Maximally symmetric hypersurfaces: isotropic cosmology.

Since there is a separate course in cosmology, we won't study cosmology here. But we do set up the stage by analyzing the spacetime one obtains from the requirements of homogeneity and isotropy.

Why? In cosmology (the study of the history, dynamics, evolution of the universe on a large scale) our observations are limited, because:

(i) can only be done from one specific point, Earth, at one specific time, now (cosmologically the fact that we have been doing astronomy for ~1000 years is still basically an instantaneous ~~observation~~ event).

(ii) can only see part of the Universe; observations are limited by

- dust and other intervening stuff
- physics, ~~as~~ es the universe is opaque before recombination
- luminosity
- only see few bandwidths of light.

Hence, to make progress it is always the case that assumptions are made ~~to stop~~ in building mathematical models ~~describing~~ of the ~~...~~ spacetime that describes the universe.

Generally/commonly two assumptions are made: approximate:

(i) Homogeneity: that there is no preferred point in ~~the~~ space

(ii) Isotropy " " " " direction " "

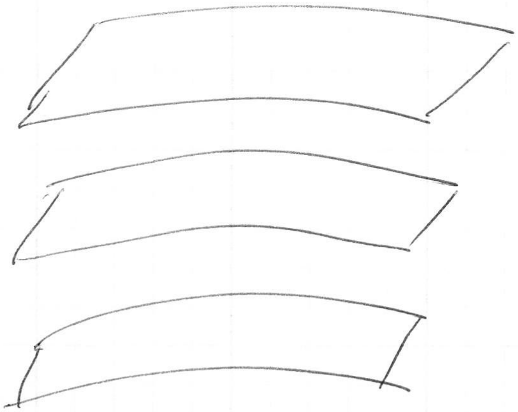
These are often ~~called~~ referred to as the <u>Copernican Principle</u> (since we would not occupy a preferred place in the Universe, just like Copernicus moved the center from Earth to Sun, now we are disposing with a center anywhere).

~~Proper~~ Notice that we are talking about <u>approximate</u> ~~sym~~ conditions. There <u>is</u> matter in the universe and it is <u>not</u> uniformly spread (there are galaxies, planets, bacteria...). But, on average, ~~using~~ smoothing over distance scales of several intergalactic ~~univers~~ lengths, the universe appears fairly smooth. So this is a starting ~~assum~~ approximation that has to be improved to account for the very interesting irregularities ⟶ whole course on cosmology.

We will restrict our attention to ~~determining~~ the spacetimes that are homogeneous & isotropic, and discuss briefly what Einstein's equations imply for them.

Technical def^n: of

Homogeneity & ~~Isotropy~~. For spacetime to be homogeneous:



Need ← foliation of spacetime by ~~space~~ 1-parameter family of spacelike surfaces Σ_t, and

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for any Σ_t ("any time"), for any two points p, q in Σ_t there is an isometry taking $p \to q$.

(Recall an isometry ϕ is $\phi: \mathcal{M} \to \mathcal{M}$ + $\phi^* g = g$).

In other words, there is some definition of time for which at each $t =$ constant hypersurface, the metric is the same at all points.

Isotropy: First define isotropy for an observer. We want to say that an observer sees same stuff in any direction. So

~~For A spacetime is to be spatially isotropic abo~~

timelike curve

$\vec{U}$ = (timelike) tangent to worldline at p.

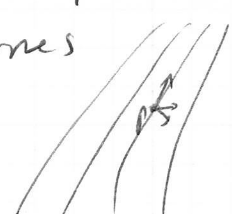
$\vec{S}_i$ = spacelike tangent vectors at p, unit magnitude, ~~tang to~~

(ie, $\vec{S}_i$ are $\perp \vec{U}$)

Isotropy at p: an isometry leaving p and $\vec{U}$ fixed but taking (rotating) $\vec{S}_1 \to \vec{S}_2$ for any pair ~~of~~ $\vec{S}_1, \vec{S}_2$.

(So there is no preferred direction, ie, no preferred spatial vector $\perp$ to $\vec{U}$).

Isotropic space: if there is a congruence of timelike curves ~~such that every point p~~ and the spacetime has isotropy at every point on these curves



← congruence.

(Wald)

Def "isotropic observers" those in this congruence

~~(So the~~

Notes:

- ~~For~~ The 2 definitions (homog & iso) required a preferred collection of subspaces.
- If spacetime is homogeneous AND isotropic, then Σ_t are $\perp$ to $\vec{U}$. For if $\vec{U}$ had a component along Σ_t, say $\vec{S}$, then this would be a spatial vector in preferred direction (we can project out the part that is not orthogonal to $\vec{U}$, to construct $\vec{S}$, a preferred vector with $\vec{S} \perp \vec{U}$).

(Actually, ~~one~~ must be careful; the previous note is true when the Σ_t and the isotropic observers ~~have~~ are unique. If not unique, one can choose $\Sigma_t \perp$ to $\vec{u}$. Example is flat Minkowski)

Now, use $\Sigma_t \to \mathcal{M}$ (embedding) to define $\boxed{h(t) = \phi^* g}$. $h_{ij}(t)$ is Riemannian (sign(+++)). (This is the same as $h = g$ restricted to act on vectors tangent to Σ_t). ~~Space~~

So consider the space Σ_t with metric h_{ij}, inverse h^{ij}.
We expect isotropy + homogeneity $\Rightarrow$ Σ_t is a 3-dim maximally symmetric space.

In fact isotropy is enough to show this (and so isotropy $\Rightarrow$ homogeneity). Consider the 3-d curvature tensor (~~field~~)

$$\bar{R}^{ij}{}_{kl} \qquad \text{(the bar for 3-D)}$$

With indexes raised as shown, this is a linear map ~~on~~ L on 2-forms

$L: \Omega^{(2)} \to \Omega^{(2)}$ ~~ω_{ij}~~ $\tilde{\omega} = \omega_{ij}\, d\tilde{x}^i \wedge d\tilde{x}^j$

$$\tilde{\omega} \to L(\tilde{\omega}) = \omega_{ij} \bar{R}^{ij}{}_{kl}\, d\tilde{x}^k \wedge d\tilde{x}^l$$

Now, defining the positive, symmetric inner product on $\Omega^{(2)}$ by

$$(\tilde{\sigma}, \tilde{\omega}) = (\tilde{\omega}, \tilde{\sigma}) = \omega_{ij}\, \sigma_{lk}\, h^{il} h^{jk}$$

then L is self-adjoint; $(\tilde{\sigma}, L\tilde{\omega}) = (L\tilde{\sigma}, \omega)$
(which follows from $\bar{R}_{ijkl} = \bar{R}_{klij}$). $\to$ there is a basis of orthonormal eigenvectors of L. By isotropy the eigenvalues must be all the same (else, special direction), so ~~$L = K\,I$~~

so $\quad L \propto I$

or

$$\bar{R}^{ij}{}_{kl} = \kappa \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k \right)$$

$$\Rightarrow \quad \bar{R}_{ijkl} = \kappa \left(h_{ik} h_{jl} - h_{il} h_{jk} \right)$$

$\Rightarrow$ ~~h_{ij} is maximally symmetric.~~

Now, this is the statement that h_{ij} is maximally symmetric if κ is constant (same everywhere on Σ_t). This follows from homogeneity. But it also follows from the Bianchy identity

$$\bar{R}_{ijkl;m} + \bar{R}_{ijmk;l} + \bar{R}_{ijlm;k} = 0$$

$$\Rightarrow \quad \kappa_{,m}(h_{ik}h_{jl} - h_{il}h_{jk}) + \kappa_{,l}(h_{im}h_{jk} - \text{rest}) + \kappa_{,k}(\,-\,-\,) = 0$$

Now contract with $h^{ik} h^{jl}$:

$$\kappa_{,m}\,(3^2-3) + \kappa_{,m}\,(1-3) + (1-3)\,\kappa_{,m} = 0$$

$$\Rightarrow \kappa_{,m} = 0$$

So, isotropy $\Rightarrow$ homogeneity <u>AND</u> h_{ij} is maximally symmetric.

Also $\quad \kappa = \dfrac{\bar{R}}{6} \quad$ a constant for each Σ_t

Finally ~~we have we~~

~~$g_{ab} = -u_a u_b + h$~~

So we have a space which admits a congruence of isotropic observers (with tangent $\vec{U}$) with a corresponding foliation by spacelike surfaces Σ_t orthogonal to $\vec{U}$, ~~a which~~ which are Riemannian 3-dim maximally symmetric spaces with metric h. The full space time has metric g, ~~If~~ and if $\vec{\xi}', \vec{\xi}$ are on Σ_t then

$$g(\vec{\xi}, \vec{\xi}') = h(\vec{\xi}, \vec{\xi}').$$

Let $\tilde{U}(\cdot) = g(\vec{U}, \cdot)$. Then clearly, if we define $h(\vec{U}, \vec{x}) = 0$

$$g = h + \lambda \tilde{U} \otimes \tilde{U}$$

however, since $g(\vec{U}, \vec{U}) = -1$, $\tilde{U}(\vec{U}) = -1$ and

$$-1 = 0 + \lambda \quad \Rightarrow \lambda = -1$$

$$\boxed{g = -\tilde{U} \otimes \tilde{U} + h}$$

In components $\quad g_{\mu\nu} = -U_\mu U_\nu + h_{\mu\nu}$

~~Choo~~ Useful coordinates:

(i) Obvious choice on each Σ_t, ie, spherical coordinates if $\bar{R} > 0$ ...

(ii) Assign a fixed spatial coordinate label to each isotropic observer ("comoving coordinates")

iii) Homogeneity $\Rightarrow$ all isotropic observers agree on proper time of Σ_t, so label Σ_t by proper time τ of isotropic observer.

$$ds^2 = -dt^2 + a^2(t) \begin{cases} d\chi^2 + \sin^2\chi \, d\Omega_2^2 & \bar{R} > 0 \\ d\chi^2 + \chi^2 d\Omega_2^2 & \bar{R} = 0 \\ d\chi^2 + \sinh^2\chi \, d\Omega_2^2 & \bar{R} < 0 \end{cases}$$

Robertson-Walker metric.

Note, there <u>is</u> a preferred set of observers: the isotropic observers. In comoving coordinates ~~the~~ the distance between fixed points p_1 & p_2 on the hypersurface Σ_t evolves with t as $a(t)$.

Einstein's Equations

~~We have $\bar{R}_{ij} = 2k\,h_{ij}$~~

Need to compute Einstein's tensor. It is fairly standard to introduce radial coordinate χ by

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

with $k = +1, 0, -1$ for $\bar{R} > 0, = 0, < 0$. Then

$r = \sin\chi, \chi, \sinh\chi$ in each case. Then

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right]$$

As usual $\Gamma_{\mu\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu})$ and $\Gamma^\mu_{\nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda}$

so

$$\Gamma_{011} = -\frac{1}{2}\frac{2a\dot{a}}{1-kr^2} = -\Gamma_{110} = -\Gamma_{101}$$

$$\Gamma_{0\theta\theta} = -\frac{1}{2}2a\dot{a}r^2 = -\Gamma_{\theta 0\theta} = \Gamma_{\theta\theta 0}$$

$$\Gamma_{0\phi\phi} = -a\dot{a}r^2\sin^2\theta = -\Gamma_{\phi\phi 0} = -\Gamma_{\phi 0\phi}$$

$$\Gamma_{111} = \frac{1}{2}a^2\frac{2kr}{(1-kr^2)^2}$$

$$\Gamma_{1\theta\theta} = -\Gamma_{\theta 1\theta} = -\Gamma_{\theta\theta 1} = -a^2 r$$

$$\Gamma_{1\phi\phi} = -\Gamma_{\phi 1\phi} = -\Gamma_{\phi\phi 1} = -a^2 r\sin^2\theta$$

$$\Gamma_{\theta\phi\phi} = -\Gamma_{\phi\theta\phi} = -\Gamma_{\phi\phi\theta} = -a^2r^2\sin\theta\cos\theta$$

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^1_{10} = \Gamma^1_{01} = \frac{\dot{a}}{a}$$

$$\Gamma^0_{\theta\theta} = a\dot{a}r^2 \qquad \Gamma^\theta_{\theta 0} = \frac{\dot{a}}{a} = \Gamma^\phi_{\phi 0}$$

$$\Gamma^0_{\phi\phi} = a\dot{a}r^2\sin^2\theta$$

$$\Gamma^1_{11} = \frac{kr}{1-kr^2}$$

$$\Gamma^1_{\theta\theta} = -r(1-kr^2) \qquad \Gamma^\theta_{1\theta} = \frac{1}{r}$$

$$\Gamma^1_{\phi\phi} = -r(1-kr^2)\sin^2\theta \qquad \Gamma^\phi_{1\phi} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \mathrm{ctg}\,\theta$$

and

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

so we have

$$R_{00} = -3\,\partial_t\left(\frac{\dot a}{a}\right) - 3\left(\frac{\dot a}{a}\right)^2 = \boxed{-3\,\frac{\ddot a}{a}}$$

$$R_{11} = \partial_t\left(\frac{a\dot a}{1-kr^2}\right) + \partial_r\left(\frac{kr}{1-kr^2}\right) - \partial_r\left(\frac{kr}{1-kr^2}\right) - \partial_r\left(\frac{2}{r}\right) + \Gamma^\rho_{\rho 0}\Gamma^0_{11} + \Gamma^\rho_{\rho 1}\Gamma^1_{11}$$

$$- \Gamma^\rho_{1\lambda}\Gamma^\lambda_{\rho 1}$$

$$= \frac{\ddot a a + \dot a^2}{1-kr^2} + 3\left(\frac{\dot a}{a}\right)\frac{a\dot a}{1-kr^2} + \frac{2}{r^2} + \frac{kr}{1-kr^2}\left(\frac{2}{r} + \frac{kr}{1-kr^2}\right) - \left[2\left(\frac{\dot a}{a}\right)\frac{a\dot a}{1-kr^2}\right.$$

$$\left. + \left(\frac{kr}{1-kr^2}\right)^2 + \frac{2}{r^2}\right]$$

$$= \frac{1}{1-kr^2}\left[\ddot a a + \dot a^2 + 3\dot a^2 + 2k - 2\dot a^2\right]$$

$$= \boxed{\frac{\ddot a a + 2\dot a^2 + 2k}{1-kr^2}}$$

$$R_{\theta\theta} = \partial_t(a\dot a r^2) + \partial_r(-r(1-kr^2)) - \partial_\theta(\mathrm{ctg}\,\theta)$$

$$+ a\dot a r^2\left(3\frac{\dot a}{a}\right) + (-r(1-kr^2))\left[\frac{kr}{1-kr^2} + \frac{2}{r}\right]$$

$$- \left[2(a\dot a r^2)\left(\frac{\dot a}{a}\right) + 2(-r(1-kr^2))\frac{1}{r} + \mathrm{ctg}^2\theta\right]$$

$$= (a\ddot a + \dot a^2)r^2 - (1-kr^2) + 2kr^2 + \frac{1}{\sin^2\theta} + 3\dot a^2 r^2 - kr^2 - 2(1-kr^2)$$

$$- 2\dot a^2 r^2 + 2(1-kr^2) - \frac{\cos^2\theta}{\sin^2\theta}$$

$$= (a\ddot a + 2\dot a^2)r^2 - 1 + 2kr^2 + \frac{1-\cos^2\theta}{\sin^2\theta} = \boxed{(a\ddot a + 2\dot a^2 + 2k)r^2}$$

$$R_{\phi\phi} = \partial_t(a\dot a r^2\sin^2\theta) + \partial_r(-r(1-kr^2)\sin^2\theta) + \partial_\theta(-\sin\theta\cos\theta)$$

$$+ (a\dot a r^2\sin^2\theta)\left(3\frac{\dot a}{a}\right) + (-r(1-kr^2)\sin^2\theta)\left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) + (-\sin\theta\cos\theta)\,\mathrm{ctg}\,\theta$$

$$- \left[2\left(\frac{\dot a}{a}\right)(a\dot a r^2\sin^2\theta) + 2(-r(1-kr^2)\sin^2\theta)\left(\frac{1}{r}\right) + 2(-\sin\theta\cos\theta)\,\mathrm{ctg}\,\theta\right]$$

$$= (\ddot a a + \dot a^2)r^2\sin^2\theta - (1-3kr^2)\sin^2\theta - \cos^2\theta + \sin^2\theta + \dot a^2 r^2\sin^2\theta$$

$$- kr^2\sin^2\theta + \cos^2\theta = \boxed{(\ddot a a + 2\dot a^2 + 2k)r^2\sin^2\theta}$$

And

$$R = g^{\mu\nu}R_{\mu\nu} = 3\frac{\ddot{a}}{a} + \frac{1}{a^2}\left[(\ddot{a}a + 2\dot{a}^2 + 2k)\cdot 3\right]$$

$$= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$$

Now Einstein's equations are $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$

or, since $-R = 8\pi G T$; $R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$

Model energy & matter in the universe by a perfect fluid. For consistency with the isotropy and homogeneity of the metric we must choose the fluid to be homogeneous and isotropic, that is the fluid is at rest in comoving coordinates:

$$U^\mu = (1,0,0,0)$$

$$T_{\mu\nu} = (\rho+p)U_\mu U_\nu + p g_{\mu\nu} \qquad T^\mu{}_\nu = (\rho+p)U^\mu U_\nu + p\delta^\mu_\nu$$

Note that $T^\mu{}_{\nu;\mu} = 0$ and for $\nu = 0$

$\Rightarrow$ recall $T^\mu{}_{\nu;\lambda} = T^\mu{}_{\nu,\lambda} + \Gamma^\mu_{\rho\lambda}T^\rho_\nu - \Gamma^\rho_{\nu\lambda}T^\mu_\rho$

so $T^\mu{}_{\nu;\mu} = T^\mu{}_{\nu,\mu} + \Gamma^\mu_{\mu\rho}T^\rho_\nu - \Gamma^\rho_{\nu\mu}T^\mu_\rho$

and so

$$T^\mu{}_{0;\mu} = -(\rho+p)_{,0} + p_{,0} + (-\rho)\left(3\frac{\dot{a}}{a}\right) - \left(3\frac{\dot{a}}{a}\right)(p)$$

so

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho+p) = 0}$$

This equation could be obtained from Einstein's, but this is simpler.

Now $T = g^{\mu\nu}T_{\mu\nu} = -(\rho+p) + 4p = 3p - \rho$

$$R_{00} = 8\pi G\left(T_{00} - \tfrac{1}{2} g_{00} T\right)$$

$$-3\frac{\ddot{a}}{a} = 8\pi G\left(\rho + \tfrac{1}{2}(3p-\rho)\right) = 8\pi G\left(\tfrac{1}{2}\rho + \tfrac{3}{2}p\right)$$

or

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

and

$$R_{11} = 8\pi G\left(T_{11} - \tfrac{1}{2} g_{11} T\right)$$

$$\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2} = 8\pi G\left[\frac{a^2}{1-kr^2}p - \frac{1}{2}\frac{a^2}{1-kr^2}(3p-\rho)\right]$$

$$= \frac{1}{1-kr^2}\, 8\pi G a^2 \left(p - \tfrac{1}{2}(3p-\rho)\right)$$

$\Rightarrow$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)$$

Eliminate $\frac{\ddot{a}}{a}$ using above, $\frac{\ddot{a}}{a}$ and $\frac{4\pi G}{3}(\rho + 3p + 3\rho - 3p) = \frac{4^2\pi G\rho}{3}$

$\Rightarrow$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho}$$

These are Friedmann equations. Metrics that obey these are called FRW metrics (Friedmann-Robertson-Walker).

We have two equations for three unknowns, $a(t)$, $\rho(t)$ and $p(t)$. But ρ and p are not independent if we know what constitutes the matter/energy in the universe. For example, a collisionless fluid (dust) has $p=0$, while radiation has $p = \frac{1}{3}\rho$. So we write an "equation of state" $p = w\rho$.

We will take w to be a fixed number, and are particularly interested in the cases

$$w = \begin{cases} 0 & \text{dust (or "matter")} \\ \frac{1}{3} & \text{radiation} \\ -1 & \text{cosmological constant.} \end{cases}$$

The last one is just the statement that if we ~~add~~ modify Einstein's equations by adding Einstein's cosmological constant Λ:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

then we can rewrite

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu} \right)$$

and think of $-\frac{\Lambda}{8\pi G} g_{\mu\nu}$ as a contribution to $T_{\mu\nu}^{(total)} = T_{\mu\nu} + T_{\mu\nu}^{(\Lambda)}$.

Then, ~~the~~ $T_{\mu\nu}^{\Lambda}$ is of the form of ~~a~~ a fluid with $\rho = +\frac{\Lambda}{8\pi G}$ and $p = -\rho$ (so $w = -1$).

Then $T^{\mu}{}_{0;\mu} = 0 \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + w\rho) = 0$ or

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \frac{d}{dt} \ln\rho = -3(1+w)\frac{d}{dt}\ln a$$

$$\boxed{\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}}$$

Note, $w = 0 \Rightarrow \rho \sim \frac{1}{a^3}$ makes sense, $\rho \sim \frac{1}{\text{volume}}$

$w = \frac{1}{3} \Rightarrow \rho \sim \frac{1}{a^4}$ ✓ $\rho \sim \frac{1}{\text{volume}} \times \text{redshift}$ (which we have not explained.

$w = -1 \Rightarrow \rho = \text{constant}$, in fact $\rho = \frac{\Lambda}{8\pi G}$.

One can then solve Einstein's equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G\rho_0}{3}\left(\frac{a_0}{a}\right)^{3(1+w)}$$

or

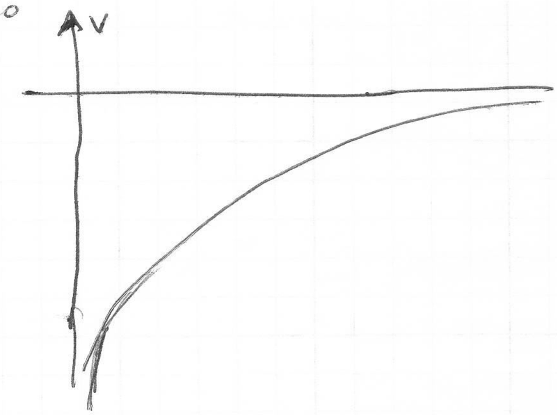
~~$\dot{a}^2 + k - \frac{8\pi G\rho_0 a_0}{3}$~~

$$\boxed{\dot{a}^2 - \frac{8\pi G}{3}\rho_0 a_0^{3(1+w)}\frac{1}{a^{1+3w}} = -k}$$

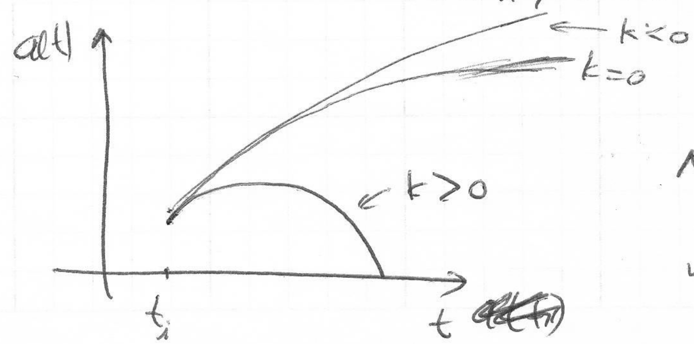
This is like the equation

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

multiplied by $\frac{2}{m}$, so ~~it is a particle~~ the solution has same time dependence as a particle in a potential $V \sim \frac{(-1)}{x^{1+3w}}$ with energy $E \sim -k$. For $1+3w > 0$ $V \to -\infty$ as $x \to 0$ and $V \to 0$ as $x \to \infty$, so



Now if $E < 0$ the motion has a ~~maximum~~ turning point at some maximum r and then eventually $r \to 0$. ~~and~~ For $E \geq 0$ the motion is unbounded provided $\dot{r} > 0$ initially. So for $w > \frac{1}{3}$ we have



Note that for $k=0$ $\dot{a} \to 0$ as $t \to \infty$ while for $k<0$ $\dot{a} > 0$ for $t \to \infty$.

Clearly it is of great (political) interest to know if the universe will expand for ever (and if so whether it will do so by slowing down to $\dot{a}=0$ asymptotically) or if it will collapse into a "big ~~g~~ crunch". Need to know $k \gtrless 0$.

Note, however, that if we start with $\dot{a}>0$ at some point, running the clock back in any case gives $a \to 0$, so it looks like the universe grew out of a singular ($a=0$) condition, or better, started small at some t_0 and quickly grew. This is called "the big bang". However, it is <u>not</u> an explosion. Recall, comoving observers are separated by fixed comoving separation. It is just that the distance (space between any two of them) is $\to 0$ as $t \to t_{\text{big bang}}$.

To figure out whether $k>0$, $k=0$ or $k<0$ in our present universe we can measure each term in the left of

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G \rho_0}{3}\left(\frac{a_0}{a}\right)^{3(1+w)} = -\frac{k}{a^2}$$

First, if this is evaluated today, then $a=a_0$ and ~~[illegible]~~ we have

$$\left(\frac{\dot{a}}{a}\right)^2\Big|_0 - \frac{8\pi G \rho_0}{3} = -\frac{k}{a_0^2}$$

Need $H_0 = \frac{\dot{a}}{a}\Big|_0$ the Hubble ~~parameter~~ constant (should be called Hubble parameter since $H = \frac{\dot{a}}{a} \neq$ constant)

and ρ_0 = energy density.

H_0 can be measured from redshift vs luminosity of standard candles (see below) while ρ_0 can be "counted"

Actually, we should be more careful to include all types of matter (different equations of state) possible in the analysis --- we have assumed one dominant type.

Write

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)} - \frac{k}{a^2}$$

You will often see this written as

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i$$

where one term, $i=k$ = curvature has $\rho_k = \frac{3}{8\pi G}\left(-\frac{k}{a^2}\right)$ ← not a real energy density! (set aside "ρ_c" for critical density).

Also dividing this by H^2,

$$1 = \sum_i \Omega_i$$

where $\Omega_i = \frac{8\pi G}{3H^2}\rho_i = \frac{\rho_i}{\rho_c}$

where $\rho_c = \frac{3H^2}{8\pi G}$ is a quantity depending only on the geometry, through H, which gives the critical value for which k changes sign: if we define $\Omega = \sum_{i\neq k}\Omega_i$ then we have

$$\Omega_k = 1 - \Omega$$

and $\Omega_k > 0, = 0, < 0$ ($k<0, =0, >0$) iff $\Omega < 1, =1, >1$.

So we need to measure all components ρ_i of ρ and compare them with ρ_c, obtained from measuring H.

Note that the different components scale differently:

$$\Omega_\Lambda \sim a^0 \qquad \Omega_k \sim \frac{1}{a^2} \qquad \Omega_{matt} \sim \frac{1}{a^3} \qquad \Omega_{rad} \sim \frac{1}{a^4}$$

If they were all similar today, then in the past, as $a \to 0$, Ω_{rad} would be dominant.

In fact, today it's found $\Omega_M \sim \frac{1}{2}\Omega_\Lambda \gg \Omega_k, \Omega_{rad}$ with $\Omega \simeq 1$.

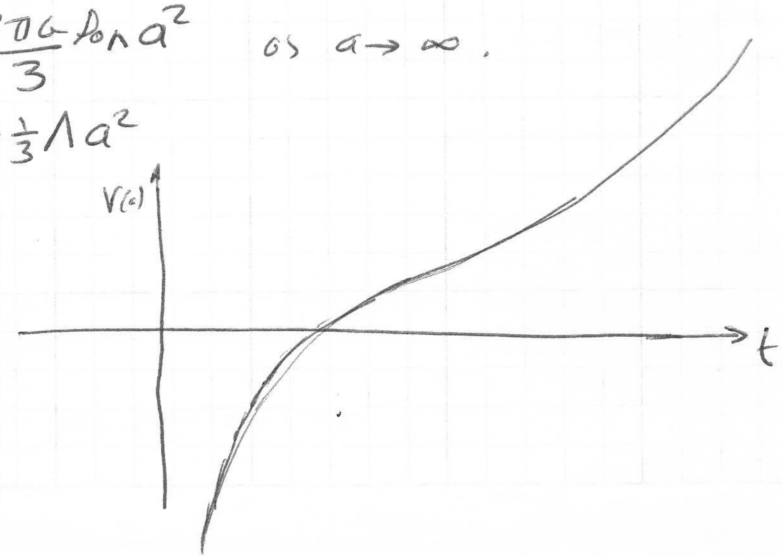
Moreover, the evolution of $a(t)$ is still given, as before, by

$$\dot{a}^2 - \frac{8\pi G}{3}\sum_i \rho_{0i}\frac{a_0^{3(1+w_i)}}{a^{1+3w_i}} = -k$$

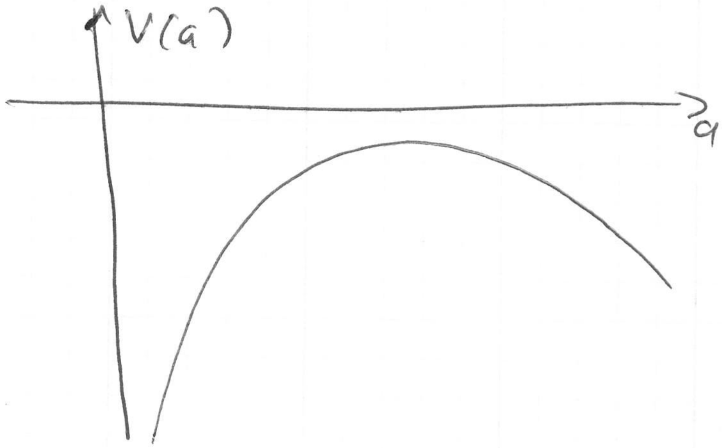
At small a, the largest w_i dominates; at large a the smallest w_i dominates. With matter, radiation and cosmological constant, we have $w_{max} = \frac{1}{3}$, $w_{min} = -1$, so the "potential" $V(a)$ has $V(a) \approx -\frac{8\pi G}{3}\frac{\rho_{0rad}\, a_0^4}{a^2}$ as $a \to 0$ and

$$V(a) \approx -\frac{8\pi G}{3}\rho_{0\Lambda} a^2 \quad \text{as } a \to \infty.$$
$$= -\frac{1}{3}\Lambda a^2$$

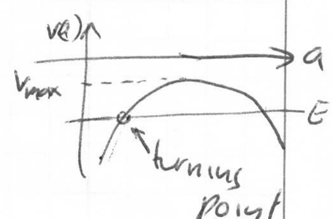
So, if $\Lambda < 0$



while, if $\Lambda > 0$



Let's look at this in more detail. For $k<0$ ("$E>0$") or $k=0$ ("$E=0$"), the "particle" motion is unbounded, describing an ever expanding universe. But for $k>0$ ("$E<0$") there is a critical value of parameters beyond which the universe recollapses. This occurs if the maximum of the potential $V(a)$ is above the energy E.



<u>Recollapse condition</u>

$$\max_a \left[-\frac{8\pi G}{3} \sum_i \rho_{0i} \frac{a_0^{3(1+w_i)}}{a^{1+3w_i}} \right] > -k$$

or multiplying by a − sign and using $\Omega_{0i} = \frac{8\pi G}{3 H_0^2} \rho_{0i}$

$$\min_a \left[H_0^2 \sum_i \frac{\Omega_{0i}\, a_0^{3(1+w_i)}}{a^{1+3w_i}} \right] < k = -H_0^2 a_0^2 \Omega_{0k}$$

or simply

$$\min_a \left[\frac{\Omega_{0Rad}\, a_0^4}{a^2} + \frac{\Omega_{0M}\, a_0^3}{a} + \Omega_{0\Lambda} a^2 \right] < -a_0^2 \Omega_{0k}$$

To simplify matters, let's ignore $\Omega_{radiation}$, since it is already negligible today. Then, taking one derivative:

$$\frac{d}{da}\left[\frac{\Omega_{0m} a_0^3}{a} + \Omega_{0\Lambda} a^2\right) = 0$$

$$\Rightarrow \quad -\frac{\Omega_{0m} a_0^3}{a^2} + 2\Omega_{0\Lambda} a = 0$$

$$\Rightarrow \quad a = \left(\frac{\Omega_{0m} a_0^3}{2\Omega_{0\Lambda}}\right)^{1/3} = a_0 \left(\frac{\Omega_{0M}}{2\Omega_{0\Lambda}}\right)^{1/3}$$

Now plug back into "potential" to find minimum

$$\text{minimum} = \frac{\Omega_{0m} a_0^3}{a_0 (\Omega_{0m}/2\Omega_{0\Lambda})^{1/3}} + \Omega_{0\Lambda} a_0^2 \left(\frac{\Omega_{0M}}{2\Omega_{0\Lambda}}\right)^{2/3}$$

$$= a_0^2\, \Omega_{0M}^{2/3} (2\Omega_{0\Lambda})^{1/3} \left(1+\tfrac{1}{2}\right)$$

and the condition for recollapse is

$$\boxed{\frac{3}{2}\Omega_{0M}^{2/3}(2\Omega_{0\Lambda})^{1/3} < -\Omega_{0k}}$$

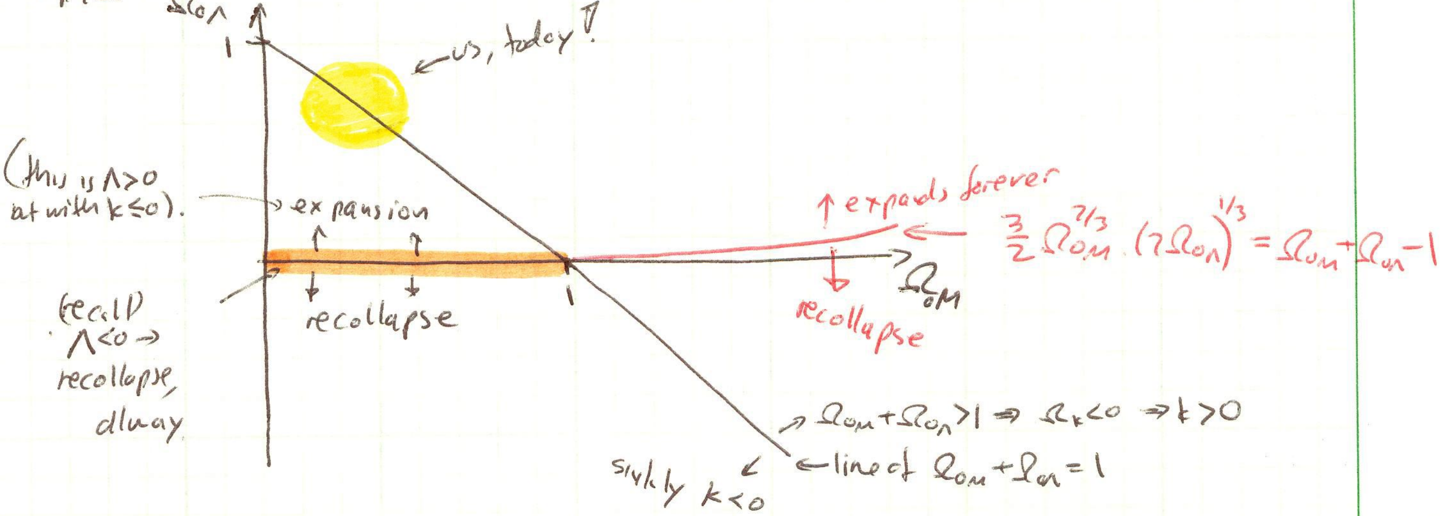
Moreover, recall that $\Omega_{0k} = 1 - \Omega_{0m} - \Omega_{0\Lambda}$, so the condition is on $\Omega_{0\Lambda}$ vs Ω_{0m}:

$$\boxed{\frac{3}{2}\Omega_{0M}^{2/3}(2\Omega_{0\Lambda})^{1/3} < \Omega_{0M} + \Omega_{0\Lambda} - 1}$$

And keep in mind that we are doing the $k>0$ case, so $\Omega_{0k}<0$ (although, the treatment has been general).

CAUTION: The solution to the inequality must be dealt with great care because of the cube root. There are two large (one positive and one negative) roots of the cubic (set the "<" to "="), and a small, positive root. Only the last is physical.

Let's put together our results in one graph: the Ω_{0M}, $\Omega_{0\Lambda}$ parameter space



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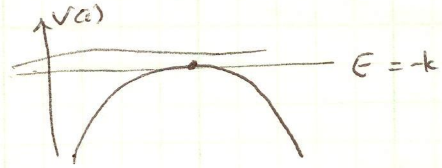
Note that there is an unstable solution to $\dot{a}^2+V(a)=-k$ with $\dot{a}=0$ and $V(a)=-k$ at the top of the hill"



That is Einstein's static universe.

Sci Am March 2005 p36 has "misconceptions about cosmology"