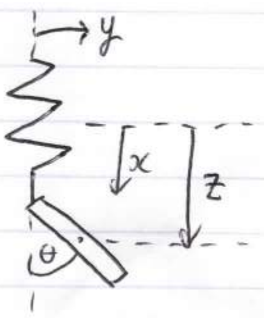


$$T = \frac{1}{2} I_{cm} \dot{\theta}^2 + \frac{1}{2} m V_{cm}^2$$

$$I_{cm} = \frac{1}{3} m l^2$$

$$V_{cm}^2 = \dot{z}^2 + \dot{y}^2$$

$$z = x + l\cos(\theta) \qquad \dot{z} = \dot{x} - l\dot{\theta}\sin(\theta)$$

$$y = l\sin(\theta) \qquad \dot{y} = l\dot{\theta}\cos(\theta)$$

$$\Rightarrow \dot{z}^2 + \dot{y}^2 = \dot{x}^2 - 2l\dot{\theta}\dot{x}\sin(\theta) + l^2\dot{\theta}^2$$

$$\Rightarrow T = \frac{2}{3} m l^2 \dot{\theta}^2 + \frac{1}{2} m\dot{x}^2 - m l \dot{x}\dot{\theta}\sin(\theta)$$

$$V = -mg(x + l\cos(\theta)) + \frac{1}{2} k x^2$$

$$\Rightarrow L = \frac{2}{3} m l^2 \dot{\theta}^2 + \frac{1}{2} m\dot{x}^2 - m l \dot{x}\dot{\theta}\sin(\theta) + mgx + mgl\cos(\theta) - \frac{1}{2} k x^2$$

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - m l\dot{\theta}\sin(\theta) \qquad (1)$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{4}{3} m l^2 \dot{\theta} - m l \dot{x}\sin(\theta) \qquad (2)$$

$$(1) \Rightarrow \dot{x} = \frac{P_x}{m} + l\dot{\theta}\sin(\theta)$$

$$(2) \Rightarrow \frac{4}{3}\dot{\theta} = \frac{P_\theta}{m l^2} + \frac{\dot{x}}{l}\sin(\theta)$$

$$\Rightarrow \dot{\theta} = \frac{3P_\theta}{4ml^2} + \frac{3\dot{x}}{4l}\sin(\theta)$$

$$\dot{x} = \frac{P_x}{m} + \frac{3P_\theta \sin(\theta)}{4ml} + \frac{3\dot{x}}{4}\sin^2(\theta)$$

$$\Rightarrow \dot{x} = \frac{4lP_x + 3P_\theta \sin(\theta)}{4ml - 3ml\sin^2(\theta)}$$

$$\& \ \dot{\theta} = \frac{6lP_x \sin(\theta) + 6P_\theta}{5ml^2 + 3ml^2\cos(2\theta)}$$

$$\Rightarrow H = \frac{6P_\theta^2 + 12lP_\theta P_x \sin(\theta) + l^2(-m(2gmx - kx^2 + 2glm\cos(\theta))(5 + 3\cos(2\theta) + 8P_x^2)}{2l^2m(5 + 3\cos(2\theta))}$$

$$\Rightarrow \dot{x} = \frac{12lP_\theta \sin(\theta) + 16l^2P_x}{2ml^2(5 + 3\cos(2\theta))} \qquad \left[= \frac{4lP_x + 3P_\theta \sin(\theta)}{4ml - 3ml\sin^2(\theta)}\right]$$

$$\dot{\theta} = \frac{12lP_x \sin(\theta) + 12P_\theta}{2ml^2(5 + 3\cos(2\theta))} \qquad \left[= \frac{6lP_x \sin(\theta) + 6P_\theta}{5ml^2 + 3ml^2\cos(2\theta)}\right]$$

$$\dot{P}_x = gm - kx$$

$$\dot{P}_\theta = \frac{-1}{ml^2(5 + 3\cos(2\theta))^2}\Big\{gm^2l^3(5 + 3\cos(2\theta))^2\sin(\theta)$$

$$+ 6\cos(\theta)(6\sin(\theta)P_\theta^2 + l(11 - 3\cos(2\theta)P_\theta P_x$$
$$+ 8l^2P_x^2\sin(\theta))\Big\}$$

Jialin Fei A53111966

Problem Set III

As assuming sound is beamed in $\hat{z}$ direction. The answer of $\nabla^2\psi + \frac{\omega^2}{c_0^2}n^2(\vec{x})\psi = 0$ equation can be expressed in.

2. a) $\psi = \psi_0 e^{ik_z z}$ $\quad \left(\frac{\partial_z \psi_0}{\psi_0}\right) \ll |k_z|$

$$\nabla\psi = (\nabla\psi_0 + ik_z\psi_0\hat{z})e^{ik_z z}$$

$$\nabla^2\psi = \nabla\cdot\nabla\psi = (\nabla^2\psi_0 + ik_z\hat{z}\cdot\nabla\psi_0)e^{ik_z z} + ik_z\hat{z}e^{ik_z z}\cdot(\nabla\psi_0 + ik_z\psi_0\hat{z})$$

As $\frac{\partial_z\psi_0}{\psi_0} \ll |k_z|$, $k_z^2 \gg \frac{\partial_z^2\psi_0}{\psi_0}$

$$= (\nabla^2\psi_0 + 2ik_z\partial_z\psi_0 - k_z^2\psi_0)e^{ik_z z}$$

$$\simeq [(\nabla_\perp^2 + 2ik_z\partial_z - k_z^2)\psi_0]e^{ik_z z}$$

perpendicular.

$$k_z^2 = \frac{\omega^2}{c_0^2} \quad (\text{zero}^{th} \text{ order})$$

plug into $\nabla^2\psi + \frac{\omega^2}{c_0^2}[1+\delta(\vec{x})]\psi = 0$

$$2ik_z\frac{\partial\psi}{\partial z} + \nabla_\perp^2\psi + \frac{\omega^2}{c_0^2}\delta(\vec{x})\psi = 0$$

b) i) Approximation: $|k_z| \gg \left|\frac{\partial_z\psi_0}{\psi_0}\right|$ (the amplitude varies slowly since \delta is small)

change in phase >> that of amp. in z-direction.

ii) The first term is a source term, the second a diffraction term, the third a scattering term.

$(\nabla_\perp^2\psi)$

iii) $\partial_z\psi$, $\delta(\vec{x})\psi$ should be in same order.

In particular, the amplitude should vary more slowly than the frequency in some comparable length scale.

c) ~~let~~ $\psi = A(\vec{x})\, e^{i\phi(\vec{x})}$

$$\partial_3\psi = [\partial_3 A + i(\partial_3\phi)A]e^{i\phi}$$

$$\nabla_\perp\psi = [\nabla_\perp A + i(\nabla_\perp\phi)A]e^{i\phi}$$

$$\nabla_\perp^2\phi = \nabla_\perp\cdot\nabla_\perp\phi = [\nabla_\perp^2 A + i\nabla_\perp\phi\cdot\nabla_\perp A + iA(\nabla_\perp^2\phi)]e^{i\phi}$$

$$+ [\nabla_\perp A + i(\nabla_\perp\phi)A]\cdot i\nabla_\perp\phi\, e^{i\phi}$$

$$= \nabla_\perp^2 A + 2i(\nabla_\perp\phi)(\nabla_\perp A) + iA(\nabla_\perp^2\phi) - |\nabla_\perp\phi|^2 A$$

$$2ik_3\partial_3 A - 2k_3(\partial_3\phi)A + \nabla_\perp^2 A + 2i(\nabla_\perp\phi)\cdot(\nabla_\perp A)$$

$$+ iA(\nabla_\perp^2\phi) - |\nabla_\perp\phi|^2 A + \frac{\omega^2}{c_0^2}\delta(\vec{x})\, A = 0$$

Real Part:

$$-2k_3(\partial_3\phi)A + \nabla_\perp^2 A - |\nabla_\perp\phi|^2 A + \frac{\omega^2}{c_0^2}\delta(\vec{x})A = 0$$

except for the 2^{nd} term, $\delta(\vec{x})$ gives rise to change in ϕ as in ~~eikonal~~ eikonal theory: $|\nabla\phi|^2 = \frac{\omega^2}{c_0^2}n^2(\vec{x})$

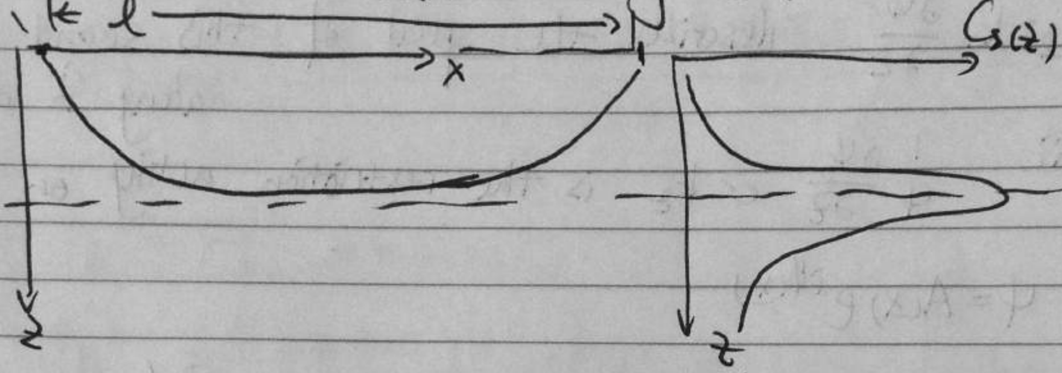
Imaginary Part:

$$2k_3\partial_3 A + 2(\nabla_\perp\phi)\cdot(\nabla_\perp A) + A(\nabla_\perp^2\phi) = 0$$

If we apply eikonal theory to a beam propagating in the z-direction, we get these equations except that here we have separated the z-direction from the perpendicular directions to make appropriate approximations corresponding to the geometry of the problem. One of these equations describes the scattering of the amplitude, the other the variation of the wave in the z-direction.

Siyang Wang

prob#3 $C_s(z)$ must have a peak to act like an acoustic wave guide.



Fermat $\Rightarrow$ $0=\delta\int_1^2 \frac{1}{C_s(z)}\,ds$

$|d\underline{x}|=ds$

$$0=\delta\int_1^2 \frac{1}{C_s(z)}\left(\frac{d\underline{x}}{ds}\frac{d\underline{x}}{ds}\right)^{1/2} ds$$

where $\frac{d\underline{x}}{ds}=\dot{\underline{x}}$ is a vector (unit direction vector)

let $L=\frac{1}{C_s(z)}\left(\frac{d\underline{x}}{ds}\frac{d\underline{x}}{ds}\right)^{1/2}$

$\dot{x}=\frac{dx}{ds}$

$|\dot{x}|=\left|\frac{dx}{ds}\right|=1$

we have $\frac{\partial L}{\partial \underline{x}}-\frac{d}{ds}\left(\frac{\partial L}{\partial\left(\frac{d\underline{x}}{ds}\right)}\right)=0$

c is short for $C_s(z)$

$$\Rightarrow |\dot{x}|\frac{\partial \frac{1}{c}}{\partial \underline{x}}-\frac{d}{ds}\left(\frac{1}{c}\frac{\dot{x}}{|\dot{x}|}\right)=0$$

$$\Rightarrow \frac{\partial\frac{1}{c}}{\partial\underline{x}}-\frac{d}{ds}\left(\frac{1}{c}\frac{d\underline{x}}{ds}\right)=0$$

$$\Rightarrow \frac{1}{c}\frac{d^2\underline{x}}{ds^2}+\left(\frac{\partial\frac{1}{c}}{\partial\underline{x}}\cdot\frac{d\underline{x}}{ds}\right)\frac{d\underline{x}}{ds}=\frac{\partial\frac{1}{c}}{\partial\underline{x}}$$

$$\Rightarrow \frac{d^2\underline{x}}{ds^2}=c\frac{\partial\frac{1}{c}}{\partial\underline{x}}-c\left(\frac{\partial\frac{1}{c}}{\partial\underline{x}}\cdot\frac{d\underline{x}}{ds}\right)\frac{d\underline{x}}{ds}$$

$$=\dot{\underline{X}}=-\frac{\partial\ln c}{\partial\underline{x}}+\left(\frac{\partial\ln c}{\partial\underline{x}}\cdot\frac{d\underline{x}}{ds}\right)\frac{d\underline{x}}{ds}\qquad(*)$$

Siyang Wang

in this 2D problem
we can get from (*) that

in X direction $\Rightarrow \ddot{X} = \frac{\partial \ln(s(z))}{\partial z} \cdot \dot{z}\dot{X}$

in z direction $\Rightarrow \ddot{z} = \frac{\partial \ln(s(z))}{\partial z} (\dot{z}^2 - 1)$

where $\frac{dz}{ds} = \dot{z}$ $\frac{d^2x}{ds^2} = \ddot{X}$

$\frac{dX}{ds} = \dot{X}$ ~~so that~~

$\frac{d^2z}{ds^2} = \ddot{z}$

we know $\dot{z}^2 - 1 \leq 0$ always

when ~~...~~ wave is approaching a wave guide (with peak velocity)

$\frac{\partial \ln(s(z))}{\partial z} > 0 \Rightarrow \ddot{z} \leq 0$

$\Rightarrow \dot{z}$ is decreasing

if $\dot{z}$ can reach approximately to 0 near velocity peak depth ~~but still positive~~
then the trajactory of wave would be
similar to the graph draw at the beginning

the path would be the wave goes to the maximum
velocity area and ~~move it~~ propagating in $\vec{z}$ direction (horizontal)
then comes up at end.
so the wave takes a least time path.
~~which is same~~

4a. $$\frac{\partial \tilde{\rho}}{\partial t} + \vec{v} \cdot \nabla \tilde{\rho} = -\rho_0 \nabla \cdot \vec{\tilde{v}}$$

Robin Heinonen

$$\rho_0 \left(\frac{\partial \vec{\tilde{v}}}{\partial t} + (\vec{v} \cdot \nabla) \vec{\tilde{v}} \right) = -c_s^2 \nabla \tilde{\rho}$$

Make the ansatz

$$\tilde{\rho} = A e^{i\phi}$$
$$\tilde{v} = \vec{B} e^{i\phi} \qquad \phi = \phi(\vec{x}, t)$$

$$\Rightarrow \quad A \dot{\phi} + A \vec{v} \cdot \nabla \phi = -\rho_0 (\vec{B} \cdot \nabla \phi)$$

$$\rho_0 \left(\vec{B} \dot{\phi} + (\vec{v} \cdot \nabla \phi) \vec{B} \right) = -c_s^2 A \nabla \phi$$

Then $$\vec{B} = \frac{-c_s^2 A}{\rho_0 (\dot{\phi} + \vec{v} \cdot \nabla \phi)}$$

so $$A (\dot{\phi} + \vec{v} + \nabla \phi) = \frac{c_s^2 A}{\dot{\phi} + \vec{v} \cdot \nabla \phi}$$

or $$(\dot{\phi} + \vec{v} + \nabla \phi)^2 = c_s^2 (\nabla \phi)^2$$

as desired.

Write $\phi = \vec{k} \cdot \vec{x} - \omega t$. Then

$$\dot{\phi} = -\omega, \quad \nabla \phi = \vec{k} \quad \text{so}$$
$$(\vec{k} \cdot \vec{v} - \omega)^2 = c_s^2 k^2$$

b. We then have $|\vec{k}\cdot\vec{v} - \omega| = c_s k$

$$\Rightarrow \omega = \vec{k}\cdot\vec{v} \pm c_s k$$

This motivates taking $\omega \to \omega + \vec{k}\cdot\vec{v}$ in the non-flowing medium ray equations.

$$\Rightarrow \frac{d\vec{k}}{dt} = -\frac{d}{d\vec{x}}\left(\omega + \vec{k}\cdot\vec{v}\right)$$

$$\frac{d\vec{x}}{dt} = \frac{d}{d\vec{k}}\left(\omega + \vec{k}\cdot\vec{v}\right)$$

c. Because of the $\frac{d}{dx}(\vec{k}\cdot\vec{v})$ term, if the component of the wind speed along the direction of propagation is spatially varying, $\vec{k}$ will change direction

In particular, if we assume there is significant vertical wind shear, i.e. $\vec{v} = v(z)\,\hat{x}$ with $\frac{dv}{dz}$ large, then

$$\frac{dk_z}{dt} \simeq -k_x \frac{\partial v}{\partial z}$$

So assuming $\frac{dv}{dz} > 0$, the sound will veer off into the ground if $\vec{k}$ is parallel with $\vec{v}$ and into the sky if $\vec{k}, \vec{v}$ are anti-parallel.

d. For this flow to be Hamiltonian,

we must have $\frac{d}{d\vec{k}}\dot{\vec{k}} + \frac{d}{d\vec{x}}\dot{\vec{x}} = 0 \qquad (*)$

But $\frac{d}{d\vec{k}}\dot{\vec{k}} = -\frac{d}{d\vec{k}}\frac{d}{d\vec{x}}\left(\omega + \vec{k}\cdot\vec{v}\right)$

and $\frac{d}{d\vec{x}}\dot{\vec{x}} = \frac{d}{d\vec{x}}\frac{d}{d\vec{k}}\left(\omega + \vec{k}\cdot\vec{v}\right)$

so since $\frac{d}{d\vec{x}}\frac{d}{d\vec{k}} = \frac{d}{d\vec{k}}\frac{d}{d\vec{x}}$

$(*)$ is satisfied.

Tom Zdyrski

5) For a general harmonic oscillator with spring constants k_x, k_y, k_z, the potential and kinetic energies are

3D

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2), \quad V = \frac{1}{2}(k_x x^2 + k_y y^2 + k_z z^2)$$

So $L = \frac{1}{2}\left(m[\dot{x}^2+\dot{y}^2+\dot{z}^2] - (k_x x^2 + k_y y^2 + k_z z^2)\right)$

or, defining the momenta $p_i = \frac{\partial L}{\partial \dot{x}_i}$ $i=1,2,3$,

$$H = p_i \dot{x}_i - L = \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + \frac{1}{2}(k_x x^2 + k_y y^2 + k_z z^2)$$

or with $p_i = \frac{\partial S}{\partial x_i}$, $H = \frac{1}{2m}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right) + \frac{1}{2}(k_x x^2 + k_y y^2 + k_z z^2)$

Then, since $\frac{\partial H}{\partial t} = 0$, we can use the time independent HJ eq. Further, $\frac{\partial S}{\partial t} = H \Rightarrow E \Rightarrow S = E(t-t_0) + f(x,y,z)$.

Then, assuming S is separable, $S = S_x(x,t) + S_y(y,t) + S_z(z,t)$,

we have

$$\left(\frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{1}{2}k_x x^2\right) + \left(\frac{1}{2m}\left(\frac{\partial S_y}{\partial y}\right)^2 + \frac{1}{2}k_y y^2\right) + \left(\frac{1}{2m}\left(\frac{\partial S_z}{\partial z}\right)^2 + \frac{1}{2}k_z z^2\right) = E$$

But the first parenthesis only depends on x,t the second on y,t, the third on z,t.

So, we can say each is a constant.

Taking only the x-component, we get

$$\frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{1}{2}k_x x^2 = E_x$$

$$\Rightarrow \frac{\partial S_x}{\partial x} = \sqrt{2m\left(E_x - \frac{1}{2}k_x x^2\right)}$$

$$\Rightarrow S_x = \int \sqrt{2m\left(E - \frac{1}{2}k_x x^2\right)}\,dx - E_x(t-t_0)$$

But we also have

$$\text{constant } -T_x = \frac{\partial S}{\partial E_x} = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - \frac{1}{2}k_x x^2}} - t$$

substituting $\sqrt{\frac{k_x}{2E_x}}\, x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$$\Rightarrow \quad t - T_x = -\sqrt{\frac{m}{2E_i}} \int \frac{\sin\theta}{\sqrt{1-\cos^2\theta}}\, d\theta = -\sqrt{\frac{2E_x}{k_x}}\sqrt{\frac{m}{2}} \int \frac{\sin\theta}{|\sin\theta|}\, d\theta$$

but $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow |\sin\theta| = \sin\theta$

$$\Rightarrow \quad t - T_x = -\sqrt{\frac{m}{k_x}} \int d\theta = -\sqrt{\frac{m}{k_x}} \cos^{-1}\left(x\sqrt{\frac{k_x}{2E_x}}\right)$$

$$\Rightarrow \quad x = \sqrt{\frac{2E_x}{k_x}} \cos\left((t-T_x)\sqrt{\frac{k_x}{m}}\right)$$ defining $\omega_x = \sqrt{\frac{k_x}{m}}$

and $T_x \omega_x = \phi_x$,

$x = \sqrt{\frac{2E_x}{k_x}} \cos(\omega_x t - \phi_x)$, or in general

$$\boxed{x_i = \sqrt{\frac{2E_i}{k_i}} \cos(\omega_i t - \phi_i) \quad \text{with } \sum_i E_i = E}$$

6a.) For cylindrical coordinates:

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)m - V(r,\phi,z)$$

So $$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{p_z^2}{2m} + V(r,\phi,z)$$

For time independence Hamilton-Jacobi equation, we have

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S_0}{\partial \phi}\right)^2 + \frac{1}{2m}\left(\frac{\partial S_0}{\partial z}\right)^2 + V(r,\phi,z) = E$$

where E is the total energy, a constant.

For separability, we hope that $V(r,\phi,z) = f_1(r) + \frac{1}{r^2}f_2(\phi) + f_3(z)$

Then $$E = \left[\frac{1}{2m}\left(\frac{\partial S_0}{\partial r}\right)^2 + f_1(r)\right] + \frac{1}{r^2}\left[\frac{1}{2m}\left(\frac{\partial S_0}{\partial \phi}\right)^2 + f_2(\phi)\right] + \left[\frac{1}{2m}\left(\frac{\partial S_0}{\partial z}\right)^2 + f_3(z)\right]$$

We could assume $S_0 = S_1(r) + S_2(\phi) + S_3(z)$

So we have

$$E = \left[\frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2 + f_1(r)\right] + \frac{1}{r^2}\left[\frac{1}{2m}\left(\frac{\partial S_2}{\partial \phi}\right)^2 + f_2(\phi)\right] + \left[\frac{1}{2m}\left(\frac{\partial S_3}{\partial z}\right)^2 + f_3(z)\right]$$

we could get:

$$\begin{cases} \frac{1}{2m}\left(\frac{\partial S_3}{\partial z}\right)^2 + f_3(z) = C_3 \\ \frac{1}{2m}\left(\frac{\partial S_2}{\partial \phi}\right)^2 + f_2(\phi) = C_2 \\ \frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2 + f_1(r) = E - \frac{C_2}{r^2} - C_3 \end{cases} \qquad (*)$$

b) Eq. (*) gives us that

$$\begin{cases} S_3 = \sqrt{2m}\int (C_3 - f_3(z))^{1/2}\,dz \\ S_2 = \sqrt{2m}\int (C_2 - f_2(\phi))^{1/2}\,d\phi \end{cases}$$

$\Phi = \int k\, d\ldots$

$$S_1 = \sqrt{2m}\int [E - \frac{C_2}{r^2} - C_3 - f_1]^{1/2} dr$$

$$S = S_1(r) + S_2(\theta) + S_3(z) - Et$$

$$\frac{\partial S}{\partial q_i} = P_i = m\frac{dq_i}{dt}$$

So $\quad dt = m\frac{dq_i}{\partial S/\partial q_i}$ —> not quite for phi

we have
$$\begin{cases} t = \sqrt{\frac{m}{2}}\int\frac{dz}{\sqrt{C_3 - f_3(z)}} + C' \\ t = \sqrt{\frac{m}{2}}\int\frac{d\phi}{\sqrt{C_2 - f_2(\phi)}} + C'' \\ t = \sqrt{\frac{m}{2}}\int\frac{dr}{\sqrt{E - C_2/r^2 - C_3 - f_1(r)}} + C''' \end{cases}$$
we could get $\vec{r} = \vec{r}(t)$

And $\quad P_i(t) = \left(\frac{\partial S}{\partial q_i}\right)(t)$

Here if $f_3(z) = 0$ then C_3 is the momentum in $\hat{z}$ direction

If, at the same time, $f_2(\phi) = 0$, that is $V = f_1(r)$

Then C_2 is just L^2, where L is the angular momentum

Generally, C_3 is corresponding to P_z, while C_2 is corresponding to L^2.

(7) (i) Relationship b/t normal vector for acoustic wavepath & profile of index of refraction.

(ii) Relate to particle motion, using e.g. for particle path.

(AKA - Monsieur Fermat & Maupertuis)

(i) Least Time : $\tau = \int_1^2 \frac{|dx|}{c(x)} = \frac{1}{c_0}\int_1^2 |dx| n(x)$

$$\delta\tau = 0 = \delta \int_1^2 \underbrace{n(x)\left(\frac{dx}{ds}\cdot\frac{dx}{ds}\right)^{1/2} ds}_{L \text{ (Ray Lagrangian)}}$$

['s' is secretly an arclength parametrization of the path x, but we can call it a dummy time].

$$\Rightarrow \quad 0 = \frac{\partial L}{\partial \underline{x}} - \frac{d}{ds}\frac{\partial L}{\partial(\partial_s x)} \quad ; \quad \text{Let } |\dot{x}| = \left[\frac{dx}{ds}\cdot\frac{dx}{ds}\right]^{1/2}$$

set $|\dot{x}| = 1$

$$0 = \frac{dn}{d\underline{x}}|\dot{x}| - \frac{d}{ds}\left(n(x)\frac{dx}{ds}\frac{1}{|\dot{x}|}\right) \rightarrow \frac{dn}{d\underline{x}} = \left(\frac{dn}{d\underline{x}}\cdot\frac{dx}{ds}\right)\frac{dx}{ds} + n\frac{d^2\underline{x}}{ds^2}$$

$$\rightarrow \frac{d^2\underline{x}}{ds^2} = \frac{1}{n}\frac{dn}{d\underline{x}} - \frac{1}{n}\left(\frac{dn}{d\underline{x}}\cdot\frac{d\underline{x}}{ds}\right)\frac{dx}{ds}$$

Not $\frac{d\underline{x}}{ds} = \hat{t}$ (unit tangent vector), $\frac{d^2\underline{x}}{ds^2} = \underline{\kappa}$ (curvature vector)

Let n_0 = unit normal vector, $\frac{d}{d\underline{x}} = \bar{\nabla}$

$$\Rightarrow \underline{\kappa} = \frac{1}{n}\bar{\nabla}n - \frac{1}{n}(\nabla n\cdot t)t = \frac{1}{n}\left(\bar{\nabla}n - (\bar{\nabla}n\cdot t)t\right)$$

$$\Rightarrow \boxed{\underline{\kappa} = \frac{1}{n}(\bar{\nabla}n\cdot n_0)n_0} \quad (\text{from } \nabla n = (\nabla n\cdot t)t + (\nabla n\cdot n_0)n_0)$$

⑦ continued #1.

(ii) Abbreviated Action / Principle of Maupertuis

(a) Pre amble 1.

Generic Least Action $\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)dt\,\delta q$

Consider 1st point fixed ~~[illegible]~~, $\delta q(t_1)=0$,

and 2nd point variable: $\delta q(t_2) \equiv \delta q$.

Letting $p = \frac{\partial L}{\partial \dot{q}}$ we get $\delta S = p\,\delta q$ from the boundary term.

[Small aside: for any choice of $q(t_2)$ ~~[illegible]~~ the actual path of motion will satisfy Lagrange's Eq. (So $\delta S = p\,\delta q$).

For $q = \underline{q}$, $\delta S = \sum_i p_i \delta q_i \rightarrow \frac{\partial S}{\partial q_i} = p_i$. $\left[S = S(q, t)\right]$.

① Note $\frac{dS}{dt} = L$

② Also, $\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i = \frac{\partial S}{\partial t} + \sum_i p_i \dot{q}_i$

$\Rightarrow \frac{\partial S}{\partial t} = -H \Rightarrow dS = \frac{\partial S}{\partial q}dq + \frac{\partial S}{\partial t}dt$

$$\underline{dS = \sum_i p_i\, dq_i - H\, dt} \qquad (43.6 \text{ L\&L})$$

Note if we had let the first coordinate be ~~[illegible]~~ variable as well, we would have

$$\underline{dS = \sum_i p_i^{(2)} dq_i^{(2)} - H^{(2)} dt^{(2)} - \sum_i p_i^{(1)} dq_i^{(1)} + H^{(1)} dt^{(1)}} \qquad (43.7 \text{ L\&L})$$

Ex (7) Continued #2

(b) Preamble 2.

Consider Least Action with t_1, $q(t_1)$, $q(t_2)$ fixed but $t_2 = t$ variable.

We would then have ~~[illegible]~~ ~~[illegible]~~ $\delta S = -H\delta t$ (using formula on last page). (43.7)

For ~~the~~ energy conserving systems $H = E \Rightarrow$ ① $\delta S + E\delta t = 0$.

Integrating (43.6), ② $S = \underbrace{\int \sum_i p_i dq_i}_{\equiv S_0 \text{ (abbreviated action)}} - E\int_{t_1}^{t} dt = \int \sum_i p_i dq_i - E(t - t_1)$

① $\delta S + E\delta t = 0$

② $S = S_0 - E(t - t_1) \rightarrow \delta S = \delta S_0 - E\delta t$

① $\rightarrow$ $\boxed{\delta S_0 = 0}$

Least Action for paths that pass through q_2 at any time & conserve energy.

(c) ~~[illegible]~~ Preamble 3.

Rewrite S in terms of q. $p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i}$, $E(q, \frac{dq}{dt}) = E$.

Medium aside:

Consider a general Lagrangian in cartesian coordinates;

$\mathcal{L} = \frac{1}{2}m\sum_i \dot{x}_i^2 + U(x)$. To transform to coordinates

defined by $x_i = f_i(\underline{q}) \rightarrow \dot{x}_i = \sum_k \frac{\partial f_i}{\partial q_k}\dot{q}_k$

$\Rightarrow \mathcal{L} = \frac{1}{2}m\sum_{i,k} a_{ik}(q)\dot{q}_i\dot{q}_k - U(q)$.

7 continued #3

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} a_{ik}(q)\,\dot q_i \dot q_k - U(q) \qquad (m \text{ in } a_{ik}).$$

$$p_i = \partial L/\partial \dot q_i = \sum_k a_{ik}(q)\,\dot q_k \ , \quad E = \frac{1}{2}\sum a_{ik}(q)\dot q_i \dot q_k + U(q)$$

$$\downarrow$$

$$dt = \left[\sum_{i,k} \frac{a_{ik}(q)\,dq_i dq_k}{2(E-U)}\right]^{1/2} \leftarrow 2(E-U) = \sum_{i,k} a_{ik}(q)\frac{dq_i dq_k}{dt^2}$$

$$S_o = \int \sum_i p_i\,dq_i = \int \sum_{i,k} a_{ik}(q)\,dq_k dq_i \left(\frac{2(E-U)}{\sum_{i,k} a_{ik}(q)\,dq_i dq_k}\right)^{1/2}$$

$$S_o = \int \left[\sum_{i,k} a_{ik}(q)\,dq_k dq_i \cdot 2(E-U)\right]^{1/2}$$

For $T = \frac{1}{2}m(dl/dt)^2$, $\boxed{S_o = \int [2m(E-U)]^{1/2}\,dl}$

(d) Actual Answer :

Not $dl^2 = dr^2$

$dl\,d\delta l = dr \cdot d\delta r$

$\Rightarrow d\delta l = \frac{dr \cdot d\delta r}{dl} = \left(\frac{dr}{dl}\right)\cdot d\delta r$

$$\delta S_o = \delta \int [2m(E-U)]^{1/2} dl = 0$$

$$= \int \left[\frac{-(\partial u/\partial r)\cdot \delta r\, dl}{2(E-U)^{1/2}} + [E-U]^{1/2} d\,\delta l\right] = 0$$

$$= \int \left[-\left(\frac{\partial u}{\partial r}\right)\cdot \frac{\delta r\, dl}{2(E-U)^{1/2}} + \underbrace{[E-U]^{1/2}\left(\frac{\partial r}{\partial l}\right)}_{u}\cdot \underbrace{d\delta r}_{du}\right] = 0$$

$$\text{IBP}: \quad \int \left[-\left(\frac{du}{dr}\right)\cdot \frac{\delta r\, dl}{2(E-U)^{1/2}} + \frac{d}{dl}\left([E-U]^{1/2}\left(\frac{dr}{dl}\right)\right)\cdot \delta r\right] + \overbrace{[E-U]^{1/2}\left(\frac{dr}{\delta l}\right)\delta r\Big|_{q_1}^{q_2}}^{0.}$$

$$\Rightarrow -\left(\frac{du}{dr}\right)\frac{1}{2[E-U]^{1/2}} - \frac{d}{dl}\left([E-U]^{1/2}\left(\frac{dr}{dl}\right)\right) = 0$$

(7) continue # 4

let $\frac{\partial u}{\partial \underline{r}} = \vec{F}$, $\frac{\partial r}{\partial \ell} = \underline{t}$,

$$\frac{\partial u}{\partial r} = -2\,[E-U]^{1/2}\,\frac{d}{d\ell}\left([E-U]^{1/2}\left(\frac{\partial r}{\partial \ell}\right)\right) = -\cancel{2[E-U]^{1/2}}\left(\frac{-\partial u/\partial r \cdot \frac{\partial r}{\partial \ell}}{\cancel{2[E-U]^{1/2}}}\right)\left(\frac{\partial r}{\partial \ell}\right) - 2[E-U]\frac{\partial^2 r}{\partial \ell^2}$$

$$\Rightarrow\ -\underline{F} = -(\underline{F}\cdot t)t - 2\,[E-U]\,\frac{d^2 r}{d\ell^2} \Rightarrow \frac{d^2 r}{d\ell^2} = \frac{F-(\underline{F}\cdot t)t}{2\,[E-U]} = \frac{(\underline{F}\cdot \underline{n}_0)\,\underline{n}_0}{2\,[E-U]}$$

$$E-U = T = \tfrac{1}{2}mv^2 \rightarrow \underline{\kappa} = \frac{(F\cdot \underline{n}_0)\underline{n}_0}{mv^2} \rightarrow \underline{mv^2\,\underline{\kappa} = (\underline{F}\cdot\underline{n}_0)\,\underline{n}_0}$$

$$\boxed{\begin{array}{ll} \text{Particles:} & \underline{\kappa} = \frac{1}{mv^2}(\underline{F}\cdot\underline{n}_0)\underline{n}_0 \\ \text{Ray:} & \underline{\kappa} = \frac{1}{n}(\nabla n\cdot n_0)n_0 \end{array}}$$

(Because Landau does ... rewrite particle eq. as $\frac{mv^2}{R} = F_n$, $R = \frac{1}{\kappa}$ radius of curvature, F_n normal force).

Robin Heinonen

8. a. $$\frac{\partial^2 \psi}{\partial x^2} + \frac{Q(x)}{\varepsilon^2}\psi = 0 \quad (*)$$

Make the ansatz $\psi = \exp\left(\frac{i}{\varepsilon}\sum \varepsilon^n \phi_n(x)\right)$

Then $$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{i}{\varepsilon}\sum \varepsilon^n \phi_n'(x)\exp\left(\frac{i}{\varepsilon}\sum \varepsilon^n \phi_n\right)\right)$$

$$= \left(\frac{i}{\varepsilon}\sum \varepsilon^n \phi_n'' - \frac{1}{\varepsilon^2}\sum_{n,m} \varepsilon^{n+m}\phi_n'\phi_m'\right)e^{\frac{i}{\varepsilon}\sum \varepsilon^n \phi_n}$$

so $(*)$ becomes

$$\frac{i}{\varepsilon}\phi_0'' + i\phi_1'' - \frac{1}{\varepsilon^2}\phi_0'^2 - \frac{2}{\varepsilon}\phi_0'\phi_1'$$

$$-\left(\phi_1'^2 + 2\phi_1'\phi_2'\right) + \mathcal{O}(\varepsilon) + \frac{Q}{\varepsilon^2} = 0$$

The $\frac{1}{\varepsilon^2}$-, $\frac{1}{\varepsilon}$- and ε^0- order equations are, respectively,

$$\begin{cases} \phi_0'^2 = Q & (1) \\ i\phi_0'' = 2\phi_0'\phi_1' & (2) \\ i\phi_1'' = \phi_1'^2 + 2\phi_0\phi_2 & (3) \end{cases}$$

b. The Helmholtz eikonal equation is $(\nabla\phi)^2 = \frac{\omega^2}{c^2}$

so (1) is the 1D eikonal equation with

$$Q = \left(\frac{\omega}{c(x)}\right)^2$$

c. From (1),

$$\phi_0 = \int \sqrt{Q(x)}\, dx$$

$$\Rightarrow \quad \phi_0'' = \frac{Q'(x)}{2\sqrt{Q(x)}}$$

substituting into (2),

$$\frac{i Q'(x)}{2\sqrt{Q(x)}} = 2\sqrt{Q(x)}\, \phi_1'$$

$$\Rightarrow \quad \phi_1' = \frac{i}{4} \frac{Q'}{Q}$$

or $\phi_1 = \frac{i}{4} \log Q$

so to order 1,

$$\psi = \frac{1}{Q^{1/4}} e^{\frac{i}{\varepsilon} \int \sqrt{Q(x)}\, dx}$$

The takeaway here is that the eikonal equation is just zeroth order WKB. This solution for ψ is the familiar WKB solution, but is in fact only the first order approximation.

d. Q should be slowly-varying for this approximation to make sense. Moreover, Q must not cross zero, so that the second term $\frac{Q}{\varepsilon^2}$ is indeed large.

Also terms in the series should be getting smaller.

Jialing Fei A53111966

Problem Set III

9. wave equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

assume an oscillatory solution to

~~n = C_0/c~~

then. $$\frac{\partial^2\psi}{\partial x^2} + \frac{\omega^2}{c^2}\psi = 0$$ as $n = C_0/c$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\omega^2}{C_0^2} n^2 (x,y,z)\psi = 0$$

for a short wavelength the wave can be treated as ray

$$\psi = A(x)\, e^{\frac{i}{\epsilon}\sum_{m=0}^{\infty} \epsilon^m \phi_m(x)}$$

$$\approx A(x)\, e^{\frac{i}{\epsilon}\phi_0(x)}$$

plug $\psi = A(x) e^{\frac{i}{\epsilon}\phi_0(x)}$ into $\frac{\partial^2\psi}{\partial x^2} + \frac{\omega^2}{C_0^2} n^2(x,y,z)\psi = 0$

$$-\frac{(\nabla\phi_0)^2}{\epsilon^2}A + \frac{i}{\epsilon}\nabla^2\phi_0 A + \frac{2i}{\epsilon}(\nabla\phi_0\cdot\nabla A) + \nabla^2 A + \frac{\omega^2}{C_0^2}n^2 A = 0$$

~~As~~ assume $A(x)$ is slowly varying $\nabla^2 A \approx 0$

$$-(\nabla\phi_0)^2 A = -\frac{\omega^2}{C_0^2}n^2 A \quad \text{(real part)}.$$

$$(\nabla\phi_0)^2 = \frac{\omega^2}{C_0^2}n^2 \quad \text{(eikonal equation)}$$

$$\left(\frac{\partial \varphi_0}{\partial x}\right)^2 + \left(\frac{\partial \varphi_0}{\partial y}\right)^2 + \left(\frac{\partial \varphi_0}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2} n^2(x,y,z)$$

to solve the ray equation

we need $\varphi_0(x) = \phi_x(x) + \phi_y(y) + \phi_z(z)$

also $n^2(x,y,z) = a(x) + b(y) + c(z)$

b) $$\left(\frac{\partial \phi_x}{\partial x}\right)^2 + \left(\frac{\partial \phi_y}{\partial y}\right)^2 + \left(\frac{\partial \phi_z}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2}\left[a(x) + b(y) + c(z)\right]$$

$$\begin{cases} \left(\frac{\partial \phi_x}{\partial x}\right)^2 - \frac{\omega^2}{c_0^2} a(x) = C_1 \\ \left(\frac{\partial \phi_y}{\partial y}\right)^2 - \frac{\omega^2}{c_0^2} b(y) = C_2 \\ \left(\frac{\partial \phi_z}{\partial z}\right)^2 - \frac{\omega^2}{c_0^2} c(z) = C_3 \end{cases}$$

$$\begin{cases} \frac{\partial \phi_x}{\partial x} = \pm\sqrt{C_1 + \frac{\omega^2}{c_0^2} a(x)} \\ \frac{\partial \phi_y}{\partial y} = \pm\sqrt{C_2 + \frac{\omega^2}{c_0^2} b(y)} \\ \frac{\partial \phi_z}{\partial z} = \pm\sqrt{C_3 \pm \frac{\omega^2}{c_0^2} c(z)} \end{cases}$$

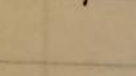
$C_1 + C_2 + C_3 = 0$

$$\phi(x) = \pm\int dx \sqrt{C_1 + \frac{\omega^2}{c_0^2} a(x)} \pm \int dy \sqrt{C_2 + \frac{\omega^2}{c_0^2} b(y)} \pm \int dz \sqrt{C_3 + \frac{\omega^2}{c_0^2} c(z)}$$

Tom Zdyrski

10) We want to find the variation of the amplitude of a sound wave with amplitude in an isothermal atmosphere.

First, we need the density profile of the planet. Taking an infinitesimal cube of air



$$0 = -gm - (P(z+dz) - P(z))dxdy$$

$$\Rightarrow g\rho = -\frac{\partial P}{\partial z}.$$

Then, the ideal gas law gives $PV = nRT$

$$\Rightarrow P = \frac{n}{V}RT = \frac{nM}{V}\frac{R}{M}T = \rho R_{sp}T \Rightarrow g\rho = -R_{sp}T\,\partial_z \rho$$

$$\Rightarrow \rho = \rho_0 e^{-z/h} \qquad h := \frac{R_{sp}T}{g}.$$

Then, taking the wave equation $\nabla^2\psi + \frac{\omega^2}{c^2}\psi = 0$ with $c \propto \frac{1}{\sqrt{\rho}}$, we take the ansatz $\psi = X(x)Y(y)Z(z)$

$\Rightarrow \frac{X''}{X} - \frac{Y''}{Y} + \frac{Z''}{Z} + \omega^2\rho_0 e^{-z/h} = 0$. Thus, the X, Y functions are just plane waves. The Z direction gives $Z'' + \omega^2\rho_0 e^{-z/h} = 0$. If we assume the small wavelength approx, $Z = A(z)e^{i\frac{\phi(z)}{\varepsilon}}$, $\varepsilon \ll 1$

$$\Rightarrow A'' + \frac{2i}{\varepsilon}A'\phi' - \frac{A}{\varepsilon^2}(\phi')^2 + \frac{i}{\varepsilon}A\phi'' + \omega^2\rho_0 Ae^{-\frac{z}{h}} = 0$$

Collecting terms in powers of ε gives:

$\frac{1}{\varepsilon^2}$: $A(\phi')^2 = 0 \Rightarrow \phi' = \text{const.}$

$\frac{1}{\varepsilon}$: $2iA'\phi' + iA\phi'' = 0 \Rightarrow 2A'\phi' = A\phi''$

1: $A'' = -\omega^2\rho_0 e^{-z/h}A$. Since we assume long wavelength

$\left(\frac{\partial A}{A} \ll k_z \Rightarrow z \ll h\right) \Rightarrow A'' = A\omega^2\rho_0(-1 + z/h)$

Substituting $u = -1 + \frac{z}{h} \Rightarrow \frac{d^2}{du^2}A = A\left(\frac{\omega}{h}\right)^2\rho u \Rightarrow \boxed{A(u) = \text{airy fun} = A(\tfrac{z}{h} - 1)}$