PHYSICS 215A: PARTICLES AND FIELDS, FALL 2013
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Assignment 2

January 27, 2014 (Due Monday February 10, 2014)

1. In class we defined an isometry as a symmetry of the metric tensor. That is, if $\phi : \mathscr{M} \to \mathscr{M}$ is a diffeomorphism, and g is the metric, then ϕ is an isometry of g if the pullback $\phi^* g$ agrees with g, $\phi^* g = g$. Moreover we saw that for continuos symmetries there is a one parameter set of diffeomorphisms ϕ_t, with $\phi_{t=0}$ the identity, and that the infinitesimal version of the isometry condition is the Killing equation, $\mathscr{L}_K g = 0$, where the Lie derivative is along the "Killing" vector field $K = \partial/\partial t$.
In class we also saw that Penrose diagrams use a conformally transformed metric, so we may want to consider some generalization of symmetry that includes the possibility of making a conformal transformation. An isometry up to a conformal transformation is a generalization of this, where the symmetry condition is replaced by $\phi^* g = \Omega^2 g$ where Ω^2 is a real positive function, $\Omega^2 : \mathscr{M} \to \mathbb{R}^+$. For a continuous symmetry, $\phi_t^* g = \Omega_t^2 g$ where clearly we need $\Omega_{t=0}^2 = 1$ for consistency.

(i) Show that the infinitesimal version of the isometry up to a conformal factor is (the "conformal Killing condition")

$$\mathscr{L}_K g = 2\omega g,$$

where $K = \partial/\partial t$ is a "conformal Killing field" and $2\omega = \partial_t \Omega_t^2|_{t=0}$. Expand, in components, to show K satisfies

$$K_{(\mu;\nu)} - \omega g_{\mu\nu} = 0, \tag{1}$$

where $K_{(\mu;\nu)} = \frac{1}{2}(K_{\mu;\nu} + K_{\nu;\mu})$ and $K_{\mu;\nu} = \nabla_\nu K_\mu$, as usual.

(ii) For n spacetime dimensions, eliminate ω from Eq. (1) (that is, solve for ω and plug back).

(iii) What is the conformal killing condition for Minkowski space (with metric $\eta = \text{diag}(-++\,+)$)? Write this as an explicit differential equation of K_μ, and do it for arbitrary number n of dimensions of spacetime.

(iv) Contracting one derivative with the conformal killing equation show that for a flat spacetime with flat $n = 2$ (and only in this space-time dimension) $\nabla^2 K_\mu = 0$. Find all the solutions of the conformal Killing equation for $n = 2$ flat Minkowski spacetime. (*Note: This is simpler that it looks. It's just the wave-equation.*)

(v) Find the most general solution to the conformal Killing condition in Minkowski space for $n > 2$ (*Hint: By judiciously taking two more derivatives of the conformal Killing equation you can limit the form of the most general solution*). How many independent conformal Killing vectors (including Killing vectors) are there? The Killing vectors correspond to translations and Lorentz transformations, as we saw in class. What do the new solutions correspond to?

(vi) The electromagnetic vector potential is naturally a 1-form, $A = A_\mu \mathrm{d}x^\mu$. The transformation under an isometry up to a conformal transformation is just the pull-back, $A' = \phi_t^* A$ and the infinitesimal transformation is then just the Lie derivative, $\delta A = A' - A = t\mathcal{L}_K A$, where $K = \partial_t$ is a conformal Killing vector field and t is infinitesimal. Specializing to Minkowski space find the variation of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (or, in terms of forms, $F = \mathrm{d}A$), and show that the action integral $S = \int d^4x\, (-\frac{1}{4}F^{\mu\nu}F_{\mu\nu})$ is invariant provided $n = 4$ and the fields vanish sufficiently rapidly at infinity that surface terms can be neglected.

2. *Hyperbolic Spaces.* We introduced in class the Poincare half-plane, $\mathbb{H}^2$, as the region $y > 0$ of the cartesian plane (hence "upper-half plane") with metric

$$ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2). \tag{2}$$

From here on we will use $a = 1$; it is easy enough to restore the factor. In this exercise we will connect this space to other definitions of Hyperbolic spaces, and in the next problem we will generalize to other dimensions.
The space $\mathbb{H}^n$ is defined as the as the n-dimensional surface $-(X^0)^2 + (X^1)^2 + \cdots (X^n)^2 = -1$ embedded in $(n+1)$-dimensional Minkowski space, that is, with metric $ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots (dX^n)^2$.

(i) For the case $n = 2$, eliminate X^0 from the equations defining $\mathbb{H}^n$ and introduce polar coordinates for X^0 and X^1 to show that the metric on $\mathbb{H}^2$ is

$$ds^2 = \frac{dr^2}{1+r^2} + r^2 d\theta^2. \tag{3}$$

What is the range of r and θ here?

(ii) Both (2) and (3) describe the same space. Find a transformation that takes you from one to the other.

(iii) For (2) the space is the upper half plane, while for (3) it is the whole plane (with a coordinate singularity at the origin). Explain the relation.

(iv) If we change the sign in the definition of the surface $-(X^0)^2 + (X^1)^2 + (X^2)^2 = 1$ (notice the "+" on the right hand side), show that the resulting metric on this 2-dimensional space is

$$ds^2 = \frac{dr^2}{1-r^2} + r^2 d\theta^2.$$

Interpret this result (*Hint: what is* $\int \frac{dr}{\sqrt{1-r^2}}$).

Note the three spaces,

$$ds^2 = \frac{dr^2}{1+kr^2} + r^2 d\theta^2.$$

for $k = +1, 0, -1$ are maximally symmetric. On to generalizing this:

3. More on Hyperbolic Spaces.

(i) Evidently the metric in (3) can be written as

$$ds^2 = d\chi^2 + \sinh^2\chi d\theta^2.$$

Find a coordinate system for $\mathbb{H}^n$ on which the metric is a direct generalization of this, namely,

$$dH_n^2 \equiv ds^2 = d\chi^2 + \sinh^2\chi d\Omega_{n-1}^2.$$

We have given it a special name (much as we do with the spherical line element $d\Omega^2$) because it is common enough that it is convenient to do so. You have already encountered it! The AdS metric we first wrote down in class had spatial sections with precisely this metric:

$$ds_{AdS}^2 = -\cosh^2\chi dt^2 + d\chi^2 + \sinh^2\chi d\Omega_{n-1}^2 = -\cosh^2\chi dt^2 + dH_n^2.$$

(ii) Find the analogs of Eqs. (2) and (3) for the n-dimensional case.

4. We defined both dS and AdS spaces (and H^n above) by giving embeddings of 4-dim surfaces in 5-dim spaces. Geodesics in these spaces can therefore be found by finding paths that render $\int ds$ an extremum, subject to staying on the 4-dim surface. Use this to find the geodesics in both dS and AdS spaces.