Gravitational Resistive Instability of an Incompressible Plasma in a Sheared Magnetic Field

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The gravitational resistive instability analysed by Furth, Killeen, and Rosenbluth and subsequent authors is examined from a new point of view, which brings out the connection with ordinary Rayleigh–Taylor instability and thermal convection. In contrast to the modes found by earlier authors, which are either sharply localized in the vertical direction or require a boundary layer, it is shown that coherent motions of arbitrary vertical extent can occur. These alternative modes are derived by first considering a simpler but related model in which resistivity is concentrated at the ends of a system of finite length. This analysis shows that such systems may be unstable even if they satisfy the Newcomb criterion. The new resistive modes do not have the usual periodic dependence along the horizontal direction of the main field, but have finite length and represent convective rolls which are twisted to conform to the field lines. The relation of these new modes to the original periodic localized modes is examined.

I. INTRODUCTION

THE analysis of resistive instabilities in a fluid supported by a sheared magnetic field, initiated by Furth, Killeen, and Rosenbluth,[1] leads to a type of normal mode in which the influence of resistivity is concentrated in a thin region about the singular magnetic surface Σ at which $\mathbf{k}\cdot\mathbf{B}_0 = 0$, (where $\mathbf{k}$ is the component of the wave vector normal to the direction of shear). This thin region plays a role similar to that of a boundary layer in hydrodynamics. In this paper we show that by adopting a different viewpoint one is led to consider an alternative class of unstable modes in which the influence of resistivity is not localized, and which do not have this "boundary-layer" characteristic.

The model investigated by Furth, Killeen, and Rosenbluth was a plane slab of incompressible fluid, in which the destabilizing effect of field curvature was represented by a fictitious gravity; this leads to three types of instability, called the rippling, tearing and gravitational modes. The gravitational or G-mode, which is the only one studied in this paper, was examined in more detail by Johnson, Greene, and Coppi,[2] and again by Coppi,[3] who showed that in a limit which corresponds to $\beta \rightarrow 0$ a set of G-modes exists for which the perturbations effectively vanish outside the resistive layer. (In the limit of zero resistivity, instabilities concentrated near the singular surface Σ had been found previously by Suydam[4] and Rosenbluth.[5])

Following Ref. 1 we consider an equilibrium situation in which the magnetic field lies in the (y, z) plane, i.e., $\mathbf{B} = (0, sxB_0, B_0)$, and there is a gravitational field in the negative x direction. Rippling and tearing modes are excluded from our analysis and the discussion is confined to gravitationally driven modes in a system with weak shear, such as the Stellarator. The low-β approximation is also frequently used.

Because the equilibrium is independent of y and z, it has been customary in stability theory to look for normal modes of the form $f(x) \exp i(k_y y + k_z z)$. In this paper we discard this assumption and adopt a more general form $f(x, z) \exp (ik_y y)$; that is we do *not* Fourier analyze in z, the direction of the main field. With this changed viewpoint, modes are found which are neither localized near a particular horizontal surface Σ, nor dependent on a boundary layer. Looked at in this way the localization of the instabilities found by Suydam[4] and others,[1–3,5] appears as a property, not so much of the physical disturbances themselves, as of their Fourier transforms. An important advantage of our approach is that it immediately brings out the connection between resistive instabilities and the convective cells of hydrodynamics.

To illustrate this more general type of mode we approach the full problem of resistive instability in a

[1] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[2] J. L. Johnson, J. M. Greene, and B. Coppi, Phys. Fluids **6**, 1169 (1963).
[3] B. Coppi, Phys. Rev. Letters **12**, 417 (1964); and Phys. Fluids **7**, 1501 (1963).
[4] B. R. Suydam, Progr. Nucl. Energy **1**, 463 (1959).
[5] M. N. Rosenbluth, in *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 31, p. 85.

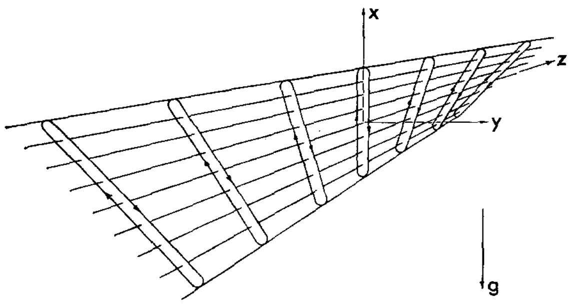

FIG. 1. Twisted slicing mode.

sheared magnetic field through a series of simpler but related problems. In Sec. III we first consider the gravitational instability of a perfectly-conducting incompressible fluid in a sheared magnetic field, contained between conducting endplates which, however, are coated with a thin insulating layer so that the field lines are *not* tied. In the limit $B_0 \rightarrow \infty$ exact solutions are found which are not of the form $\exp(ik_z z)$ and which therefore do not fit within the framework of the stability analysis of Suydam[4] and Newcomb.[6] These modes may be unstable even if the system satisfies the Newcomb stability criterion which strictly applies only to an infinite system. They represent twisted interchange motions in which fluid filaments or "flux tubes" move as rigid bodies. As each filament rises or falls it rotates about a vertical axis to keep aligned with the local magnetic field at each height x and so avoid distortion of the field (Fig. 1). These modes, which exist in finite systems but have no analogue in infinite systems, may be important in experiments but it is difficult to decide this as the real boundary conditions are much more complex than in our model. Their importance in the present context is that they are due to resistive layers at the ends, and so provide a prototype for the true resistive instabilities which are examined in Secs. IV–VI.

We begin the discussion of resistive instabilities proper with the gravitational instability of a resistive fluid in a magnetic field *without* shear but with the field lines tied to conducting end plates (Sec. IV), and later examine what happens to these motions when a weak shear is imposed (Sec. V). When the shear is zero, but the field lines are tied at the ends, resistive gravitational modes occur which take the form of a "slicing" motion, in which alternate vertical sheets of fluid move up and down. The sheets are parallel to the unperturbed magnetic field and the most dangerous modes have small longitudinal wavenumber k_z (so that the field is only slightly distorted), small vertical wavenumber k_x, and large transverse wavenumber k_y. This corresponds to long thin convective cells ('slices') which extend the full height of the fluid. Such modes might occur in the unstable sectors of "$\int dl/B$ stable" devices.[7] They form a special case of convective rolls,[8] modes which have been identified by Danielson with the penumbral filaments observed in sunspots.[9]

If now a weak shear is imposed on the magnetic field in this system, very similar motions are still possible, but with the convective cells twisted so that their surfaces remain everywhere approximately parallel to the field lines. This "twisted slicing" motion is illustrated in Fig. 1. As a fluid filament rises or falls it must now rotate about a vertical axis (just as in the model problem with resistance confined to layers at each end), in such a way that it always lies along the local direction of $\mathbf{B}$, since this minimizes the field distortion.

Finally we consider what happens as the length of the system is increased indefinitely. In the nonsheared case, tying at the ends becomes ineffective and the most dangerous modes have $k_z \rightarrow 0$, i.e., they represent interchange motion of infinitely long filaments, and the growth rate is independent of B and η. When shear is present the moving filaments cannot become infinitely long, since they are constrained by the field to rotate as they rise or fall and the rotational kinetic energy would increase without limit if the mode length adjusted itself to the length of the system. In fact the mode length l automatically adjusts itself to give a balance between the rotational kinetic energy and the dissipative loss due to motion across the field. The growth rate is then approximately that for a system with no shear and finite length $\Lambda = l$. It turns out that $l \sim \eta^{-\frac{1}{3}}$ so that the growth rate $p \sim \eta^{\frac{1}{3}}$ in agreement with Ref. 1.

The chain of argument thus leads to unstable modes of quite a different character to the localized modes found hitherto. It is natural to ask how these 'twisted slicing' modes, with finite length l and arbitrary height, are related to the G-modes of Ref. 1 and others, which have finite height h and unlimited length, and this question is examined in Sec. VI. The two types of mode have the same growth rate and there is a relation connecting the height h of the G-mode with the length l of our mode. Each of our twisted slicing modes is in fact a linear superposition of the localized G-modes introduced by

[6] W. A. Newcomb, Ann. Phys. (N. Y.) **10**, 232 (1960).

[7] H. P. Furth and M. N. Rosenbluth, Phys. Fluids **7**, 764 (1964).

[8] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London, 1961).

[9] R. E. Danielson, Astrophys. J. **134**, 275 (1961).

Coppi,[5] one for every value of x. This superposition is possible because a system with weak shear is almost degenerate and localized G-modes centered at different heights have almost identical growth rates. If the degeneracy of the G-modes were exact there would be no unique normal mode and any combination of degenerate modes would be a mode with identical growth rate. In an actual case the growth rate of the G-modes centered at different points varies only by a small fraction; e.g., $\sim 10^{-3}$ for a typical Stellarator field. The individual components in any combination of these modes will therefore not increase at precisely the same rate—and for this reason we shall call the combination a "quasi-mode"—but it will take many e-folding periods for a significant discrepancy to occur and by this time the instability should be out of the linear phase.

II. THE GRAVITATIONAL MODEL

The plane incompressible fluid model used in this paper is intended to describe pure gravitational or G-modes; we eliminate rippling and tearing modes[1] by assuming the resistivity η and the shear s to be uniform. A uniform gravitational field g is directed downwards, and the unperturbed density distribution is Rayleigh–Taylor unstable, increasing linearly with height x according to

$$\partial \rho_0/\partial x = \alpha \rho_0. \tag{2.1}$$

There is a uniform horizontal magnetic field $(0, 0, B_0)$, together with a transverse field $B_y = sxB_0$ (where $s = \text{const}$), so that the field direction changes with height. (We take $s = 0$ in Sec. IV.) The fluid is contained between perfectly conducting rigid walls at $x = \pm H$, where the boundary conditions for perturbed variables are $v_x = B_x = E_z = E_y = 0$, and is unbounded in the y direction. The shear is assumed to be weak, so that $sH \ll 1$. In Secs. III and IV we impose boundary conditions at $z = \pm L$, which will be discussed later; elsewhere the system is assumed to be infinitely long.

The linearized equations are

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}_0 + \frac{1}{4\pi}(\nabla \times \mathbf{B}_0) \times \mathbf{B} + \rho \mathbf{g}, \tag{2.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0) + \frac{\eta}{4\pi}\nabla^2 \mathbf{B} = (\mathbf{B}_0 \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{B}_0 + \frac{\eta}{4\pi}\nabla^2 \mathbf{B}, \tag{2.3}$$

$$\partial \rho/\partial t = -(\mathbf{v} \cdot \nabla)\rho_0 = -v_x \alpha \rho_0, \tag{2.4}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{v} = 0, \tag{2.5}$$

where $\mathbf{B}_0 = (0, sxB_0, B_0)$. In equilibrium the weight of the fluid and the force $(4\pi)^{-1}(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0$ are balanced by the fluid pressure P_0. We shall treat ρ_0 as uniform in the inertial term of (2.2) (Boussinesq approximation) and also in (2.4).

In Secs. III and IV we deal with the full set of equations (2.2)–(2.5) while the principal approximation which we shall make in Secs. V and VI is to neglect the term $\partial \mathbf{B}/\partial t$ in (2.3), which can be shown to be of order $\beta = 8\pi P_0/B_z^2$ (for the modes discussed in this paper), compared to the term $\eta \nabla^2 \mathbf{B}/4\pi$. Equations (2.2)–(2.5) can then be combined to give an equation for the vertical velocity v_x;

$$p^2 \rho_0 \eta \nabla^2 v_x + pB_0^2\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)^2 v_x - \alpha g \rho_0 \eta \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) v_x = 0, \tag{2.6}$$

where we have assumed a time dependence $\sim \exp(pt)$.

As previously remarked, we shall solve (2.6) without making a Fourier analysis in the z direction. Suppose however that one does assume a dependence

$$v_x \sim f(x) \exp(pt + ik_y y + ik_z z),$$

and sets

$$\bar{k}^2 = k_y^2 + k_z^2; \qquad k_z + sxk_y = sXk_y,$$

then (2.6) becomes

$$\left\{\frac{\partial^2}{\partial X^2} - \frac{B_0^2 s^2 k_y^2}{p\eta\rho_0} X^2 + \bar{k}^2\left(\frac{\alpha g}{p^2} - 1\right)\right\} v_x = 0, \tag{2.7}$$

which is Weber's equation in the vertical coordinate X. As shown in Sec. VI; it leads to a set of G-modes localized near the point $X = 0$, in terms of which the quasi-modes of Sec. V may be expanded. These localized G-modes are *related* to the G-modes originally found by Furth, Killeen, and Rosenbluth—for example the growth rates and characteristic widths are the same—but they are not identical; in fact they are the modes investigated by Coppi.[3] The distinction may be explained as follows. Consider the full set of equations (2.2)–(2.5), without the approximation $\partial \mathbf{B}/\partial t \ll \eta \nabla^2 \mathbf{B}$ and assume a dependence $\sim \exp(ik_z z)$, then the equations are of fourth order in $\partial/\partial x$ and symmetric about $x = 0$. Each eigenvalue is doubly degenerate[2] in the limit $\eta \to 0$ and so to each eigenvalue p_n of the growth rate p there is an eigenfunction S_n in which v is symmetric, and another A_n in which v is antisymmetric.

As $\beta \to 0$, the even eigenfunctions I become localized[3] within the resistive layer near $X = 0$, and the external part of the solution vanishes; these are the eigenfunctions for which the approximation $\partial \mathbf{B}/\partial t \approx 0$ is valid and which are derived from (2.7). On the other hand, in Ref. 1 the assumption is made that $B_x \approx$ const within the resistive layer ($\psi \approx$ const in their notation); this rules out all S-modes (for which B_x is antisymmetric).

III. RAYLEIGH-TAYLOR INSTABILITY OF AN IDEAL FLUID IN A SHEARED FIELD

In this section we demonstrate, by means of a simple example, that interchange modes exist in a sheared system of finite length which are not of the form exp $(ik_z z)$. Although the fluid conductivity is assumed here to be perfect, this type of interchange may be regarded as a prototype of the twisted slicing mode in a resistive fluid (to be discussed in Sec. V), and it shows some analogies with more general types of instability in sheared systems with both finite and zero η.

As a preliminary, consider a system without shear, that is to say one with uniform field, zero resistivity, and perfectly conducting rigid endplates at $z = \pm L$. However the endplates are imagined to be coated with a thin perfectly insulating layer, so that the lines of force are *not* tied and interchanges can occur. These motions are, in fact, resistive instabilities but the resistivity is here concentrated at the endplates instead of being uniformly distributed. The boundary conditions to be applied at $z = \pm L$ are

$$B_z = v_z = (\nabla \times \mathbf{B})_z = 0. \qquad (3.1)$$

We consider normal modes with $k_z = 0$, for which Eqs. (2.2)–(2.5) show that all components of $\mathbf{B}$, together with v_z, are everywhere zero and (3.1) is satisfied identically. The magnetic field has no effect on these motions, and arbitrary two-dimensional interchanges can occur with

$$v_x(x, y, z) = v_x(x, y, 0), \quad v_y(x, y, z) = v_y(x, y, 0), \qquad (3.2)$$

$$v_z = 0, \qquad \partial v_x/\partial x + \partial v_y/\partial y = 0.$$

The growth rate of a normal mode is

$$p = (\alpha g)^{\frac{1}{2}} \, |k_y|/(k_x^2 + k_y^2)^{\frac{1}{4}}.$$

It may now be conjectured that the imposition of shear on this system, by introducing an extra transverse component $B_{y0}(x)$, could not prevent instability, since the volume of each flux tube $\int dl/B = \int dz/B_0$ remains unaltered. Interchange motions should still occur freely, but the flux tubes must *twist* during the interchange to follow the local field whose direction changes with height. In the limit $B_0 \rightarrow \infty$ the motion again becomes two-dimensional, and may be defined by the values of v_x, v_y on the midplane. The gravitational energy released by any interchange depends only on v_x and is unaffected by the shear, but the kinetic energy increases with the length of the system L because of the transverse velocity due to twisting. Therefore we may expect p to be decreased by increased length, or increased shear, but we certainly should not expect shear to stabilize the system. In the remainder of this section we show that these conjectures are correct.

As in Sec. II we assume $B_{y0} = sxB_0$. The unperturbed current $\mathbf{j}_0$ is in the z direction and is uniform, the force $\mathbf{j}_0 \times \mathbf{B}_0$ being balanced by pressure. Some further justification of the boundary conditions is now needed, since $\mathbf{j}_0$ has to pass to the conducting endplates across a layer which we have assumed to be an insulator. However it is consistent to assume a very large voltage drop across this layer in the unperturbed state, i.e., its conductivity may be made so small that it can be treated as an insulator so far as the perturbed variables are concerned, while still permitting the equilibrium $\mathbf{j}_0$.

The component $u \equiv v_x$ satisfies a fourth-order equation derived from (2.2)–(2.5) namely:

$$p^2 \rho_0 \nabla^2 u = \frac{B_0^2}{4\pi}\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)\nabla^2\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)u + \alpha g \rho_0\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u. \qquad (3.3)$$

This equation can be solved in the limit $B_0 \rightarrow \infty$ by expanding u in powers of $1/B_0^2$. That is we write $u = u_0 + u_1 + \cdots$. Then

$$D_0 u_0 = 0, \qquad D_0 u_1 = D_1 u_0, \cdots,$$

etc, where

$$D_0 = B_0^2\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)\nabla^2\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right), \qquad (3.4)$$

and

$$D_1 = \rho_0\left\{p^2\nabla^2 - \alpha g\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right\}, \qquad (3.5)$$

are self-adjoint operators. We also introduce a twisted coordinate system adapted to the unperturbed magnetic field, $\xi = x$, $\chi = y - sxz$, $\zeta = z$, then

$$\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y} = \frac{\partial}{\partial \zeta}, \qquad (3.6)$$

and ξ, χ are constant along the lines of force. Any function $f(\xi, \chi)$ which is independent of ζ is then a solution of $D_0 u_0 = 0$.

Now the next equation in the sequence, $D_0 u_1 = D_1 u_0$, can possess a solution u_1 only if $D_1 u_0$ is orthogonal to all solutions of $D_0 u_0 = 0$. This imposes a further constraint on u_0 which must now satisfy the equations

$$D_0u_0 = 0, \qquad \int_{-L}^{L} D_1u_0\, d\zeta = 0. \tag{3.7}$$

Solutions of the pair of equations (3.7) exist of the form $u_0 = u_0(\xi) \exp(ik\chi)$, if

$$d^2u_0/d\xi^2 + (A\xi^2 + B)u_0 = 0, \tag{3.8}$$

where

$$A = k^2s^2(\alpha g/p^2 - 1), \quad B = k^2(\alpha g/p^2 - 1 - \tfrac{1}{3}s^2L^2). \tag{3.9}$$

In order to fit the boundary conditions $u_0 = 0$ at $x = \pm H$, the quantity $u_0^{-1}\, \partial^2 u_0/d\xi^2$ must be negative in some range, which means that

$$(\alpha g/p^2 - 1) > \tfrac{1}{3}s^2L^2/(1 + s^2H^2), \tag{3.10}$$

therefore $A > 0$ and by a *real* transformation (3.8) can be reduced to a form of Weber's equation,

$$d^2u_0/dw^2 + (\tfrac{1}{4}w^2 - a)u_0 = 0. \tag{3.11}$$

The solutions of this equation are tabulated[10] but it is unnecessary to solve it in detail; we simply remark that for any finite values of s, k, H, L it is possible to find a real positive value of $p^2/\alpha g$ such that the solution of (3.11) satisfies the boundary conditions. In other words the system is always unstable for $\alpha g > 0$, as was physically obvious from the argument with which we introduced this section. In the limit $B_0 \to \infty$, the function u_0 represents the complete solution (since $u_1 \to 0$), and the growth rate is independent of the magnitude of the field, depending only on its form. For finite B_0 there would be a complicated correction due to bending of the field lines by the moving fluid.

We observe, then, that in a system of finite length with perfect conductivity but in which lines of force are not tied at the ends, there are instabilities even when the shear is sufficient to stabilize the corresponding infinitely long system, i.e., even when the Newcomb criterion[6] is satisfied. The effect of field shear on these modes is to introduce a constraint which determines the shape of the fluid motion. The growth rate is lowered because this constraint induces rotational kinetic energy, but the stability *criterion* is unaltered.

IV. RESISTIVE INSTABILITY IN UNIFORM FIELD

In the perfect conductivity example discussed in the preceding section the introduction of shear did not alter the fundamental character of the Rayleigh–Taylor interchange modes; it simply twisted them to conform to the field lines and slowed down their growth rate. We expect shear to have a similar effect when the resistivity is distributed uniformly throughout the system instead of being concentrated into thin layers at the ends. Accordingly it is useful to first consider interchange-like motions in a resistive system with zero shear, but with *perfectly conducting* endplates at $z = \pm L$. Some care must be taken with the boundary conditions, since the equations are of fourth order in d/dz and will therefore not in general have sinusoidal solutions in a bounded region. We choose $B_z = v_x = v_y = 0$ at $z = \pm L$ but place no restriction on v_z. These conditions are equivalent to tying the tubes of force at the ends and preventing any transverse fluid displacement there. It can be shown that v_z is any case very small at $z = \pm L$, so that the precise choice of boundary conditions should have little effect.

Solutions then have the form

$$v_x \sim \cos\frac{m\pi z}{2L}\cos\frac{l\pi x}{2H}\begin{Bmatrix}\cos k_yy\\ \sin k_yy\end{Bmatrix}, \tag{4.1}$$

and correspondingly for other components, but we shall assume a dependence $\exp(pt+ik_xx+ik_yy+ik_zz)$ and represent the influence of the ends by the requirement $k_z \geq \pi/2L$. In particular the case $k_z = 0$ is now forbidden. The finite height is represented by $k_x \geq \pi/2H$, and we set $k^2 = k_x^2 + k_y^2 + k_z^2$, $\bar{k}^2 = k_y^2 + k_z^2$. Equations (2.2)–(2.5) then yield

$$p^3 + \frac{\eta k^2p^2}{4\pi} + \left(\frac{B_0^2k_z^2}{4\pi\rho_0} - \alpha g\frac{\bar{k}^2}{k^2}\right)p - \frac{\alpha g\eta\bar{k}^2}{4\pi} = 0. \tag{4.2}$$

It can be proved that no overstable modes exist, so we need only consider instabilities with real p. Also since the fluid is to be stable for $\eta = 0$ we must assume $B_0^2k_z^2/4\pi\rho_0 > \alpha g$, which for a real plasma is a condition on the ratio β of particle to magnetic pressure. We shall suppose $\beta \ll 1$ and therefore drop the second term in the bracket. Finally, the first term of (4.2) must be negligible if $p < c_Ak_z$, where $c_A = B_0/(4\pi\rho_0)^{\frac{1}{2}}$ is the Alfvén speed, i.e., if the growth rate is less than the frequency of an Alfvén wave of wavelength $\approx L$. Then the approximate dispersion relation is

$$p^2 + B_0^2k_z^2p/\eta k^2\rho_0 - \alpha g\bar{k}^2/k^2 = 0. \tag{4.3}$$

When p is small one root of this equation is

$$p \sim (\alpha g\rho_0/B_0^2k_z^2)\eta\bar{k}^2 \sim (\alpha g\rho_0L^2/B_0^2)\eta\bar{k}^2, \tag{4.4}$$

and to apply this to a real plasma we identify g in

[10] J. C. P. Miller, *Tables of Weber Parabolic Cylinder Functions* (Her Majesty's Stationery Office, London, 1955).

terms of pressure P_0 and radius of curvature R_c by $g \sim 2P_0/\rho_0 R_c$ so that

$$p \sim \lambda(\beta\eta \bar{k}^2/4\pi), \tag{4.5}$$

where $\lambda = 8\alpha L^2/\pi^2 R_c$ is a purely geometric factor of order unity. We can assume $k_x, k_z \ll k_y$, then the growth rate is almost independent of k_x and is proportional to k_y^2. The pattern of this instability is that of a "slicing" motion in which alternate thin vertical layers, parallel to the magnetic field, are moving up and down. The growth rate p increases as the transverse thickness of the slices becomes smaller, but eventually it is necessary to include the first term of (4.3), and $p \rightarrow (\alpha g)^{\frac{1}{2}}$ as $k_y \rightarrow \infty$.

The physical reason why fluid can move across the field in this way is the following. Imagine two vertical layers $\simeq k_y^{-1}$ apart, with the fluid moving up in one and down in the other, and suppose that the density perturbation has reached an amplitude $A \cos(\pi z/2L)$, so that the vertical fluid displacement $\epsilon \simeq A/\alpha$. At each stage equilibrium would be maintained if the field lines were displaced by a distance

$$\delta\epsilon \simeq A g \rho_0 / B_0^2 k_z^2,$$

so that the weight of the fluid would be balanced by tension in the field. This generates a transverse field $B_x \sim \sin(\pi z/2L)$ with opposite sign in the two layers. Within a time $\simeq (\eta k_y^2)^{-1}$ this B_x field disappears by transverse resistive diffusion so that the motion can proceed; the growth rate is therefore

$$\frac{1}{\epsilon}\frac{d\epsilon}{dt} \simeq \frac{\alpha g \rho_0}{B_0^2 k_z^2}\eta k_y^2,$$

in agreement with (4.4).

It is worth noting that while the fluid motion is vertical, the field diffusion is horizontal and can be made arbitrarily fast by choosing a large k_y (i.e., thin "slices"). There is no relation between the large scale length (k_x^{-1}) of the fluid motion and the small scale length (k_y^{-1}) over which the field diffusion occurs and one can thus retain large-scale eddies in the x direction while allowing $k_y \rightarrow \infty$; in this limit the field has no influence on the motion at all. (In practice the rapid short-wavelength slicing motion would be limited by viscosity or some nonfluid effect such as finite Larmor radius.)

In this simple model *no instabilities have been found in which p depends on fractional powers of* η, as in the modes obtained in Refs. 1–3. Nevertheless we assert that G-modes with $p \sim \eta^{\frac{1}{3}}$ are generically the same as the slicing mode discussed in this section, which has $p \sim \eta$. The reason for this, as will be shown in Sec. V, is that in a sheared field the slicing mode becomes twisted and as a result it automatically takes up a length which is proportional to $\eta^{-\frac{1}{3}}$. It can be seen from (4.4) that if we put $L \sim \eta^{-\frac{1}{3}}$ then we do indeed get a growth rate $p \sim \eta^{\frac{1}{3}}$ as found in Ref. 1 and others.

V. TWISTED SLICING MODES

In this section we examine the final model, that is a plasma slab of infinite extent in the z direction, with finite resistivity and finite shear. Guided by the simpler situations discussed in the previous sections we look for modes which are similar to those of Sec. IV, but are twisted to follow field lines like those of Sec. III.

Since the presence of shear will not increase the growth rate, $p \ll \eta k^2$ as in Sec. IV, and the $\partial \mathbf{B}/\partial t$ term can be omitted. We can therefore start from Eq. (2.6) and introduce the twisted coordinate system used in Sec. III, $\xi = x, \chi = y - sxz, \zeta = z$.

We transform Eq. (2.6) into these new coordinates and look for solutions of the form

$$v_x = v(\zeta) \exp(ik_x\xi + ik_y\chi), \tag{5.1}$$

then Eq. (2.6) gives

$$(1 + \epsilon^2 q)\frac{1}{k_y^2}\frac{\partial^2 v}{\partial \zeta^2} - 2\epsilon^2 q i s \xi \frac{1}{k_y}\frac{\partial v}{\partial \zeta} - \epsilon^2\Bigg[q(1 + s^2\xi^2) + \frac{p^2}{\alpha g}s^2\zeta^2 - \frac{p^2}{\alpha g}\left(\frac{k_x^2}{k_y^2} - 2s\zeta\frac{k_x}{k_y}\right)\Bigg]v = 0, \tag{5.2}$$

where

$$\epsilon^2 = \alpha g \rho_0 \eta / p B_0^2, \quad q = p^2/\alpha g - 1\,. \tag{5.3}$$

The results of the earlier sections, particularly Sec. IV, Eq. (4.4), lead us to expect a "twisted slicing" mode for which

$$p \simeq (\alpha g \rho_0 \eta / B_0^2) k_y^2 L^2, \tag{5.4}$$

where L is the length of the mode in the z, or ζ, direction. For such a mode $\epsilon = (k_y L)^{-1}$ and in the limit $k_y L \gg 1$ and $k_x/k_y \ll 1$, Eq. (5.2) reduces to

$$\frac{d^2 v}{d\zeta^2} - \frac{p\rho_0\eta}{B_0^2}(sk_y)^2\zeta^2 v + \frac{p\rho_0\eta k_y^2}{B_0^2}\left(\frac{\alpha g}{p^2} - 1\right)v = 0, \tag{5.5}$$

which can again be transformed into Weber's equation

$$\left(\frac{d^2}{dw^2} - \tfrac{1}{4}w^2 + a\right)v = 0, \tag{5.6}$$

(but note that this is now an equation for the dependence *along* the field), with

$$\tfrac{1}{4}w^2 = \frac{\zeta^2}{2\Delta^2}, \quad \Delta = \left(\frac{B_0}{sk_y}\right)^{\frac{1}{2}}\left(\frac{1}{p\rho_0\eta}\right)^{\frac{1}{4}}, \tag{5.7}$$

and

$$2a = \frac{(p\rho_0\eta)^{\frac{1}{2}}}{sB_0} k_y\left(\frac{\alpha g}{p^2} - 1\right). \quad (5.8)$$

In a long system the solutions of (5.5) are therefore

$$v = v_n(\zeta) = \exp(-\zeta^2/2\Delta^2)H_n(\zeta\sqrt{2}/\Delta), \quad (5.9)$$

where H_n are the Hermite polynomials

$$H_n(w) = (-1)^n \exp(\tfrac{1}{2}w^2) \frac{d^n}{dw^n} \exp(-\tfrac{1}{2}w^2), \quad (5.10)$$

and the growth rate of this mode is (for $\alpha g \gg p^2$ as in Sec. III)

$$p_n = \left(\frac{\eta k_y^2}{4\pi}\right)^{\frac{1}{3}} (\alpha g)^{\frac{2}{3}}\left(\frac{4\pi\rho_0}{B_0^2 s^2}\right)^{\frac{1}{3}}(2n+1)^{-\frac{2}{3}}, \quad (5.11)$$

which is the same as that found in Ref. 1 for *G*-modes. [As $k_y \to \infty$ we also find that $p \to (\alpha g)^{\frac{1}{2}}$ as before.] The half-width of the disturbance in ζ or z, i.e., the length of the mode along the field, is

$$\Delta_n = \left(\frac{4\pi}{\eta k_y^2}\right)^{\frac{1}{3}} \frac{1}{(\alpha g)^{\frac{1}{6}}} \left(\frac{B_0^2}{4\pi\rho_0}\right)^{\frac{1}{3}} \frac{1}{s^{\frac{1}{3}}} (2n+1)^{\frac{1}{3}}. \quad (5.12)$$

The vertical wavenumber k_x has no effect on the growth rate or the length Δ of the mode, provided only that $k_x \ll k_y$. Hence we can replace the x dependence of our modes by any arbitrary dependence $g(x)$ so long as this is slowly varying compared to the width of the slices (k_y^{-1}). We thus obtain modes of the form

$$v_x(x, y, z) = g(x)v_n(z) \exp[ik_y(y - sxz)], \quad (5.13)$$

where $v_n(z)$ is the appropriate Hermite function (5.9) with growth rate given by (5.11).

These modes represent a motion (Fig. 1) of the expected type. It is specified at one plane z = const by the function $g(x)$ (which is arbitrary provided it is slowly varying and vanishes at $x = \pm H$), and in this plane takes the form of "convective rolls" as in ordinary hydrodynamics; its form at any other value of z is determined by the fact that the flow pattern is almost constant along any field line ($y - sxz$ = constant). The rolls thus get twisted as one moves along z but at the same time the flow velocity also decays slowly away in z because of the term $v_n(z)$. For the fastest growing mode, $v_0(z)$ is a simple gaussian curve with characteristic width Δ. Higher modes have an oscillatory z dependence. Since $\Delta \sim \eta^{-\frac{1}{3}}$ the length (in z) of the twisted slices increases indefinitely as $\eta \to 0$.

The relation of this twisted slicing mode to those found in Sec. IV is now apparent. In Sec. IV the length L of the slices was set by the position of the endplates; in an infinitely long resistive plasma such as we consider in this section the length is set by the resistivity according to (5.7). If we identify the length L of Sec. IV with this "natural" length Δ then the growth rate of the two types of slicing modes are in agreement.

Physically the natural length Δ is set by a compromise between (a) rate of release of gravitational potential energy, (b) rate of resistive dissipation, and (c) rate of increase of kinetic energy. A feature of the twisted slicing motion is that in order to reduce (b) the fluid motion must follow the field lines which means that the fluid elements must rotate about a vertical axis as they rise or fall; most of the kinetic energy (c) is in this rotation and it is in order to keep this energy finite that the modes must have finite length in z. They achieve this finite length at the expense of some increase in (b), hence the length increases as $\eta \to 0$.

VI. DECOMPOSITION INTO SPATIALLY PERIODIC NORMAL MODES

We have established in the previous sections the existence of twisted slicing modes in a resistive fluid supported by a sheared magnetic field. These modes are of quite a different character to the *G*-modes found in Refs. 1 and 3 for the same problem, and one naturally asks how the two types are related.

To determine this we first re-examine the *G*-modes. These are spatially periodic in z (unlike our modes which have a definite length Δ) and are of the form

$$v_x = v_g(x) \exp(pt + ik_y y + ik_z z),$$

where $v_g(x)$ satisfies Eq. (2.7), i.e.,

$$\left\{\frac{\partial^2}{\partial X^2} - \frac{B_0^2 s^2 k_y^2}{p\eta\rho_0} X^2 + \bar{k}^2\left(\frac{\alpha g}{p^2} - 1\right)\right\}v_g(X) = 0, \quad (6.1)$$

with $\bar{k}^2 = k_y^2 + k_z^2$, $sXk_y = sxk_y + k_z$.

If we put

$$\tfrac{1}{4}w^2 = \frac{X^2}{2\delta^2}, \qquad \delta = \frac{(p\eta\rho_0)^{\frac{1}{4}}}{(B_0 s k_y)^{\frac{1}{2}}}, \quad (6.2)$$

this can be again reduced to Weber's equation and leads to eigenfunctions

$$v_g(X) = u_n(X) = \exp(-X^2/2\delta^2)H_n(X\sqrt{2}/\delta), \quad (6.3)$$

and eigenvalues p_n satisfying

$$\bar{k}^2\left(\frac{\alpha g}{p_n^2} - 1\right) = \frac{B_0 s k_y}{(p_n\eta\rho_0)^{\frac{1}{2}}} (2n+1). \quad (6.4)$$

These are *G*-modes which are highly localized around $X = 0$, i.e., around $x = -k_z/sk_y$; they have a half-width in the x direction of order δ, and $\delta \to 0$ as

$\eta \to 0$. Thus they shrink in x as the twisted slices grow in z. They form a subset of the complete set of G-modes for the problem and are related to, but not identical with, the original modes of Ref. 1.

Although these G-modes are different in character from the twisted slicing modes, their growth rates p are almost exactly the same; the only difference is that in (6.4) the term $\bar{k}^2 = k_y^2 + k_z^2$ replaces the term k_y^2 in (5.11). This means that G-modes which have the same k_y, but are localized at different heights x by reason of having different k_z, will also have slightly different growth rates. However if the shear is small this difference in growth rate is also small; in fact two G-modes of the same k_y, but with their k_z chosen so that they are localized at heights x_0 apart, have growth rates which differ only by

$$\delta p/p = \tfrac{2}{3}k_z^2/k_y^2 = \tfrac{2}{3}(sx_0)^2. \qquad (6.5)$$

For a Stellarator $(sx_0)^2$ is to be identified with $(\iota r_0/2\pi R_0)^2$, where ι is the rotational transform and r_0 and R_0 are the minor and major radii; this is typically of order 10^{-3}. This means that all G-modes with the same k_y but varying x_0 have almost identical growth rates. It also follows from (6.4) that

$$\frac{p\delta^2}{\eta} = \frac{\alpha g \rho_0}{B_0^2 s^2}(2n+1)^{-1} \approx \beta(2n+1)^{-1}, \qquad (6.6)$$

which justifies our neglect of the term $\partial \mathbf{B}/\partial t$ within the localized region near $X = 0$. For n even the field is confined to this region, but for n odd it extends beyond the neighborhood of $X = 0$ and invalidates the approximation.

Now consider a combination of periodic G-modes, all having the same value of k_y and $n = 0$, but centered at different heights x_0. This can be achieved by taking a spread of values of k_z and leads to

$$u(x, y, z, t) = \exp(ik_y y) \int f(k_z)\, dk_z \cdot \exp[ik_z z - (x - x_0)^2/2\delta_0^2] \exp(pt), \qquad (6.7)$$

where $f(k_z)$ is the weight function which we take to be slowly varying. The center of each constituent mode x_0 is related to k_z by $x_0 = -k_z/sk_y$, and p also depends on k_z but is again slowly varying. To evaluate (6.7) we note that the integrand is of the form of a highly localized function $\exp[(x-x_0)^2/2\delta_0^2]$ multiplied by slowly varying functions which can therefore be replaced by their values at $x = x_0$. Then if $f(k_z)dz = -g(x_0)dx_0$,

$$u(x, y, z, t) = \delta\sqrt{2}\,\pi g(x) \exp[ik_y(y - sxz) - (sk_y z\delta_0)^2/2 + p(0)(1 + \tfrac{2}{3}(sx)^2)t], \qquad (6.8)$$

which has the form of the twisted slicing mode studied in Sec. V, except for the weak dependence of growth rate on x. The arbitrary function $g(x)$ may be chosen to fit the boundary conditions.

The expression (6.8) does not therefore represent an *exact* normal mode, because it has no precise time dependence. Nevertheless, because sx is small it will behave like a normal mode for all practical purposes and may call it a *quasi-mode*. With $\Delta p/p \sim 10^{-3}$ the different components would keep in step for 1000 e-folding times, an enormously long period. Probably even $\Delta p/p \sim 10^{-1}$ would allow components to hold together until the disturbance was out of the linear regime. The essential point of our argument is that because the localized G-modes are *almost degenerate* any combination of them is itself "almost" a true mode, and for weakly sheared systems such as the Stellarator the distinction between a true mode and a quasi-mode is imperceptible for many hundreds of e-folding periods.

The relation between the "twisted slicing" quasi-modes and the periodic modes is analogous to the relation between compound nucleus states and scattering states, or between a wavepacket in a slightly dispersive medium and infinite periodic waves. Although the scattering states or the periodic waves may be mathematically a more exact description than the compound nucleus or the wavepacket, the latter may be more useful in practice. In the case of instabilities such as we are discussing it is especially unrealistic to ask for precise normal modes; any mode which preserves its form for many growth periods is permissible since after many periods the system is in any case out of the linear phase. In this respect the fact that the concept of convective cells retains its usefulness in the nonlinear phase of ordinary convective instability may suggest a similar utility of the twisted convective cells in the present problem.

The importance of "quasi-modes" for plasma loss is the following. The growth rate of conventional G-modes suggests that they would have a significant effect on plasma containment, however, the localized nature of these modes makes it difficult to understand exactly how they contribute to plasma loss. For low β the growth rate $p \sim \beta\eta/h^2$, where h is a measure of the mode height, and if we assume that each G-mode corresponds to an eddy of height h and velocity hp, the eddy diffusion coefficient is $\sim\beta\eta$ which is of the same order as the classical diffusion. However the present results show that the G-modes may be combined *coherently* into extended quasi-modes which can greatly enhance the plasma loss.