

# PHYS 201 Mathematical Physics, Fall 2015, Midterm Solutions

1. An infinitely long conducting cylinder of unit radius and potential 0 is placed next to an infinite conducting plate of potential  $V$ . The axis of the cylinder is parallel to the plate and at a distance of  $5/4$  units from it. Solve Laplace's equation and find the potential in the relevant domain. Follow these steps:
  - a. Is the domain simply connected or doubly connected?
  - b. Based off (a), find a bilinear mapping of the form  $f(z) \propto \frac{z-\alpha}{\alpha z-1}$  ( $\alpha$  real) to a domain where the problem can be solved easily.
  - c. Obtain the potential in the transformed region and hence find the potential in the original region.

## Solution:

- a. In 2D, the domain is a unit circle, assumed to be centered at the origin, and a vertical line at  $\text{Re}(z) = 5/4$ . The domain is doubly connected i.e., between any two points, there are two paths that cannot be smoothly deformed to the other. The problem can therefore be solved easily by finding a mapping to an annular region.
  - b. Any mapping of the form  $\frac{z-\alpha}{\alpha z-1}$  with  $\alpha$  real maps a unit circle onto itself (can check by substituting  $z = e^{i\theta}$ ). To find the value of  $\alpha$  that maps the vertical line to another circle, plug in  $z = 5/4 + iy$  for real  $y$  and make sure  $|f(z)|$  is a constant.  $\alpha = 2$  works.
  - c. Laplace's equation can be easily solved in the annular region. The real potential is of the form  $\phi(r, \theta) = A + B \ln r$ . The complex potential is therefore  $\Phi(\zeta) = A + B \ln \zeta$ . Plug in  $\zeta = f(z)$  to get the potential in the original domain.
2. Answer any two of the three parts below:
    - i. Using contour integration, evaluate

$$I = \text{PV} \int_{-1}^1 \frac{dx}{x+b} \sqrt{\frac{1-x}{1+x}}$$

for  $-1 < b < 1$ . You may leave the answer in terms of assigned labels (for instance,  $\alpha_1$  and  $\alpha_2$  as the roots of a quadratic without writing down the explicit form).

ii. Show that, if  $-\pi < \alpha < \pi$ ,

$$\int_0^\infty \frac{\cosh \alpha x}{\cosh \pi x} dx = \frac{1}{2} \sec \frac{\alpha}{2}$$

iii. Using contour integration, evaluate

$$\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx$$

**Solution:**

i. Plug in  $x = \cos \theta$  and we get

$$I = \int_0^\pi \frac{\sin \theta}{\cos \theta + b} \tan \frac{\theta}{2} d\theta$$

Notice that this equation is even, so we can equate  $2I$  to the integral from  $-\pi$  to  $\pi$ . It can be solved like most other trigonometric integrals: substitute  $z = e^{i\theta}$  and we have

$$2I = \oint_{|z|=1} \frac{i(z-1)^2}{z(z^2+2bz+1)} dz$$

The roots of the quadratic in the denominator,  $\alpha_1$  and  $\alpha_2$ , lie on the unit circle i.e.,  $|\alpha_1| = |\alpha_2| = 1$  and satisfy  $\alpha_1\alpha_2 = 1$ . There are thus three poles:  $z = 0, \alpha_1, \alpha_2$ . The latter two contribute only half the residue since they lie on the contour. Either way, the sum of the residues from these two poles cancel out, as you can check. The residue at zero is simply  $i$ , and therefore  $I = -\pi$ .

ii. Use a rectangular contour with diagonally opposite vertices  $-R$  and  $R+i$  as  $R \rightarrow \infty$ . The integrals on the lateral sides of the rectangle go to zero because  $-\pi < \alpha < \pi$  and therefore the leading term of the denominator dominates on both sides. There is a simple pole at  $i/2$  with residue  $\frac{\cos(\alpha/2)}{\pi i}$ . Adding up the contributions from the two horizontal sections of the contour, we get  $2(1+\cos \alpha)I = 2\pi i \frac{\cos \alpha/2}{\pi i}$ , which gives the required answer.

iii. Split the integral as

$$2I = \int_{-\infty}^\infty \frac{\ln(x+i)}{1+x^2} dx + \int_{-\infty}^\infty \frac{\ln(x-i)}{1+x^2} dx$$

For the first integral, close the contour using an upper semicircle. For the second, use a lower semicircle. We get  $I = \pi \ln 2$ .

3. Find the leading behavior as  $x \rightarrow \infty$  of

$$I = \int_0^\infty e^{xt} t^{-t} dt$$

**Solution:** The integral can be written as  $I = \int_0^\infty e^{xt - t \ln t} dt$ . The maximum in the exponent of the integrand is dependent on  $x$ , which is a cue to look for a substitution that avoids this.  $t = se^x$  does the trick and we get  $I = e^x \int_0^\infty e^{-e^x s \ln s} ds$ . The minimum of  $s \ln s$  occurs at  $s = 1/e$  and the double derivative at that point is  $e$ . Expanding the exponent up to second order, we get the leading behavior as  $I \sim \sqrt{2\pi} e^{e^{x-1}} e^{\frac{x-1}{2}}$ .

4. The modified Bessel function has the integral representation

$$K_\nu(x) = \frac{1}{2} \int_{-\infty}^\infty dt \exp[-x \cosh t + \nu t]$$

Show that under the conditions  $x \rightarrow \infty$ ,  $p/x \rightarrow \infty$  ( $p$  and  $x$  are both real),

$$K_{ip}(x) \sim \sqrt{2\pi} (p^2 - x^2)^{-1/4} e^{-p\pi/2} \sin \phi(x)$$

where

$$\phi(x) = \sqrt{p^2 - x^2} + p \ln \left( \frac{p + \sqrt{p^2 - x^2}}{x} \right) + \frac{\pi}{4} + \text{lower order terms}$$

Follow these steps:

- Show that the saddle points satisfy  $\sinh t = ip/x$ . Find all such points (write down the real and imaginary components of the saddle points).
- Identify the steepest descent/ascent directions at the saddle points.
- Show that there exists a constant phase contour that begins at  $t = -\infty + i0$  and passes through all the saddle points in the upper left and right quadrants. Draw the constant phase contour.
- Explain why it is that although there are infinite number of saddle points, only two contribute to the leading behavior.
- Derive the formula for  $K_{ip}(x)$  by collecting the contributions from the saddle points.

**Solution:**

- a. The integral is of the form  $K_{ip}(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{h(t)} dt$ . The saddle point satisfies  $h'(t) = 0$  which trivially gives  $\sinh t = ip/x$ . To find the real and imaginary components of the saddle points, substitute  $t = u + iv$ . We have  $\sinh(u + iv) = \sinh u \cos v + i \cosh u \sin v = ip/x$  and so  $\sinh u = 0$  or  $\cos v = 0$ . The former option is not possible because if  $u = 0$ , then  $\sin v = p/x$  is unbounded. So we have  $\cos v = 0$  with  $\sin v = 1$  so that  $v = (4n + 1)\pi/2$ . We also have  $u = \pm \cosh^{-1} p/x$ .
- b. The constant phase contour is also the steepest descent/ascent contour. To find this, notice that the constant phase contour around a saddle point  $t_0$  satisfies  $\text{Im}(h(t)) = \text{Im}(h(t_0))$ . Locally, we have  $h(t) \approx h(t_0) + \frac{h''(t_0)}{2}(t - t_0)^2$  (we just need to find the constant phase contour locally since the largest contribution to the integral is from around the saddle points). The direction in which the imaginary part of  $\frac{h''(t_0)}{2}(t - t_0)^2$  is zero gives the steepest ascent/descent direction.  $h''(t_0) = -x \cosh t_0 = -xi \sinh(u_0)$  where  $u_0 = \pm \cosh^{-1} p/x$ . For positive  $u_0$ , to make the second derivative term real and get the Gaussian-like steepest descent, the direction is at a  $45^\circ$  angle i.e.,  $t = t_0 \pm se^{i\pi/4}$  and for negative  $u_0$ , the direction is at a  $-45^\circ$  angle i.e.,  $t = t_0 \pm se^{-i\pi/4}$ .
- c. The constant phase contour begins from  $-\infty + i0$  and going up passes through the saddle points in the upper left quadrant at an angle of  $-45^\circ$  (or  $-135^\circ$ ) as mentioned in part b. The contour then extends to the upper right quadrant and forms a symmetric contour going down and passing through all the saddle points at  $45^\circ$  (or  $135^\circ$ ).
- d. At the saddle points,  $h(t_0) = -x \cosh t_0 + ipt_0 = -xi \sinh(u_0) + ipu_0 - pv_0$ . The real part of the integrand is  $e^{-pv_0}$  and therefore only the saddle points with the smallest  $v_0$  matter for the leading behavior. There are two of them:  $t_+ = (\cosh^{-1} p/x, \pi/2)$  and  $t_- = (-\cosh^{-1} p/x, \pi/2)$ .
- e. To get the required answer, add the contribution to  $h(t)$  at the two saddle points along with the Gaussian factor at each one. Some identities which might help are  $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ ,  $\sinh \cosh^{-1}(x) = \sqrt{x^2 - 1}$  and  $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}$ .