Symmetries, Isometry, Killing Vectors

$\phi : \mathcal{M} \to \mathcal{M}$ a diffeomorphism, T a tensor.

ϕ is a symmetry of T if

$$\boxed{\phi^* T = T}$$

T symmetric

Some symmetries are discrete. But for continuous symmetries there is a one parameter set of diffeomorphism ϕ_t, and then T is symmetric iff

$$\boxed{\mathcal{L}_U T = 0}$$

T symmetric, continuous symmetry.

(Clearly U generates the curve, $U = \frac{\partial}{\partial t}$).

Note that one can choose coordinates locally so that t itself is one of the coordinates. In such coordinates

$$\mathcal{L}_U T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = \partial_t T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}$$

so $\mathcal{L}_U T = 0 \Rightarrow$ all components of T are independent of t.

(Converse is obviously true!)

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(a diffeomorphism ϕ ...)

An isometry is a symmetry of the metric tensor,

$$\phi^* g_{\mu\nu} = g_{\mu\nu}$$

~~If~~ A vector field $\vec{K}$ that generates an isometry is called a Killing vector field:

$$\boxed{\mathcal{L}_K g_{\mu\nu} = 0}$$

continuous isometry, Killing vector

then $\vec{K}$ satisfies

$$K_{(\mu;\nu)} = 0$$

One can show the ~~reverse~~ reverse: if $K_{(\mu;\nu)} = 0$ then $\phi_t^* g_{\mu\nu} = g_{\mu\nu}$ where ϕ_t is generated by ~~m~~ $K = \frac{\partial}{\partial t}$. This is done by integration (see Hawking & Ellis).

Again, one can choose local coordinates that include t, and then $g_{\mu\nu}$ is independent of t.

Now, in first quarter (Schutz, 7.4) we saw that the geodesic equation can be written in terms of $\vec{p} = m\vec{U}$ as

$$m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta}\, p^\nu p^\alpha$$

so if $g_{\nu\alpha}$ is independent of one coordinate (say "t"), then the corresponding p_β is conserved, $\frac{dp_t}{d\tau} = 0$

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This can be done in a covariant language as follows: assume p_μ satisfies geodesic equation:

$$p^\mu p^\nu{}_{;\mu} = 0 \qquad (\nabla_p p = 0)$$

Then

$$p^\mu \nabla_\mu (p^\nu K_\nu) = p^\mu p^\nu \nabla_\mu K_\nu + K_\nu p^\mu \nabla_\mu p^\nu = 0$$

But LHS is just $\frac{d}{d\tau}(p^\nu K_\nu)$ so $\boxed{p^\nu K_\nu}$ is constant along particle path $\rightarrow$ a conserved quantity, as before.

<u>Exercise</u>: If $K_{\mu_1 \cdots \mu_r}$ is a Killing tensor, i.e., it satisfies

$$\nabla_{(\mu} K_{\mu_1 \cdots \mu_r)} = 0,$$

show that $K_{\mu_1 \cdots \mu_r} p^{\mu_1} \cdots p^{\mu_r}$ is conserved.

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We can see this more generally with our Killing field technology:

let
$$P^\mu = T^{\mu\nu} K_\nu$$

then
$$P^\mu{}_{;\mu} = T^{\mu\nu}{}_{;\mu} K_\nu + T^{\mu\nu} K_{\nu;\mu}$$

$$= T^{\nu\mu}{}_{;\mu} K_\nu + \tfrac{1}{2} T^{\mu\nu} K_{(\nu;\mu)} = 0$$

So the vector P^μ is a "conserved current". ~~By Gauss's the~~

<u>Example</u>: In flat space (which is highly symmetric):

Killing vectors
$$\vec{p}_{(\alpha)} = \frac{\partial}{\partial x^\alpha} \qquad \text{(a vector for each } \alpha = 0,1,2,3)$$

and
$$\vec{m}_{(i)} = x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0} \qquad i = 1,2,3$$

$$\vec{l}_{(ij)} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \qquad i,j = 1,2,3$$

10 - isometries generate 10 parameter Lie-Group of isometries of flat spacetime, the "inhomogeneous Lorentz group"

To see how this works ~~choose obvious coordinates~~ write components out: $\vec{p}_{(0)}{}^\mu = (1,0,0,0)$ $p^\mu_{(1)} = (0,1,0,0)$ -- $(p^\mu_{(\alpha)} = \delta^\mu_\alpha)$

So $p^{(\alpha)}_{\mu,\nu} = 0$ trivially for all (α).

Less trivial: $m^\mu_{(1)} = (x^1, x^0, 0, 0) \rightarrow m_{(1)\mu} = (-x^1, x^0, 0, 0)$

$$m_{(1)\mu,\nu} = \begin{pmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{so} \quad m_{(1)(\mu,\nu)} = 0$$

etc.

Then $P^\mu = T^{\mu\nu} K_\nu$ gives conservation of $E, \vec{P}$, $\vec{J}$ and ? (like $tP + xE$?). DON'T KNOW

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It is clear from the example that ~~manifolds~~ space-times may admit ~~more~~ several (or none) killing vectors.

Since ~~trans for~~ symmetry transformations generally form groups (group multiplication = composition of transformations, ie $\phi_t \circ \phi_u$). and these are continuous transformations generated by $\vec{K}$'s, we expect there to be ~~a~~ Lie groups, & the $\vec{K}$'s to form Lie algebras. This is indeed the case, with the Lie bracket being just the commutator, ie.

$$[K_1, K_2] = \mathcal{L}_{K_1} K_2$$

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Maximally Symmetric Spaces

Spaces with high degree of symmetry are easier to analyze.

What is the highest degree of symmetry?

Consider $\mathbb{R}^n$ – Euclidean space. Then we had

n translations

$\frac{1}{2}n(n-1)$ rotations

$= \frac{1}{2}n(n+1)$ symmetries in total

~~Rot~~ Symmetry under rotations at a point p is called "isotropy" at p

Symmetry under translation is called "homogeneity" of the spacetime.

This is as much as we can have, and we define a

$\boxed{\text{"maximally symmetric space" = one with } \tfrac{1}{2}n(n+1) \text{ killing vector fields}}$

Let's find them.

At $p \in M$ choose locally inertial coordinates, so that $g_{\mu\nu}$ is given by $\eta_{\mu\nu}$. Obviously (by construction) this is invariant under local Lorentz transformations. But isotropy means, in this coordinates, at this point p, $R_{\mu\nu\rho\lambda}$ should also be invariant,

$$R_{\mu\nu\rho\sigma} \propto \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}$$

only tensor with proper symmetries and invariant.

NOTE: A local Lorentz transformation acts only on $T_p(M)$, ie, it is a change of basis vectors $\vec{E}^{(a)}$. It is these vectors that are used to define the components of $\mathbb{R}$.

If we write this as

$$\cancel{R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}(g_{\rho\mu} g_{\sigma\nu}}$$

$$R_{\mu\nu\rho\sigma} = \kappa\,(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$$

since this is a tensorial relation it holds at P in any coordinate system. But then use homogeneity $\Rightarrow$ it holds everywhere on M with same constant κ.

Contracting indices

$$\boxed{R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})}$$

So, in particular, the Ricci scalar is a constant (should be obvious by homogeneity: same R everywhere).

A maximally symmetric space is determined by

- dimension
- signature
- R
- additional topological consideration (global issues).

Warm-up:

$n = 2$, $\eta = (++)$ ($n=2$ almost trivial, since only one component of $R^\rho{}_{\sigma\mu\nu}$).

$R > 0$ the sphere S^2 $\quad ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ $\quad R = \frac{2}{a^2}$

$R = 0$ is $\mathbb{R}^2$ $\quad ds^2 = dx^2 + dy^2$

$R < 0$ less familiar, the hyperboloid H^2 $\quad ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2)$ $\quad y > 0$

Exercise: For H^2 show

(i) $R = -\frac{2}{a^2}$ (ii) The distance between y_1 & y_2 along $x =$ constant is $a \ln \frac{y_2}{y_1}$

(iii) Geodesics satisfy $(x - x_0)^2 + y^2 = b^2$ for x_0, b constants.

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Now do $n=4$ with $(-+++)$

$R>0$ de Sitter space

$R=0$ (flat) Minkowski space

$R<0$ anti-de Sitter space

Study these. Study casual structure too.

<u>Minkowski Space-time</u>: (trivial, but will help us understand key concepts for other spacetimes)

$$ds^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

$$= -(dt)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$(r \geq 0)$
$(0 \leq \theta \leq \pi)$
$(0 \leq \phi < 2\pi)$

Null coordinates:

$$v = t + r$$
$$w = t - r$$

$\infty > v \geq w > \infty$

$$ds^2 = -dv\,dw + \tfrac{1}{4}(v-w)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$v = \text{const}$ and $w = \text{const}$ are null hypersurfaces.

Can we change coordinates to have only finite ranges? Let

$$W = \arctan w \qquad W < V$$
$$V = \arctan v \qquad \text{and both in } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

Now

$$ds^2 = \frac{1}{\omega^2}\left[-4\,dV\,dW + \sin^2(V-W)(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

where $\omega \equiv 2\cos W \cos V$

Finally write $T = V + W$ $\quad R = V - W$ $\qquad 0 \leq R < \pi$, $|T| + R < \pi$

so

$$ds^2 = \frac{1}{\omega^2}(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2)$$

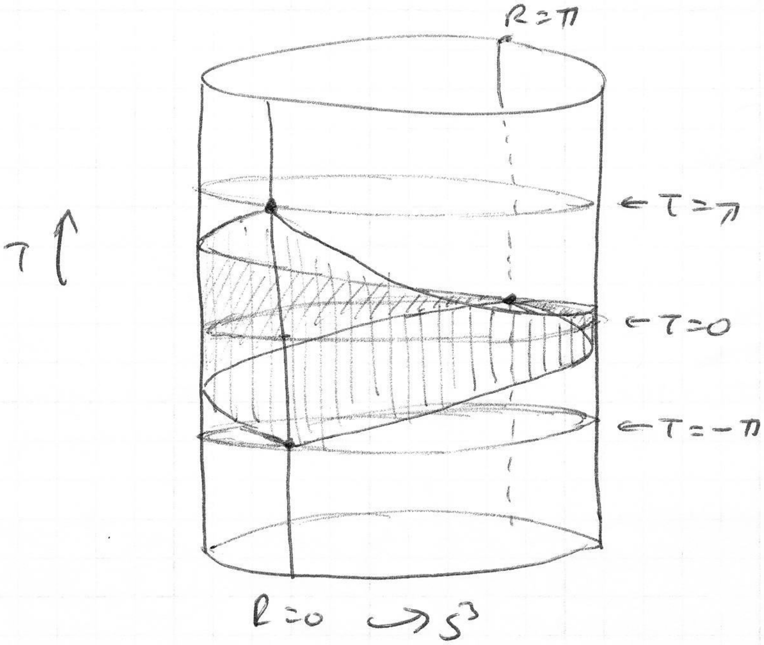
with $\omega = \cos T + \cos R$ (kind of irrelevant for us).

$$ds^2 = \frac{1}{\omega^2}\, ds_E^2$$

where $ds_E^2 = -dT^2 + dR^2 + \sin^2 R\, d\Omega^2$ "the metric for Einstein's static universe"

So Minkowski space is conformal to (a part of) the Einstein static universe

(A conformal transformation is a local change of scale; $\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}$)



Conformal Diagrams (or Penrose Diagrams)

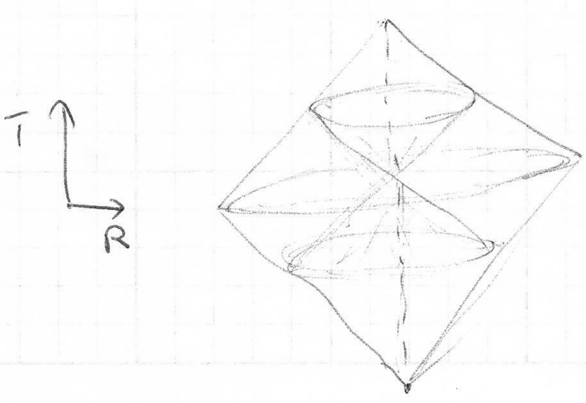
Space-time diagram for space-time (M,g). It has a "time" coordinate and a "radial" coordinate, with light-cones always at 45°. Also infinity is at finite coordinate distance (so we can fit it on a page).

Conformal transformations leave light-cones invariant

(if $d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = 0$ then $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0$)

They are useful in displaying the causal structure of spacetime.

Drawing a circle for each sphere in θ, φ



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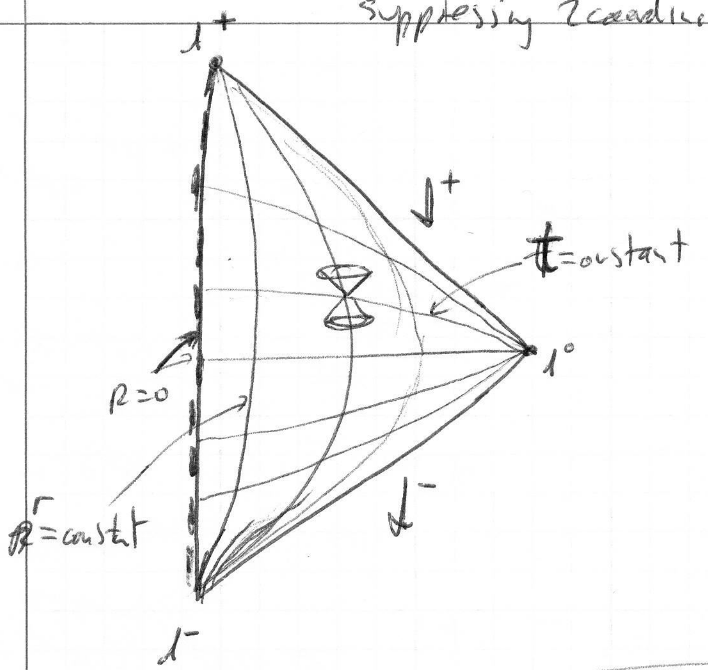
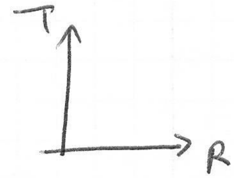
suppressing 2 coordinates (always done when with spherical symmetry):





Note: light cones always  everywhere (i.e. 45°)

SKIP IN CLASS

The constant t, r surfaces are obtained from

$$T = V + W = tg^{-1} v + tg^{-1} w = tg^{-1}(t+r) + tg^{-1}(t-r)$$

$$R = V - W \qquad\qquad = tg^{-1}(t+r) - tg^{-1}(t-r)$$

More easily: for t = constant, eliminate r $\Rightarrow$ $w + v = 2t$ fixed

$\Rightarrow \quad \Rightarrow \quad \tan V + \tan W = 2t$

$\Rightarrow \quad \tan(\frac{1}{2}(T+R)) + \tan(\frac{1}{2}(T-R)) = 2t$

etc.

---

i^+ = future timelike infinity

i^- = past ✓ ✓

i^0 = spatial infinity

$\mathscr{I}^+$ = ("scri-plus") future null infinity

$\mathscr{I}^-$ = past ✓ ✓

Features:

(i) light cones at 45°

ii) i's are points, $\mathscr{I}$'s are surfaces (null) with topology $R \times S^2$

iii) timelike geodesics: from i^- to i^+; spacelike from i^0 to i^0

null geodesics from $\mathscr{I}^-$ to $\mathscr{I}^+$

Some obvious things are more obvious in the Penrose diagram.

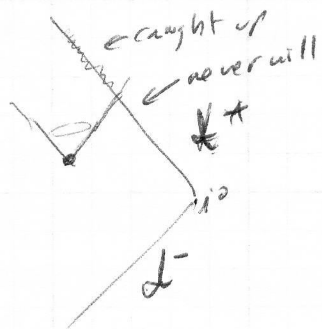
- i^+ is in the future lightcone of any event
- i^- is in the past lightcone of any event

i.e. you can reach any point, no matter how far, with a signal if you are willing to wait enough

- i^0 is neither in the future nor past light cone of an event

i.e, you can not reach space-like infinity with a signal in finite time.

If you are willing to wait an infinite time, then a signal can reach spatial infinity, but will not catch up with other signals emitted by you earlier:

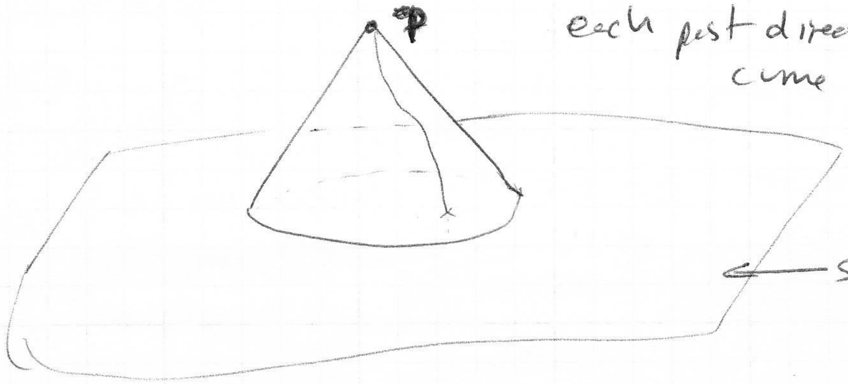


Cauchy Surfaces

$D^+(S)$: "future Cauchy development" of S :

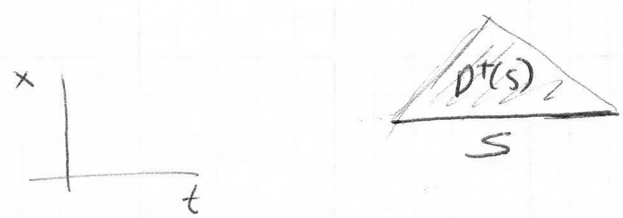
If S is a space-like 3-surface then

$D^+(S) = \{ p \in \mathcal{M} \mid$ each past directed inextendible non-spacelike curve through p intersects $S \}$

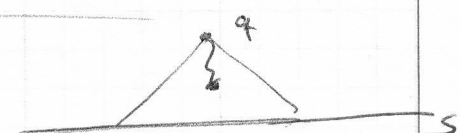


each past directed ~~timelike curve~~ non-spacelike curve through p intersects S.
$\Rightarrow p$ is in $D^+(S)$

so in Minkowski 1+1 space



Notes:



- inextendible so that we avoid:
- non-spacelike, we want the part of space that can be causaly affected by S.

~~Similarly define $D^-(S)$, "future directed curves" giving~~

Since signals can only travel on non-spacelike curves, if $p \in D^+(S)$ then knowing data (value of fields and first derivatives, or particle velocities, etc) on S is enough to predict them at p.

Similarly, if we want to evolve back into the past, but have information only on S, we can only infer the state in $D^-(S)$ (defined by "future" $\rightarrow$ "past" in def above).

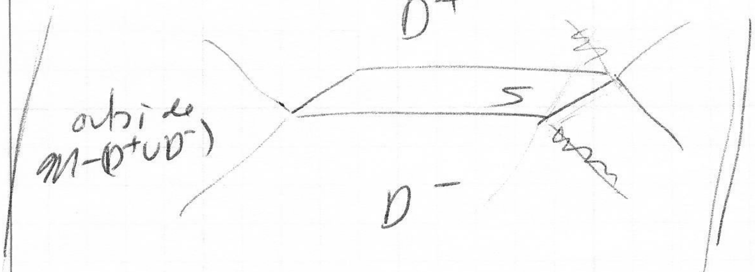
If $D^+(S) \cup D^-(S) = \mathcal{M}$

S is called a <u>Cauchy surface</u>

→ In words, ~~#~~ every inextendible non-spacelike curve in $\mathcal{M}$ intersects S. ~~S is Cauchy~~

In Minkowski space $t=0$ is a Cauchy surface. In fact $t = c$ = a constant is a collection of Cauchy surfaces that cover the whole of $\mathcal{M}$.

Not every spacelike surface in (Minkowski space) is Cauchy. Clearly extendible surfaces are not

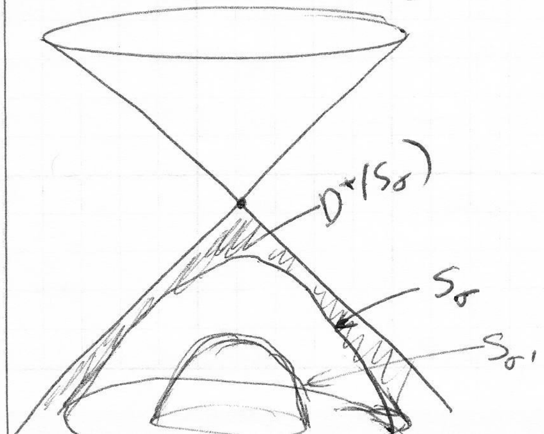


More interestingly, some inextendible surfaces are not Cauchy. For example the surfaces

$$\cancel{-(x^0)^2 + (x^1)^2 +}$$

$$-t^2+x^2+y^2+z^2=\sigma \in \mathbb{R}$$ *

are spacelike if $\sigma<0$. Let S_σ be the surface with $t<0$



These S_σ are not Cauchy, ~~and their~~ but are inextendible spacelike. The collections fills the past lightcone of the origin.

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de Sitter space-time

This is $R > 0$. Note that this means ($R = \text{const}$)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{1}{4} R g_{\mu\nu} \qquad R = \text{const} > 0$$

Comparing with Einstein's equation, (8.8) in Schutz

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = k T^{\mu\nu}$$

we see that either $k T^{\mu\nu} = -\frac{R}{4} g^{\mu\nu}$ (a weird fluid??)

or $\Lambda = \frac{1}{4} R > 0$. That is, de Sitter spacetime is a solution to Einstein's equations with a ~~cosmological~~ positive cosmological constant, but no matter.

Since we have observed $\Lambda \neq 0$, with $\Lambda \sim \rho_{\text{dark matter}}$, and since Λ is const while ρ is decreasing with the (observed) expansion of the universe, soon ρ will be negligible and the future of the universe will be described by ~~apt~~ (approximately) de Sitter space time.

It is defined by embedding the hyperboloid ~~(5 dimensions)~~

$$-U^2 + X^2 + Y^2 + Z^2 + W^2 = \alpha^2$$

defined in 5-dim Minkowski space, $ds_5^2 = -dU^2 + dX^2 + dY^2 + dZ^2 + dW^2$

~~in 4 dimensions~~

Let

$$U = \alpha \sinh(t/\alpha)$$

$$W = \alpha \cosh(t/\alpha) \cos\chi$$

$$X = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi$$

$$Y = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi$$

$$Z = \alpha \cosh(t/\alpha) \sin\chi \cos\theta$$

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Then

$$ds^2 = -dt^2 + \alpha^2 \cosh^2\left(\tfrac{t}{\alpha}\right)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

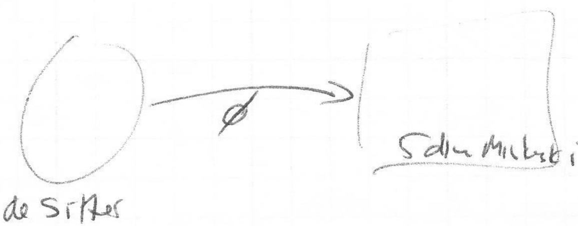
(Exercise: but not for class time!)

(1) $-U^2 + X^2 + Y^2 + Z^2 + W^2 = -\alpha^2\sinh^2\left(\tfrac{t}{\alpha}\right) + \alpha^2\cosh^2\left(\tfrac{t}{\alpha}\right)\left[\cos^2\chi + \sin^2\chi(\cos^2\theta + \sin^2\theta(\cdots))\right]$

$= \alpha^2$ ✓✓

(2)

$$g_{\mu\nu} = \frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu} g_{ab} \quad \text{is a pull back} \quad \phi^*: g \to \phi^* g$$



ϕ given by the above eqs, i.e., $U = \alpha\sinh(t/\alpha)$ etc.

So $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu} g_{ab}\, dx^\mu dx^\nu$

$$= -\frac{\partial U}{\partial x^\mu}\frac{\partial U}{\partial x^\nu}dx^\mu dx^\nu + \frac{\partial W}{\partial x^\mu}\frac{\partial W}{\partial x^\nu}dx^\mu dx^\nu + \cdots + \frac{\partial Z}{\partial x^\mu}\frac{\partial Z}{\partial x^\nu}dx^\mu dx^\nu$$

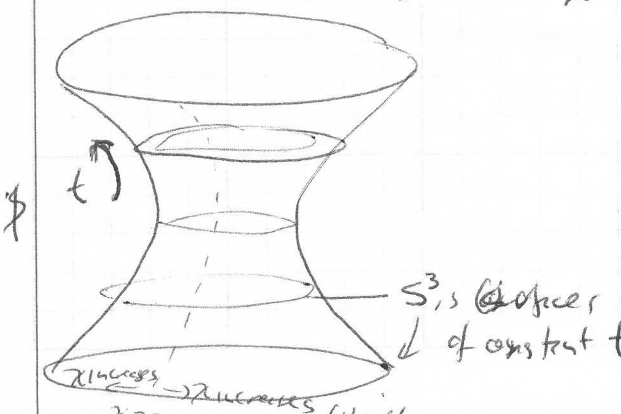
$$= -\cosh^2\frac{t}{\alpha}\,dt^2 + \sinh^2\frac{t}{\alpha}(\cos^2\chi + \sin^2\chi(\cos^2\theta + \cdots))\,dt^2$$

$$+ \alpha^2\cosh^2\frac{t}{\alpha}\left[\sin^2\chi + \cos^2\chi\left(\cos^2\theta + \sin^2\theta\,(\cos^2\phi + \sin^2\phi)\right)\right]d\chi^2$$

$$+ \alpha^2\cosh^2\frac{t}{\alpha}\sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Note $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ metric on a 2sphere

Similarly $d\Omega_3^2 = d\chi^2 + \sin^2\chi\, d\Omega_2^2 \to$ metric on a 3sphere



de Sitter space: spatial 3-sphere that shrinks to a minimum radius α, then re-expands.

topology: $\mathbb{R}^1 \times S^3$

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Another coordinate system that is common is

$$\hat{t} = \alpha \log\left(\frac{w+u}{\alpha}\right) \qquad \hat{x} = \frac{\alpha x}{w+u} \qquad \hat{y} = \frac{\alpha y}{w+u} \qquad \hat{z} = \frac{\alpha z}{w+u}$$

restricted to the hyperboloid (you can simply write ~~a~~ $\hat{t}, \ldots, \hat{z}$ as a function of t, x, θ, ϕ). In terms of these

$$ds^2 = -d\hat{t}^2 + e^{2\hat{t}/\alpha}\left(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2\right)$$

NOT NEEDED [To see this, write

$$w+u = \alpha e^{\hat{t}/\alpha}$$

$$x = \hat{x} e^{\hat{t}/\alpha}$$
$$y = \hat{y} e^{\hat{t}/\alpha}$$
$$z = \hat{z} e^{\hat{t}/\alpha}$$

and insist on the hyperboloid, $(w-u)(w+u) + r^2 = \alpha^2 \Rightarrow w-u = \dfrac{\alpha^2 - \hat{r}^2 e^{2\hat{t}/\alpha}}{\alpha e^{\hat{t}/\alpha}}$ or

$$w-u = \alpha e^{-\hat{t}/\alpha} - \frac{1}{\alpha}(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) e^{\hat{t}/\alpha}$$

Now proceed with the pull-back of η^{ab}:

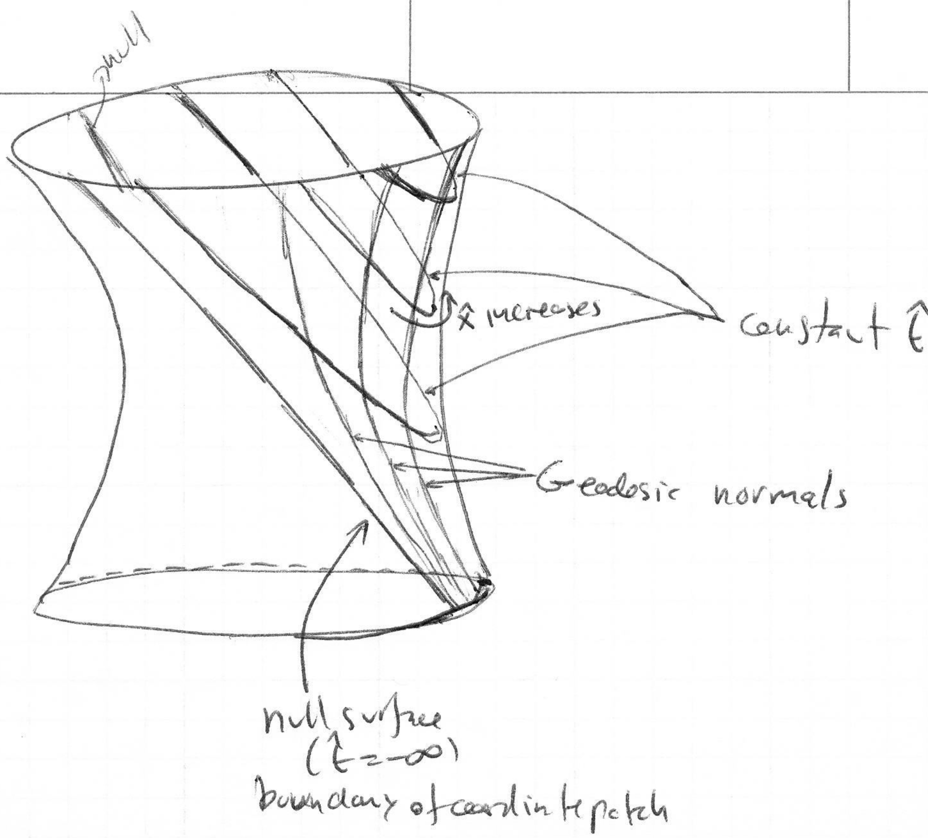
$$ds^2 = d(w+u)\,d(w-u) + dx^2 + dy^2 + dz^2 = (e^{\hat{t}/\alpha} d\hat{t})\left[\left(-e^{-\hat{t}/\alpha} - \frac{\hat{r}^2}{\alpha^2} e^{\hat{t}/\alpha}\right) d\hat{t} - \frac{\hat{\vec{r}} \cdot d\hat{\vec{r}}}{\alpha} e^{\hat{t}/\alpha}\right]$$
$$+ \left(d\hat{\vec{r}} - \frac{\hat{\vec{r}}}{\alpha} d\hat{t}\right)^2 e^{2\hat{t}/\alpha}$$

cross terms cancel!]

This coordinates cover only the region

$$w + u \geq 0$$

of the hyperboloid



[check $w + u = 0 \iff \alpha \sinh\left(\frac{t}{\alpha}\right) + \alpha \cosh\left(\frac{t}{\alpha}\right) \cos\chi = 0$

$$\Rightarrow \quad \cos\chi = -\tanh\left(\frac{t}{\alpha}\right)$$

As $t \to \mp\infty$, $\tanh\left(\frac{t}{\alpha}\right) \to \mp 1$ so $\cos\chi \to \pm 1$ or $\chi \to 0$ or π]

Penrose diagram for de Sitter:

~~Let~~ ~~$t'=$~~ Change coordi fr t to t' by

$$\tan\left(\tfrac{1}{2}t' + \tfrac{\pi}{4}\right) = e^{t/\alpha}$$

with $t' \in (-\pi/2, \pi/2)$

Not needed [then

$$dt^2 \frac{e^{2t/\alpha}}{\alpha^2} = \left(\frac{1/2}{\cos^2(\frac{1}{2}t' + \frac{\pi}{4})}\right)^2 dt'^2$$

or

$$dt^2 = \frac{\alpha^2}{4} \frac{1}{\cos^4(\frac{t'}{2}+\frac{\pi}{4})} \frac{\cos^2}{\sin^2} (dt')^2 = \frac{\alpha^2}{4\cos^2\sin^2} dt'^2$$

$$= \frac{\alpha^2}{\sin^2(t'+\frac{\pi}{2})} dt'^2$$

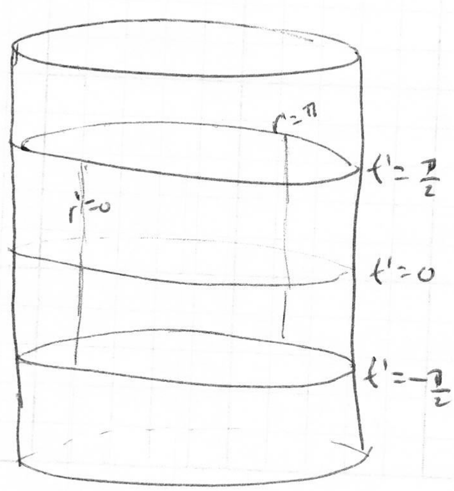
and

$$\cosh\frac{t}{\alpha} = \frac{1}{2}\left(\tan + \frac{1}{\tan}\right) = \frac{1}{2}\,\frac{\sin^2 + \cos^2}{\sin\cos} = \frac{1}{\sin(t'+\frac{\pi}{2})}$$ so]

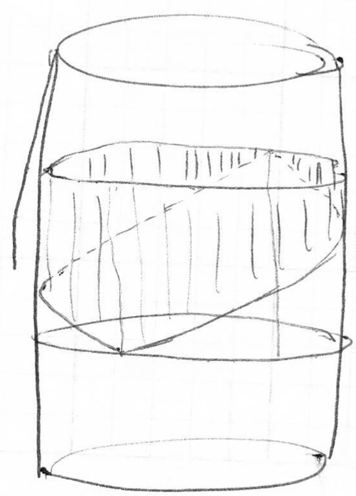
$$ds^2 = \frac{\alpha^2}{\sin^2(t'+\frac{\pi}{2})}\, d\bar{s}^2$$ ($d\bar{s}^2 = ds_E^2$ in previous notation)

where $d\bar{s}^2 = -dt'^2 + d\chi^2 + d\Omega_2^2 = -dt'^2 + d\Omega_3^2$

So de-Sitter is conformal to the metric $d\bar{s}^2$ = Einstein Static ~~&~~ familiar from Minkowski. Now



and



steady state universe (Lemaitre?)

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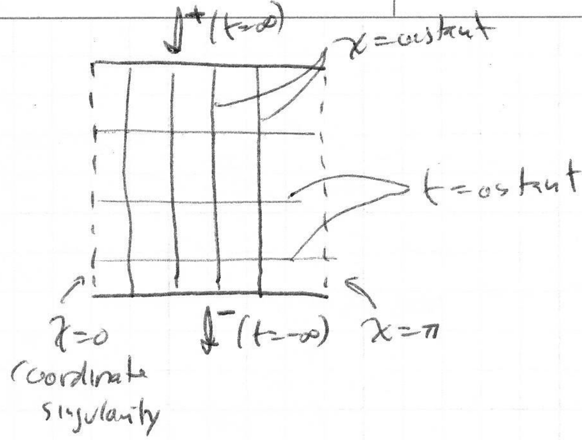
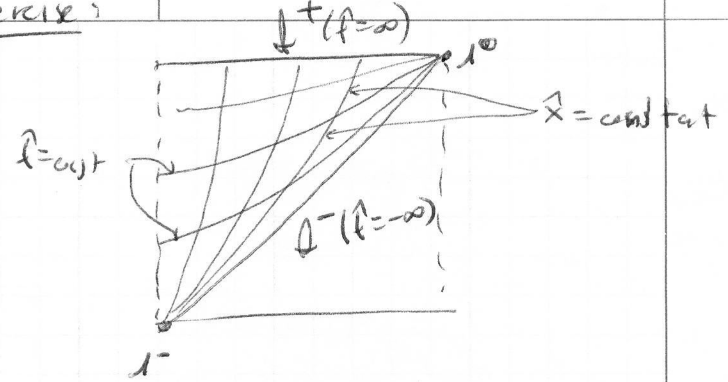
Exercise:



de-Sitter



Steady-state universe of Bondi + Gold, and Hoyle (circa 1948)

$\mathcal{I}^\pm$ spacelike future/past infinity.

Horizons: (NEXT PAGE)

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