.4 and 0.5 poses no problems: it is $\frac{1}{10}$.

As an example, consider the gaussian distribution $P(x) = Ne^{-(x-\mu)^2/2\sigma^2}$ where N is some constant to be determined by imposing the normalization condition 1.6. Then (exercise to the reader)

- $N = (2\pi\sigma^2)^{-1/2}$
- $\langle x \rangle = \mu$
- $\langle x^2 \rangle = \mu^2 + \sigma^2$
- $\mathrm{Var}[x] = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2$

2 Counting Modes

As an ingredient in our calculation of the black body spectrum, we need to know how many modes there are in some range of wavelengths $\lambda \to \lambda + d\lambda$. We know that the solutions to maxwell's equations in the absence of charge leads to the wave equation $c^2\nabla^2 E = \partial_t^2 E$ which admits as solutions plane waves subject to some boundary conditions. Assuming an oscillatory form for the field $E(r) \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ this equation reduces to the time independent differential equation

$$\partial_x^2 E + \partial_y^2 E + \partial_z^2 E = -\underbrace{(\omega/c)^2}_{k^2} E \tag{2.1}$$

By making the separation of variables guess: $E(x, y, z) = X(x)Y(y)Z(z)$ we can reduce this to three ordinary differential equations

$$\begin{aligned} \partial_x^2 X + k_x^2 X = 0 \\ \partial_y^2 Y + k_y^2 Y = 0 \\ \partial_z^2 X + k_z^2 Z = 0 \end{aligned} \tag{2.2}$$

where $k_x^2 + k_y^2 + k_z^2 = k^2$. In the case of modes inside of a cavity, we will have standing waves which are constrained to vanish on the boundary. This boundary condition allows the solution

$$E(x, y, z) \sim \sin(k_x x)\sin(k_y y)\sin(k_z z) \tag{2.3}$$

and setting $E = 0$ at $x = L$ gives us the condition on k that $k = n\pi/L$. These $k's$ satisfy $k_x^2 + k_y^2 + k_z^2 = k^2$, which is the equation specifying the surface of a sphere in 3 dimensions.

If we imagine a 3-dimensional coordinate system with tick marks every π/L steps in all directions, then each point on this grid represents a possible mode. We can see that the 'volume' occupied by each mode is $(\pi/L)^3$. The volume of the space between k and $k + dk$ is $\frac{1}{2}\pi k^2 dk$. This is one eight of what you would normally expect because we restrict ourselves to a single octant of the three dimensional space where all $k \geq 0$.

Then the number of modes between k and $k + dk$ is the volume divided by the volume per mode. We also multiply by two for the two polarization vectors:

$$N(k)dk = \frac{\pi k^2 dk}{(\pi/L)^3} = \frac{V k^2 dk}{2\pi^2} \tag{2.4}$$

Finally, use the relation $k = \frac{2\pi}{\lambda}$ and divide by the volume to obtain the result we want.

$$N(\lambda)d\lambda = \frac{8\pi}{\lambda^4} d\lambda \tag{2.5}$$