Homework April 21, 2015 (to be returned on April 28)

We want to determine mean and variance of the allele frequency variation Δp in situations where the environment is fluctuating very rapidly. Viabilities are $w_{11}^{(t)} = 1 + Y_1^{(t)}$; $w_{22}^{(t)} = 1 + Y_2^{(t)}$ and $w_{12}^{(t)} = 1 + \left(Y_1^{(t)} + Y_2^{(t)}\right)/2$ for the homozygous A_1A_1, A_2A_2 and the heterozygous A_1A_2 or A_2A_1, respectively. The Y's are random variables supposed all independent and the index t refers to the generations.

1) From the general formula that we derived during the lectures for the variation Δp of the frequency p of allele A_1 under random mating and selection with the previous viabilities, show that:

$$\Delta p^{(t)} = \frac{p^{(t)}q^{(t)}}{2}\frac{Y_1^{(t)} - Y_2^{(t)}}{1 + p^{(t)}Y_1^{(t)} + q^{(t)}Y_2^{(t)}}\,. \tag{1}$$

where $\Delta p^{(t)}$ indicates the difference between the values at the generations $t+1$ and t.

2) Suppose that the Y's are small and expand (1) to the second order.

3) Mean and variances of the Y's are: $\langle Y_1^{(t)}\rangle = \mu_1$; $\langle Y_2^{(t)}\rangle = \mu_2$ and $\mathrm{Var}\,Y_1^{(t)} = \mathrm{Var}\,Y_2^{(t)} = \sigma^2$, with the means μ_i of the same order as the variance σ^2. Determine the expression of the mean $\langle\Delta p^{(t)}\rangle$ and $\mathrm{Var}\,\Delta p^{(t)}$.

The previous calculation was done under the hypothesis of additivity. To generalize to non-additive situations, one can suppose that viabilities are $w_{11}^{(t)} = W\left(1 + Y_1^{(t)}\right)$; $w_{22}^{(t)} = W\left(1 + Y_2^{(t)}\right)$ and $w_{12}^{(t)} = W\left(1 + \frac{Y_1^{(t)} + Y_2^{(t)}}{2}\right)$, where the Y's are again random variables, supposed all independent and the function $W(x)$ is a convex function, e.g. the Michaelis-Menten curve discussed during the course $W(x) = x(1+a)/(a+x)$.

1) Supposing the Y's small and having means and variances as above, calculate the geometric means, e.g. $w_{11}^g(t) = \left(\prod_{i=0}^{t-1} w_{11}^i\right)^{1/t}$, and show that the condition for the heterozygote to be fitter than both homozygotes is

$$|\mu_1 - \mu_2| < \frac{\sigma^2}{2W'(1)}\left(W'^2(1) - W''(1)\right)\,, \tag{2}$$

where $W'(1)$ and $W''(1)$ are the first and second derivatives of $W(x)$ at $x = 1$.

2) Obtain the condition (2) of coexistence for the explicit form of W given above, recover the additive limit and discuss whether the general case gives a range for coexistence larger or smaller than in the additive case.