NOTES ON BASIC TURBULENCE

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PHYS 235 Course

"When I meet God, I'm gonna ask him two questions. Why Relativity? And why Turbulence? I believe he will have an answer for the first."

– Warner Heisenberg

Summary:

In this set of notes, basics of 3D, 2D and pipe flow turbulence have been discussed. States dominated by large spectral fluxes, correspond to turbulent cascades, in which nonlinear interactions couple source and sink at different scales by a sequence of local transfer events. Famous Kolmogrov cascade, as well as, 3D forward cascade and 2D inverse cascade has been discussed. Prandtl theory for pipe flows and similarity and differences and relevency to fusion also dissc

* Basic Turbulence:

⇒ Considering Navier-Stokes Momentum Eqn:

Assuming Incompressibility
$\nabla . v = 0$

$$\frac{\partial v}{\partial t} + v.\nabla v \mp \nu \nabla^2 v = -\nabla P/\rho_0 + \tilde{f}_{ext}$$

$$\Rightarrow \frac{\partial}{\partial t} \langle \tilde{v}^2 \rangle + \langle v. \cancel{\nabla} . \nabla v \rangle + \nu \underbrace{\langle (\nabla v)^2 \rangle}_{\text{viscous dissipation}} = \underbrace{\langle \tilde{f} . \tilde{v} \rangle}_{\text{energy input}}$$

Note that:
$v . \nabla v = \nabla (v^2/2) - v \times u$

⇒ And dissipative force and power are:

$$F_d \sim \rho_0 A v^2 \underbrace{C_D(Re)}_{\sim Re^0} \Rightarrow \frac{P_d}{M} \sim \frac{F_d v}{M} \sim C_D(Re) \frac{v^3}{l}$$

Reynolds Number
$Re \sim \frac{v(l)\, l}{\nu}$

- Macro-scale: P_d independent of Re
- Micro-scale: $\langle \tilde{f} . v \rangle \sim \nu \langle \underbrace{(\nabla v)^2}_{\sim 1/r} \rangle$ singular flow shear

The question of turbulent cascades:

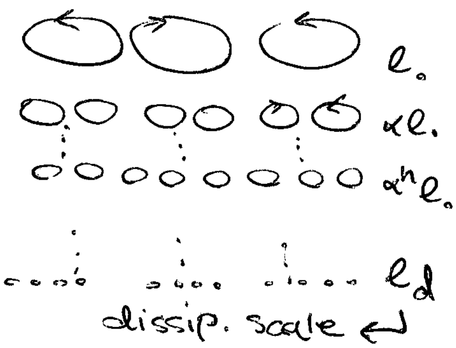
- How to form singular ∇v
- Observed inertial range

Different turbulent structures have been disscussed:

- K41 (Kolmogrov 41)
- 2D Turbulence
- Pipe flow

Energy flow in scale space $\Leftrightarrow$ K41 Cascade

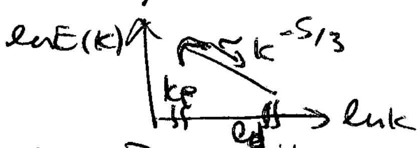
- Assumptions for Kolmogorov's theory of High Reynolds number turbulence:
  - $\rightarrow$ Spatial Homogeneity
  - $\rightarrow$ Isotropy
  - $\rightarrow$ Self_similarity
  - $\rightarrow$ locality of interaction
    - $\hookrightarrow$ large scales can't distort/destroy small scales.



Constant Energy throughput for all inertial range scales:

$$\epsilon \sim v_0^3/\ell_0 \sim v(\ell)^3/\ell \Rightarrow v(\ell) \sim \epsilon^{1/3}\ell^{1/3}$$

$$E(k) \sim \epsilon^{2/3} k^{-5/3}$$



Data structure of flow field.

Also Random-Walk is rough (non_differentiable)

$\delta x^2 \sim t$

$\frac{d}{dt}\delta x \sim 1/t^{1/3}$

$v(\ell) = |v(r+\ell) - v(r)|$ (rough in space)

$$\frac{|v(r+\ell) - v(r)|}{|\ell|} \sim \frac{\epsilon^{1/3}}{\ell^{2/3}} \rightarrow \begin{cases} -\nabla v \text{ diverges as scaler} \\ -\text{Cutoff at dissp. scale} \end{cases}$$

$\Rightarrow$ Richardson Theory of particle Seperation:

How the distance between two particle grows in time in a turbulent flow.

$$\frac{d\ell}{dt} = v(\ell) \sim \epsilon^{1/3}\ell^{1/3} \Rightarrow \frac{d\ell}{\ell^{1/3}} = \epsilon^{1/3}dt \Rightarrow \boxed{\ell^2 \sim \epsilon^2 t^3}$$

Richardson (Relative Seperation)

Richardson Seperation growth vs. Brownian Motion

$\ell \sim t^{3/2}$ $\qquad$ $\ell \sim t^{1/2}$

↳ Larger eddies support larger speed, seperation process is self-accelerating.

Review Article [Fakoviech, Gawedsk Vergassola, 2001]

To find dissipation scale, we balance the eddy straining rate $\frac{|v(r+\ell)-v(r)|}{\ell} \sim \frac{\epsilon^{1/3}}{\ell^{2/3}}$

with viscous dissipation rate ν/ℓ^2

$$\underbrace{1/\tau(\ell)}_{\text{decay rate}} \sim \underbrace{\frac{\nu}{\ell^2}}_{\text{diffusion rate}} \Rightarrow \frac{\epsilon^{1/3}}{\ell^{2/3}} \sim \frac{\nu}{\ell^2} \Rightarrow \boxed{\ell_d = \frac{\nu^{3/4}}{\epsilon^{1/4}}}$$

Kolmogorov Microscale

Integral Scale (Outer scale)
less than this no lon[ger] eddy flows be carrie[d]

$$\ell_o \sim \int \frac{dr\, v(x) v(x+r)}{|v(x)|^2}$$

Why does energy cascade self-similarity to small scales?

Vorticity + Vortex Dynamics
↳ Enstrophy

Energy $E = \frac{1}{2}\rho\langle v^2\rangle$

Enstrophy $\Omega = \langle \omega^2 \rangle$

vorticity $\omega = \nabla \times v$

3D Turbulence
↓
Enstrophy generatio[n]

2D Turbulence
↓
Enstrophy conservati[on]

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{\nabla P}{\rho} + \nu \nabla^2 v$$

$$\frac{\partial v}{\partial t} = -\nabla\left(\frac{P}{\rho} + \frac{v^2}{2}\right) + v \times \omega + \nu \nabla^2 \omega$$

$$\frac{\partial \omega}{\partial t} = \underbrace{\nabla \times (v \times \omega)}_{-v\cdot\nabla\omega + \omega\cdot\nabla v} + \nu \nabla^2 \omega$$

$$\Rightarrow \frac{d\omega}{dt} - \nu \nabla^2 \omega = \omega \cdot \nabla \omega$$

$\xleftrightarrow[\text{alike}]{\text{look}}$ $\frac{\partial B}{\partial t} - \mu \nabla^2 B = B \cdot \nabla v$

$$\frac{\partial}{\partial t}\langle w^2\rangle = \langle w\cdot(w\cdot\nabla v)\rangle - \nu\langle(\nabla w)^2\rangle$$

· Enstrophy Production

How to estimate it?

$$\frac{d}{dt}\langle w^2\rangle \sim \langle w^3\rangle - \nu\langle(\nabla w)^2\rangle$$

↳ signal of explosive growth

finite time singularity!

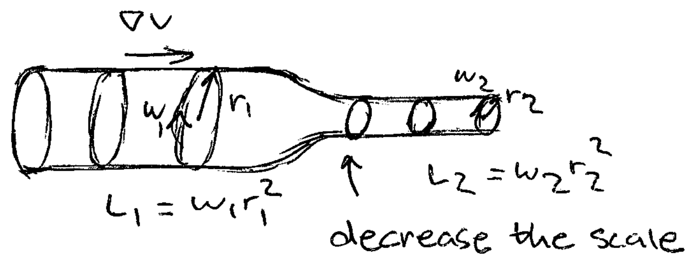
Finite time singularity of Enstrophy?

↓

Clay's Prize!

+

Vortex Tube Stretching



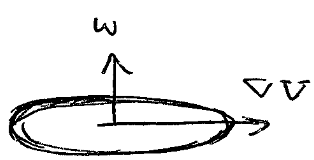
Idea:

Maximum Entropy?

Minimum Enstrophy?

⇒ <u>2D Turbulence</u>:

- Is very important paradigm for plasma turbulent.



- Vortex tube stretching doesn't exist here ($w\cdot\nabla v = 0$) ⇒ Vorticity conserved locally

$$\frac{\partial}{\partial t} w + v\cdot\nabla w - \nu\nabla^2 w = 0$$

or by respresenting v using a stream function ($v = \nabla\varphi\times\hat{z}$):

Hasegawa-Wakatani

$\nabla\cdot J = 0$; $\frac{d}{dt} n = 0$

$\partial_t\nabla^2\varphi + v\cdot\nabla\nabla^2\varphi + \nu\nabla^2\nabla^2\varphi = CD_{\parallel}(n-\varphi)$ $\xleftrightarrow{\text{resemble}}$ $\partial_t\nabla^2\varphi + v\cdot\nabla\nabla^2\varphi - \nu\nabla^2\nabla^2\varphi = 0$

$\partial_t n + v\cdot\nabla n - D\nabla^2 n = CD_{\parallel}(n-\varphi)$ limits of HW

$C = v_{th}^2/\nu_{ee}$

Hence in 2D Turbulence:

- Modulos V, E, Ω are conserved.
- → No vortex tube stretching!
- Vortex Merger → like Biot-Savart Law

Energy:

$$E = \iint d^2x \frac{v^2}{2} = \iint d^2x \frac{|\nabla\varphi|^2}{2}$$

Enstrophy:

$$\Omega = \iint d^2x \frac{\omega^2}{2} = \iint d^2x \frac{|\nabla^2\varphi|^2}{2}$$

Existance of two conserved quantities

⇓

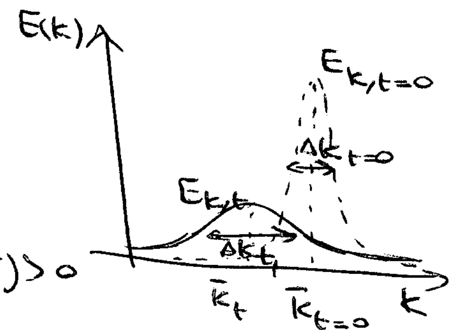
Theory of dual-cascade (Kraichnan, 67):

- Self-similar local enstrophy cascade toward high k's.
- Self-similar local energy flux cascade toward low k's (inverse cascade).

Inverse Cascade??

Turbulent act toward

broadening Δk^2: $\partial_t(\Delta k^2) > 0$



Δk: varience of distribution

$\bar{k}$: centroid of distribution

$$\Delta k^2 = \frac{\int dk\,(k-\bar{k})^2 E(k)}{\int dk\, E(k)} = \frac{\int dk\,(\overset{①}{k^2} - \overset{②}{2k\bar{k}} + \overset{③}{\bar{k}^2})E(k)}{E_0}$$

E_0: conserved energy

①: $\int dk\, k^2 E(k) = \Omega_0$ conserved enstrophy

②: $-2\bar{k}\dfrac{\int dk\, k E(k)}{E_0} = \bar{k}\dfrac{E_0}{E_0}$

③: $\int dk\, \bar{k}^2 E(k) = \bar{k}^2 E_0$

$$\Rightarrow \partial_t \langle (\Delta k)^2 \rangle = \partial_t\left(\frac{\Omega_0 - \bar{k}^2 E_0}{E_0}\right) = -\partial_t \bar{k}^2$$

$$\Rightarrow \partial_t \langle (\Delta k)^2 \rangle = - \partial_t k^{-2}$$

$\downarrow$ since this in >0 $\longrightarrow$ <0 $\Rightarrow$

- Spectral Broadening
- Downshift of Spectral centroid

$\downarrow$

Idea of inverse cascade!

Constructing spectra using scaling arguments:

$\Rightarrow$ Enstrophy Cascade:

$$kE(k) \sim \langle v^2 \rangle \Rightarrow \frac{1}{\tau_{cascade}} = \frac{1}{\tau_{\substack{eddy \\ turnover}}} = \frac{v(l)}{l} = k(kE(k))^{1/2}$$

Also $k^3 E(k)$ corresponds to Enstrophy density:

Enstrophy Dissipation Rate: $\eta = \frac{k^3 E(k)}{\tau_{cascade}}$

$$\Rightarrow \eta = (k^3 E(k))^{3/2} \Rightarrow \boxed{E(k) = \eta^{2/3} k^{-3}}$$

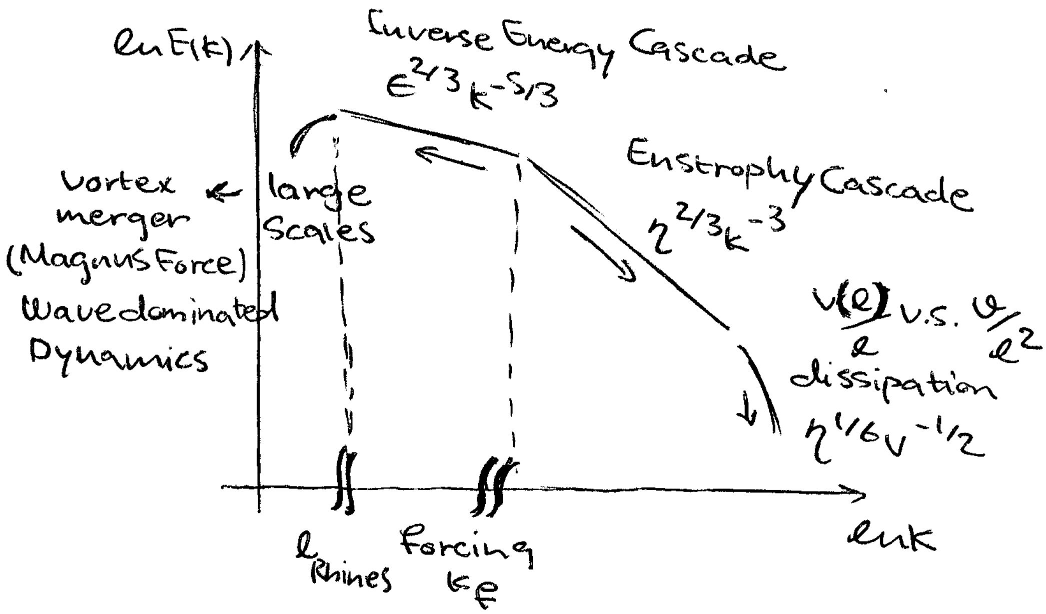
Forward Cascade

$\Rightarrow$ Inverse Energy Cascade:

Balancing energy dissipation rate with flow rate of energy to larger scales:

$$\epsilon = \frac{kE(k)}{\tau_{cascade}} \Rightarrow \epsilon = k^{5/2} E(k)^{3/2}$$

$$\Rightarrow \boxed{E(k) = \epsilon^{2/3} k^{-5/3}}$$ Inverse Cascade



Rossby Wave:

$$\partial_t \nabla^2 \varphi + v \cdot \nabla \nabla^2 \varphi = \beta \partial_x \varphi + \mu \nabla^2 \nabla^2 \varphi + \dots$$

$$\Rightarrow \omega_k = -\frac{\beta k_x}{k^2}$$

Strong dispersion at large Scales

Drag at large scales:

$$\partial_t \omega + v \cdot \nabla \omega = -\mu \omega + \nu \nabla^2 \omega$$

↳ Drag → Botton Ekman Effects
→ Toroidal Effects

Rhines lenght Scale:

Cross-over
Transition lenght Scale in which eddy dominated dynamics will become wave interaction dynamics.

- Wave-dominated: $\omega_k \tau_c|_k > 1$
- eddy-dominated: $\omega_k \tau_c|_k < 1$

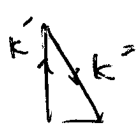
$$k\tilde{v} \text{ vs. } \frac{\beta k_x}{k^2} \Rightarrow k^2_{cross\text{-}over} = \frac{\beta}{\tilde{v}}$$

$$\Rightarrow \ell_{Rhines} = \sqrt{\tilde{v}/\beta}$$

inverse cascade vs. ZF

$$T^{wave} \sim \delta(\omega_k - \omega_{k'} - \omega_{k''})$$

↳ Two Rosby Waves + One Zonal Flow



$k \to k_x \to 0$
ZF interaction

Rhines Mechanism:

- Drift-wave $\ell_{ZF} \sim \ell_1$
- Lack of clear Self-Similarity range
- Lack of dispersion at large Scales

Dispersion of particle pairs (Richardson's Problem) for 2D turbulence:

particle distance ℓ: $\frac{d\ell}{dt} = v(\ell)$

$\ell > k_\epsilon^{-1}$: $v(\ell) = \epsilon^{1/3} \ell^{1/3} \Rightarrow \boxed{\ell^2 \sim \epsilon t^3}$

Inverse Cascade
(Same as K41)
(grows super diffusively)

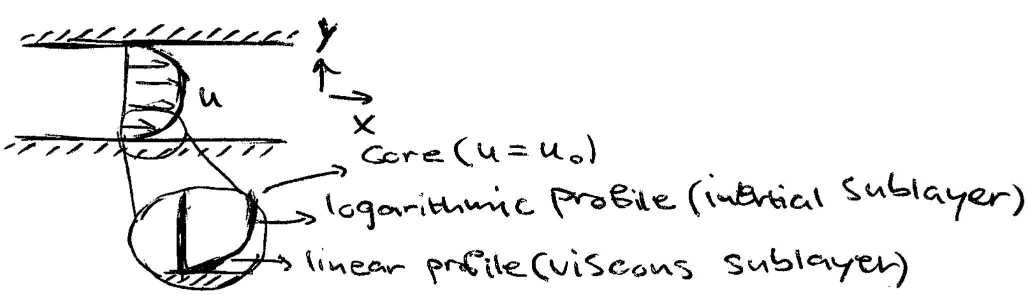
$\ell < k_\epsilon^{-1}$: $v(\ell) = (k E(k))^{1/2} \Rightarrow E(k) = \eta^{2/3} k^{-3}$

$\Rightarrow \frac{d}{dt} \ell = \eta^{1/3} \ell$ Forward Cascade
(grows exponantially in time)

$\Rightarrow$ Turbulence in Pipes & channels:

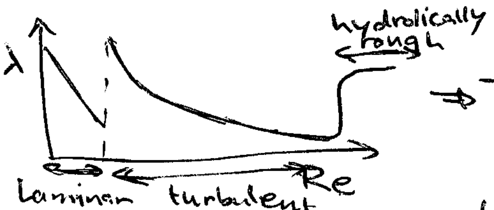
Up to now
- homogenous flow in periodic box
- cascade in scale space (Kolmogrov)

For pipes
- inhomogenous flow
- Momentum transport in a boundary layer (Prandtl)



Similarity to K41: pipe flow turbulent menifests an element of universility.

Poisseulle Prandtl
Laminar vs turbulent:



resistance: $\lambda = 2a \frac{\overbrace{\Delta P / \ell}^{\text{Pressure drop}}}{\underbrace{\frac{1}{2} \rho u^2}_{\text{mean flow energy}}}$

Characteristics of logarithmic profile:

- profile consistency over wide Re Range
- Logarithmic profile is universal (Prandtl "Law of the wall")
- Turbulence resistance curve also universal!

Driven by → turbulent mixing of cross-stream shear of mean flow by Reynolds stress.

Pipe flow
↓
Simple example of flux-driven turbulence
↓
Also tokamaks, solar conventions, etc

Turbulent Energy Production:

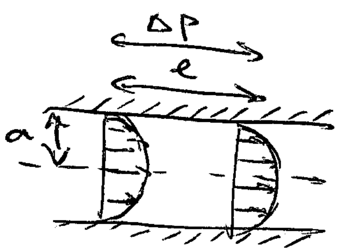
$$P = - \underbrace{\langle \tilde{v}_y \tilde{v}_x \rangle}_{\text{Reynold Stress}} \underbrace{\frac{d}{dy} v_x(y)}_{\text{cross-stream shear}}$$

→ Viscous Sublayer:

no-slip boundary condition:

$\lim_{x \to 0} f(x) \to 0$

Momentum flux to wall ⇒ Stress on the wall
↓
must balance pressure drop! (ΔP)



Force on wall $\simeq \rho u_*^2 A_{wall}$

Force on fluid $\simeq \Delta P \, A_{flow}$

u_* : friction velocity

ρu_* : wall stress

$\rho u_*^2 = \Gamma_f$: momentum flux

$$\rho u_*^2 (2\pi a l) = \Delta P (\pi a^2) \Rightarrow \underbrace{u_* = \left[\left(\frac{\Delta P}{2\rho}\right)\left(\frac{a}{2l}\right)\right]^{1/2}}_{\text{charactristic velocity for pipe flow}}$$

In viscous sublayer

$Re = u_* y < 1$

Viscous sublayer width: $\underbrace{y_d = \nu u_*^{-1}}_{\text{charactristic scale for pipe flow}}$

Satisfies relation $(Re < 1)$ & $u(y) \sim u_* \, y / y_d$

→ Inertial Sublayer (log law of the wall):

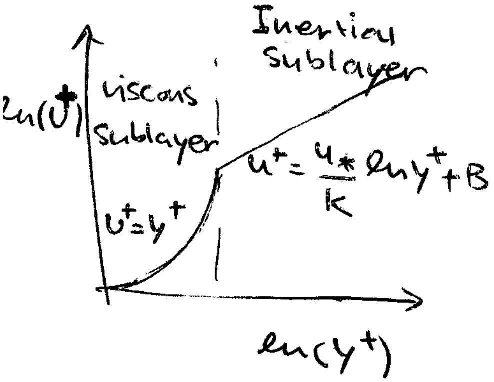
Assumption: Scale inveriance on $y_d < y < a$
↓
emperical!!

flow gradient is independent in inertial Scale:

$$\frac{du(y)}{dy} \simeq \frac{u_*}{y}$$

$$\Rightarrow \boxed{u(y) = \frac{u_*}{k}\ln(y/y_d) + \text{const.}}$$ Log Law of the wal

$k \sim 0.4$ constant ↵
von-karman constant (universal constant)



$U^+ = \frac{u(y)}{u_*}$; $y^+ = y/y_d$

$\langle \tilde{u}\tilde{v}\rangle = -\nu_T \partial_y u$

$\Rightarrow \tau = -(\nu + \nu_T)\partial_y u$

Reynolds Averaged Navier-Stokes (RANS):

Shear Stress: $\tau = -\langle \tilde{u}\tilde{v}\rangle + \nu\,\partial_y u$ $\quad \begin{cases} u = v_y \\ v = v_x \end{cases}$

Momentum in x: $\frac{1}{\rho}\partial_x P = -\partial_y \langle \tilde{u}\tilde{v}\rangle + \nu\,\partial_{yy} u$

$$= -\partial_y \tau$$

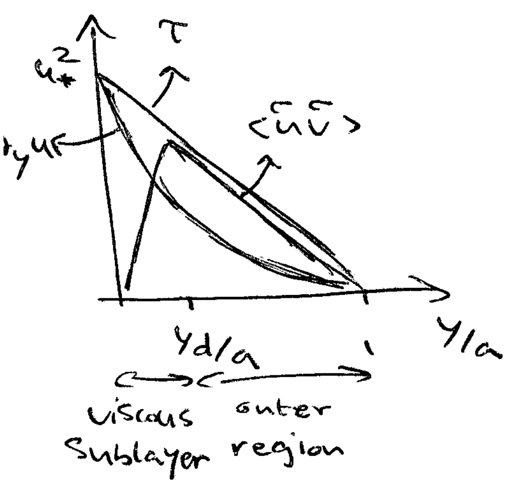
$$\Rightarrow \left(\frac{1}{\rho}\partial_x P\right) y = -\langle \tilde{u}\tilde{v}\rangle + \nu\,\partial_y u + \xcancel{\text{const.}}^{\,-u_*^2}$$

B.C.: $\partial_y u|_{y=0} = -u_*^2$

B.C. $y = a \Rightarrow$ statistically symmetric $\Rightarrow$

$$\langle \tilde{u}\tilde{v}\rangle\big|_y = -\langle \tilde{u}\tilde{v}\rangle\big|_{2a-y} \Rightarrow \langle \tilde{u}\tilde{v}\rangle\big|_a = 0$$

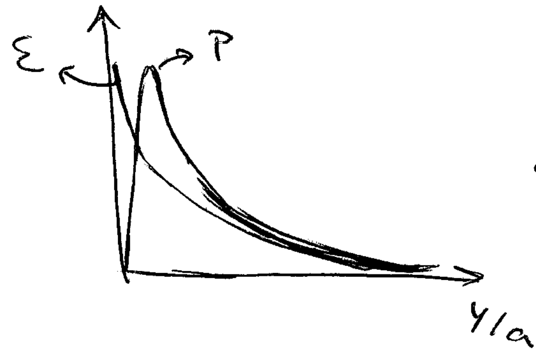
$$\Rightarrow \tau = -\langle \tilde{u}\tilde{v}\rangle + \nu\,\partial_y u = u_*^2(1 - y/a)$$



Now if are calculating energy dissipation:

$$\varepsilon = \nu \int dv\, |\nabla u|^2$$

We will have:



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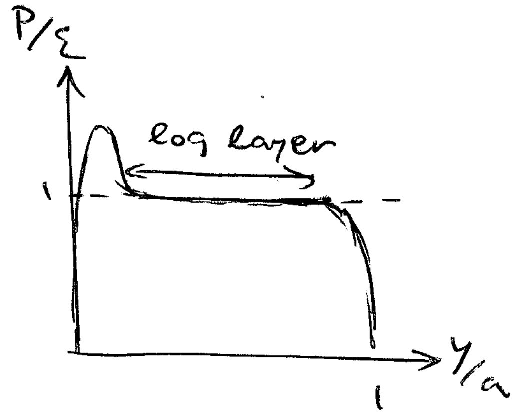


We have turbulence energy as:

$$\frac{\partial E}{\partial t} = P - \varepsilon = u_* \ell \left(\frac{u_*^2}{y^2} \right) - \frac{u_*^3}{\ell}$$

ℓ: characteristic lenghtscale of turbulence

$$P \to \nu_T \left(\frac{\partial u(y)}{\partial y} \right)^2$$

turbulent viscosity

Note that: $\frac{\tau_w}{\rho} = \langle \tilde{u}_y \tilde{u}_x \rangle = \nu_T \partial_y u$

Stationary sublayer ⇒ $\ell \sim y$

↓

turbulent lenghtscale in distance from wall!

"Mixing lenght theory always works... Provide you know the mixing lenght..."
– P.D.

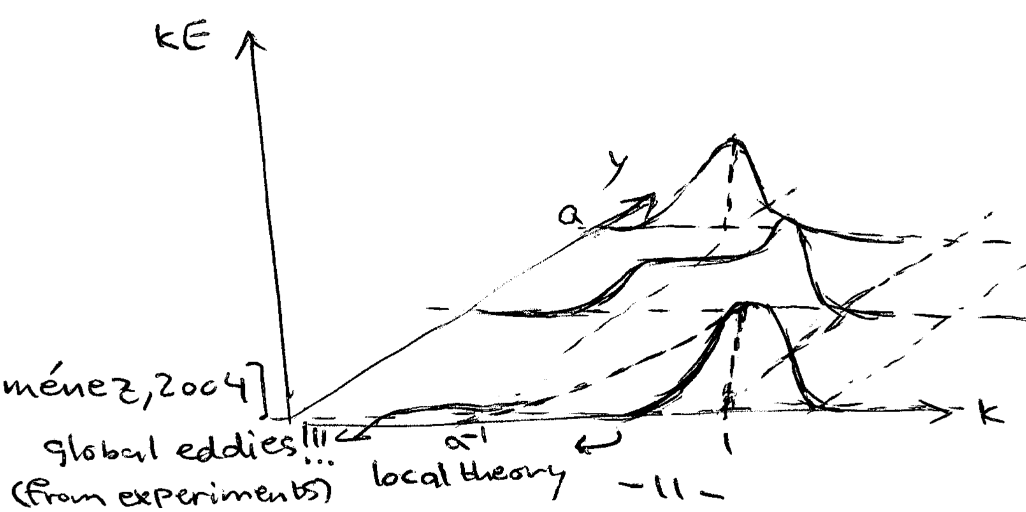
$$\Rightarrow \boxed{\nu_T = u_* y}$$

eddy viscosity or Turbulent viscousity

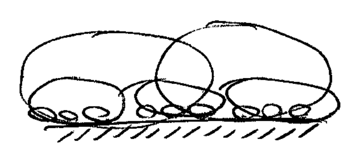
↓

rate of turbulent transport of momentum

Spectra for wall turbulence:



[Jiménez, 2004]



↓

observation showed local theory doesn't work!

↓

Non-locality?

→ Approach to self-similarity:

Again, we can formulate the problem "mean velocity gradient" which is locally self-similar and seemingly "universal".

- Dimensionless function $y u_*^{-1} \frac{\partial u(y)}{\partial y}$ is determined by dimensionless parameters in problem:

$$\frac{y}{u_*} \frac{\partial u(y)}{\partial y} = F\left(\frac{y_d}{y}, \frac{y}{a}\right)$$

In inertial sublayer and high Re

$$\Downarrow$$

$$y/y_d \gg 1 \quad \& \quad y \ll a$$

assuming complete Re number similarity

$$\downarrow$$

$$y_d/y \to 0 \quad \& \quad y/a \to 0$$

$$\Rightarrow \frac{y}{u_*} \frac{\partial u(y)}{\partial y} = F(0,0) \to \text{const.}$$

$$\uparrow$$

log law of the wall

Additional References:

[Pope, Turbulent flows]

[Tennekes & Lumley, A First course in Turbulence]

→ Conclusion and practical Issues:

The parallel between Prantdl and K41

has been mentioned before. See below for details of comparisson:

	Inertial Range Spectrum (K41)	Pipe flow turbulence (Prantdl)
Self-similarity	in scale intertial range spectrum $v(\ell)$	in space inertial sublayer profile $v(y)$
Scales & invariance	ℓ_0, ℓ_n, ℓ_d $\ell \to$ scale space	$a, y, \nu/u_*$ $y \to$ real space
layers	$\begin{cases}\text{inertial range}\\ \text{dissipation range}\end{cases}$	$\begin{cases}\text{inertial sublayer}\\ \text{viscous sublayer}\end{cases}$
Throughput	$\epsilon = \frac{v(\ell)^2}{\tau(\ell)}$	$u_*^2 = \nu_T \frac{\partial u}{\partial y}$
Rate	$1/\tau(\ell) \sim v(\ell)/\ell$ (eddy turnover)	$\nu_T y^{-2} \sim u_*/y$
Balance	$v(\ell) \sim \epsilon^{1/3} \ell^{1/3}$	$\partial u(y)/\partial y \sim u_*/y$ (log profile)
Universality	universal spectral scaling	universal profile
Dissipation lenghtscale	$\ell_d = \nu^{3/4} \epsilon^{-1/4}$	$y_d = \nu u_*^{-1}$
Fit constant	Kolmogrov Constant	Von-Karman Constant

In fusion, Prantdl attacked problem better!

→ Practical Issues:

- Parallel between Prandtl and K41 doesn't exist in dealing with rigorous results.

triple Moment: $\begin{cases} K41: \langle \delta v^3(\ell) \rangle = -\frac{4}{5} \epsilon \ell & \boxed{4/5 \text{ law}} \\ P32: \langle \delta v^3(\ell) \rangle \simeq u_*^3 \simeq \epsilon y \end{cases}$

No theory available at this time!

- Resistance law v.s. pipe flows

$$y_d < y \leq a \xrightarrow{y \to a} u \simeq \frac{u_*}{K} \ln\left(\frac{u_* a}{\nu}\right) \; *$$

$$u_* = \left(\frac{a}{\ell} \frac{\Delta P}{2\rho}\right)^{1/2} \overset{(*)}{\Rightarrow} u = \left(\frac{a \Delta P}{2\rho \ell K^2}\right)^{1/2} \ln\left(a\left(\frac{a \Delta P}{2\rho \ell}\right)/\nu\right)$$

using $Re = \dfrac{2au}{\nu}$ and $\lambda = \dfrac{2a\Delta P/\ell}{\frac{1}{2}\rho u^2}$ we can rewrite friction law as:

$$\begin{cases} 1/\sqrt{\lambda} = .88 \ln(Re\sqrt{\lambda}) - .85 \\ Re = 2au/\nu \; ; \; \lambda = (2a\Delta P/\ell)/(\frac{1}{2}\rho u^2) \end{cases}$$

good fit to pipe flow data??

References:

[1] Diamond, P. H., Itoh, S., Itoh, K., "Modern Plasma Physics" (2007)

[2] PHYS 218C Lecture Notes, by Prof. Diamond

[3] MAE 214A Lecture Notes, by Prof. DelAlamo

[4] Jiménez, J., "Turbulence and vortex dynamics" (2004)

[5] Pope, S.B., "Turbulent Flows" (2000)