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PHYS 200B - HW # 3

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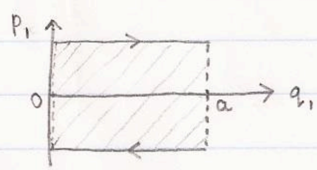
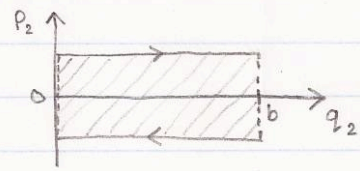
1. (a) A particle is confined in a 2D box of length a, b. The motion in phase space, projected onto the q_1, p_1 and q_2, p_2 planes respectively, looks like:





The dashed vertical lines are the (instantaneous) collisions of the particle with the walls of the box. Motion in one direction is completely independent of motion in the other.

Action variables:

$$I_1 = \frac{1}{2\pi}\oint p_1\, dq_1 = \frac{|p_1|\, a}{\pi}. \qquad \text{Similarly,} \quad I_2 = \frac{|p_2|\, b}{\pi}$$

The Hamiltonian, as a function of I_1 and I_2, is

$$H_0 = \frac{p_1^2}{2M} + \frac{p_2^2}{2M} = \frac{\pi^2}{2M}\left[\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2}\right] = E_1 + E_2$$

The angle variables are

$$\theta_1 = \frac{\pi}{a}\begin{cases}(2la + q_1) & \text{for } lT_1 < t < (l+\tfrac{1}{2})T_1 \\ (2la - q_1) & \text{for } (l-\tfrac{1}{2})T_1 < t < lT_1\end{cases}$$

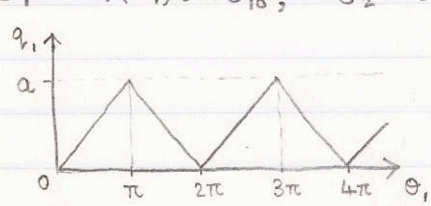
(l integer), where $T_1 = 2Ma/|p_1| = 2\pi/\omega_1$ is the period of motion in the first direction (time taken to return to $q_1 = 0$ after one cycle), and similarly for θ_2.

We have:

$$\dot{I}_1 = \dot{I}_2 = 0$$

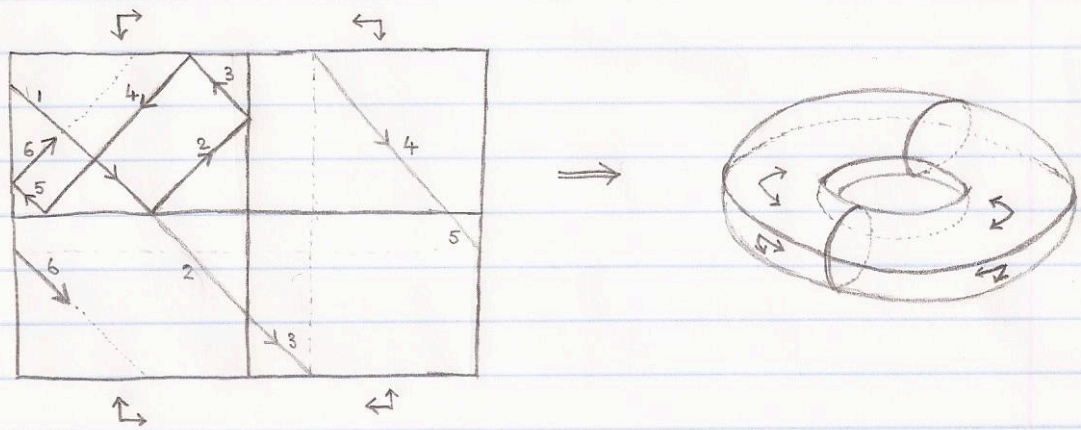
$$\dot{\theta}_1 \equiv \omega_1 = \frac{\partial H_0}{\partial I_1} = \frac{\pi^2 I_1}{Ma^2}, \qquad \dot{\theta}_2 \equiv \omega_2 = \frac{\partial H_0}{\partial I_2} = \frac{\pi^2 I_2}{Mb^2}$$

$$\Rightarrow \theta_1 = \omega_1(I_1)t + \theta_{10}, \quad \theta_2 = \omega_2(I_2)t + \theta_{20}$$



Since ω_j are linearly increasing functions of I_j, the flow on the nested tori is <u>sheared</u>.

In order to properly visualize the mapping from motion in real space to motion on a torus in action-angle space, the following construction is useful. The real motion occurs in the top-left box. The other quadrants are virtual, reflected trajectories, which show motion on the torus:



(b) $H(\vec{\theta}, \vec{I}) = H_0(\vec{I}) + \epsilon H_1(\vec{\theta}, \vec{I})$, where $H_1(\vec{\theta}, \vec{I}) = \sum_{m,n} V_{mn}(\vec{I}) e^{i(m\theta_1 - n\theta_2)}$

Naive perturbation theory:

$$\dot{I}_1 = -\frac{\partial H}{\partial \theta_1} = -\epsilon \frac{\partial H_1}{\partial \theta_1} = -\epsilon \sum_{m,n} i m V_{m,n}(\vec{I}) e^{i(m\theta_1 - n\theta_2)},$$

$$\dot{\theta}_1 = \frac{\partial H}{\partial I_1} = \omega_1(I_1) + \epsilon \sum_{m,n} \frac{\partial V_{m,n}}{\partial I_1} e^{i(m\theta_1 - n\theta_2)}, \quad \text{etc.}$$

These can be integrated iteratively:

$$\theta_1^{(0)}(t) = \omega_1(I_1^{(0)})t \equiv \omega_{10}t, \quad \theta_2^{(0)}(t) = \omega_2(I_2^{(0)})t \equiv \omega_{20}t$$

$$\Rightarrow I_1^{(1)}(t) = I_1^{(0)} - \epsilon \sum_{m,n} i m V_{m,n}(\vec{I}^{(0)}) \frac{e^{i(m\omega_{10} - n\omega_{20})t}}{i(m\omega_{10} - n\omega_{20})},$$

and so on...

The small divisor problem is already visible here. We would also encounter it, in the same way, if we tried to do "canonical" perturbation theory (attempting to transform to new action-angle variables $\vec{\theta}', \vec{I}'$ using some S).

(c) Let $m\omega_1 - n\omega_2 = 0$ for some m, n and some I_1, I_2

Define $\phi_1 = m\theta_1 - n\theta_2$ and $\phi_2 = \theta_2$ and corresponding J_1, J_2

The transformation to these variables is generated by

$$F(\vec{\theta}, \vec{J}) = (m\theta_1 - n\theta_2) J_1 + \theta_2 J_2$$

$$\Rightarrow I_1 = \frac{\partial F}{\partial \theta_1} = mJ_1 \quad , \quad I_2 = \frac{\partial F}{\partial \theta_2} = -nJ_1 + J_2$$

$$\Rightarrow H_0(\vec{J}) = \frac{\pi^2}{2M}\left[\frac{m^2 J_1^2}{a^2} + \frac{(J_2 - nJ_1)^2}{b^2}\right]$$

Using the commensurability or "resonance" condition,

$$m\omega_1 - n\omega_2 = 0 \quad \Rightarrow \quad \frac{\pi^2}{M}\left[m\frac{I_1}{a^2} - n\frac{I_2}{b^2}\right] = 0$$

$$\Rightarrow \quad \frac{m^2 J_1}{a^2} + \frac{n^2 J_1}{b^2} - \frac{nJ_2}{b^2} \quad ,$$

it is easy to verify that $\dot{\phi}_1 = \tilde{\omega}_1(J_1) = \frac{\partial H_0}{\partial J_1} = 0$, as intended.

Now, the problem to solve is described by the Hamiltonian

$$H(\vec{\phi}, \vec{J}) = H_0(\vec{J}) + \epsilon H_1(\vec{\phi}, \vec{J}) \quad ,$$

where

$$H_1 = \sum_{r,s} V_{r,s}(I_1, I_2)\, e^{i(r\theta_1 - s\theta_2)} \quad ; \quad r\theta_1 - s\theta_2 = r\frac{(\phi_1 + n\phi_2)}{m} - s\phi_2$$

$$\Rightarrow H_1(\vec{\phi}, \vec{J}) = \sum_{r,s} V_{r,s}(mJ_1, J_2 - nJ_1)\, \exp\left\{i\left[\left(\frac{r}{m}\right)\phi_1 + \left(\frac{rn - sm}{m}\right)\phi_2\right]\right\}$$

Effect an action-angle transformation only for the fast variables $(\phi_2, J_2) \rightarrow (\phi_2', J_2')$, with the slow variables (ϕ_1, J_1) unchanged, such that ϕ_2' is cyclic and $H = H(\phi_1, J_1; J_2')$ only.

This transformation is generated by the abbreviated action $S(\phi_1, \phi_2, J_1, J_2')$.
S satisfies the Hamilton-Jacobi equation:

$$H\left(\phi_1, \phi_2, \frac{\partial S}{\partial \phi_1}, \frac{\partial S}{\partial \phi_2}; \epsilon\right) = H_0\left(\frac{\partial S}{\partial \phi_1}, \frac{\partial S}{\partial \phi_2}\right) + \epsilon H_1\left(\phi_1, \phi_2, \frac{\partial S}{\partial \phi_1}, \frac{\partial S}{\partial \phi_2}\right)$$

$$= \alpha(\phi_1, J_1, J_2'; \epsilon)$$

where α will become the new Hamiltonian.

We expect that

$$S = S_0 + \epsilon S_1 + \cdots \qquad \text{and} \qquad \alpha = \alpha_0 + \epsilon \alpha_1 + \cdots$$

<u>To zeroth order in ϵ:</u>

$$\alpha_0(J_1, J_2') = H_0(J_1, J_2') \quad ; \quad S_0 = \phi_1 J_1 + \phi_2 J_2'$$

<u>To first order in ϵ:</u>

$$\alpha_1(\phi_1, J_1, J_2') = \left(\frac{\partial H_0}{\partial J_2'}\right)\left(\frac{\partial S_1}{\partial \phi_2}\right) + H_1(\phi_1, \phi_2, J_1, J_2')$$

$$= \tilde{\omega}_2(J_2')\left(\frac{\partial S_1}{\partial \phi_2}\right) + H_1(\phi_1, \phi_2, J_1, J_2')$$

We can expand S_1 and H_1 as fourier series of the form

$$S_1 = \sum_m S_{1m}(\phi_1, J_1, J_2')\, e^{im\phi_2} \quad , \quad H_1 = \sum_m H_{1m}(\phi_1, J_1, J_2')\, e^{im\phi_2}$$

We can similarly consider α_1 to have such a series, wherein all the terms for $m > 0$ are zero. Matching fourier coefficients, we get

$$\alpha_1(\phi_1, J_1, J_2') = H_{10}(\phi_1, J_1, J_2') = \overline{H_1(\phi_1, J_1; \phi_2, J_2')}$$

where the bar denotes an average over ϕ_2 only.

We could go on to get S_1 by matching $m \geq 1$ modes, but why bother?

To first order, then,

average over ϕ_2 only.

$$H(\phi_1, J_1; J_2') = H_0(J_1; J_2') + \epsilon\, \overline{H_1(\phi_1, J_1; \phi_2, J_2')}$$

$$\therefore\ H(\phi_1, J_1; J_2') = \frac{\pi^2}{2M}\left[\frac{m^2 J_1^2}{a^2} + \frac{n^2 J_1^2}{b^2} - \frac{2n J_1 J_2'}{b^2} + \frac{(J_2')^2}{b^2}\right]$$
$$+ \epsilon \sum_{r,s} V_{r,s}(mJ_1, J_2' - nJ_1)\, e^{i(r/m)\phi_1}\, \delta_{rn-sm}$$

The Kronecker delta appears because if $(rn-sm)$ is nonzero, the function will oscillate as ϕ_2 varies and therefore average to zero.

Since m, n, r and s are all integers, we have

$rn - sm = 0 \Rightarrow rn = sm \Rightarrow r = mp\ \&\ s = np,$

where p is an integer. (i.e. only harmonics of (m,n) contribute to the sum).

$$\therefore\ H(\phi_1, J_1; J_2') = H_0(J_1, J_2') + \epsilon \sum_p V_{pm,pn}(\cdots)\, e^{ip\phi_1}$$

If we keep only the $p = 0, \pm 1$ harmonics, we obtain:

$$H(\phi_1, J_1; J_2') = H_0(J_1, J_2') + \epsilon V_{0,0}(\cdots) + 2\epsilon V_{m,n}(\cdots)\cos\phi_1$$

(since $V_{-r,-s} = V_{r,s}$ to ensure H_1 is real)

By assumption, J_1 and J_2' are near resonance, and J_2' is a constant of motion by construction. At a resonance,

$$\tilde{\omega}_1(J_{10}) = \left(\frac{\partial H_0}{\partial J_1}\right)_{J_{10}} = 0 \quad , \quad (J_{10} \text{ is a function of } J_2')$$

Expanding about this resonance (in J_1, for a fixed J_2'), and ignoring constant terms,

$$H_{eff} = \frac{1}{2}\left(\frac{\partial^2 H_0}{\partial J_1^2}\right)_{J_{10}} (J_1 - J_{10})^2 + 2\epsilon V^{\circ}_{m,n}\cos\phi_1$$

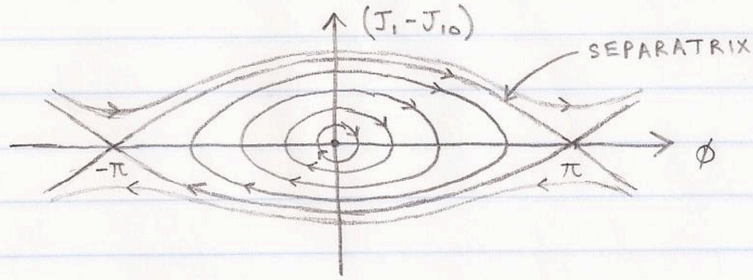
$$\Rightarrow\ H_{eff} = \frac{1}{2 I_{eff}}(J_1 - J_{10})^2 - K_{eff}\left[\cos\phi_1 - 1\right]$$

where $V^{\circ}_{m,n} \equiv V_{m,n}(mJ_{10}, J_2' - nJ_{10})$, $\quad K_{eff} = -2\epsilon V^{\circ}_{m,n}$,

and $I_{eff} = \dfrac{1}{(\partial^2 H_0/\partial J_1^2)_{J_{10}}} = \left[\dfrac{\pi^2}{M}\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)\right]^{-1}$

This is of the form of the Hamiltonian of a simple pendulum.

Islands arise when phase trajectory goes from $-\pi$ to π:



Island width is easily determined by equating H_{eff} (which is a constant of motion $= E_{eff}$) at $\phi = 0$ and at $\phi = \pi$ for motion along the separatrix:

$$\frac{(J_1 - J_{10})^2}{2 I_{eff}} = 2K_{eff} \Rightarrow (\Delta J_1)_{max} = |J_1 - J_{10}|_{\phi=0} = 2\sqrt{I_{eff} K_{eff}}$$

$$\Rightarrow (\Delta J)_{max} = 2\left[\frac{-2\epsilon M V^0_{m,n}}{\pi^2 (m^2/a^2 + n^2/b^2)}\right]^{1/2}$$

Note that this is really the island "half-width".

(d) If the perturbing potential is a hard wall function in space, the physical interpretation is that some obstacle was placed into the box which scatters the particle. It is not clear how the "strength" of such a perturbation may be varied continuously in a perturbation parameter ε.

③ a) SUMMARIZE THIS STORY IN YOUR OWN WORDS.

The overarching question is this: How common is integrability? To answer this question, we consider what happens if we apply a perturbation to a Hamiltonian H_0 which is integrable:

$$H(\vec{I},\vec{\theta}) = H_0(\vec{I}) + \epsilon H_1(\vec{I},\vec{\theta}).$$

If the orbits are still restricted to tori, then $H(\vec{I},\vec{\theta}) = K(\vec{J})$, and we can expand the Hamilton-Jacobi equation $H(\frac{\partial S}{\partial \vec{\theta}}, \vec{\theta}) = K(\vec{J})$, just as we did in class.

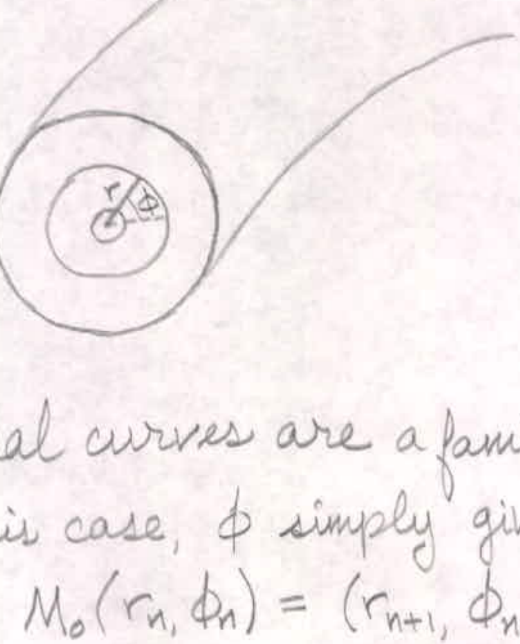
Following this course, we solve for the generating function to first order in ϵ,

$$S(\vec{J},\vec{\theta}) = \vec{J}\cdot\vec{\theta} + i\epsilon \sum_{\vec{m}} \frac{H_{\vec{m}}(\vec{J})}{\vec{m}\cdot\vec{\omega}_0(\vec{J})} \exp(i\vec{m}\cdot\vec{\theta}) + O(\epsilon^2),$$

where $H_{\vec{m}}$ denotes the Fourier coefficients of H_1 and $\vec{\omega}_0(\vec{J}) \equiv \frac{\partial H_0}{\partial \vec{J}}$ is the winding frequency. For this result to be sensible, we require the sum over $\vec{m}$ to converge, but in the event that $\vec{m}\cdot\vec{\omega}_0 = 0$, the condition is evidently not satisfied. This is the <u>problem of small denominators</u>, and $\vec{m}\cdot\vec{\omega}_0 = 0$ is said to define the <u>resonant tori</u> of the unperturbed system.

The preceding analysis suggests that non-resonant tori do survive small perturbations; the KAM theorem confirms this hypothesis rigorously. What, then, becomes of the resonant tori, for which perturbation theory failed?

Here Ott's treatment diverges from our discussion in class. Following Ott, we consider an integrable Hamiltonian system with two pairs of action-angle variables. Here's a cross section with $\theta_2 = \text{const.}$:



The cross-sectional curves are a family of concentric circles. (In this case, ϕ simply gives θ_1.) We define a map $M_0(r_n, \phi_n) = (r_{n+1}, \phi_{n+1})$,

$$\begin{cases} r_{n+1} = r_n, \\ \phi_{n+1} = [\phi_n + 2\pi R(r_n)] \bmod 2\pi, \end{cases}$$

where $R(r) = \frac{\omega_1}{\omega_2}$, $\vec{\omega}_0 = (\omega_1, \omega_2) = \left(\frac{\partial H_0}{\partial I_1}, \frac{\partial H_0}{\partial I_2}\right)$. How do the concentric circles behave under this map?

Note that, on a resonant torus, $R(r) = \frac{\omega_1}{\omega_2}$ is rational. Suppose that $R(\hat{r}) = \frac{p}{q}$, so that $\hat{r}$ corresponds to a resonant torus. (p and q have no common factors, of course.) Then M_0^q (the q^{th} power of M_0) returns every point on the circle of radius $\hat{r}$ to its original location, since we are modding out by 2π.

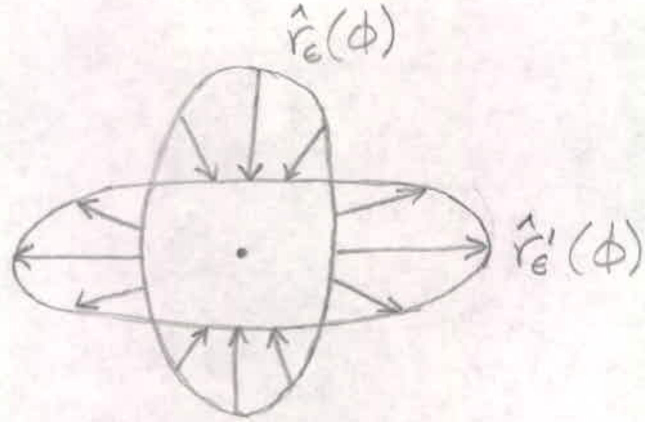
With this observation in mind, we now introduce a perturbation to produce a new area-preserving map M_ϵ satisfying

$$\begin{cases} r_{n+1} = r_n + \epsilon g(r_n, \phi_n), \\ \phi_{n+1} = [\phi_n + 2\pi R(r_n) + \epsilon h(r_n, \phi_n)] \bmod 2\pi. \end{cases}$$

What happens to the circle of fixed points at $\hat{r}$ under this new map?

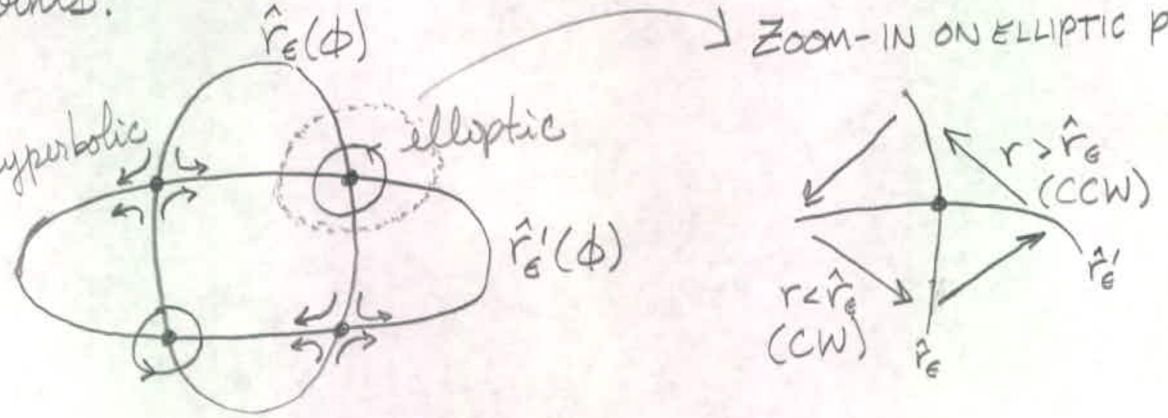
To answer this question, suppose that $R(r)$ is a smoothly increasing function of r in the vicinity of $\hat{r}$. We can choose a circle at $r_+ > \hat{r}$ which is rotated CCW, and a circle at $r_- < \hat{r}$ which is rotated CW, by M_0^q. The circle $\hat{r}$ is not rotated at all.

For small enough ϵ, M_ϵ^q likewise maps points on r_+ CCW and points on r_- CW, although now the radial coordinate may change. It follows that there must be a curve $\hat{r}_\epsilon$ between r_+ and r_- and close to $\hat{r}$ on which points are mapped purely radially by M_ϵ^q. We will denote the curve obtained by applying M_ϵ^q to $\hat{r}_\epsilon$ by the symbol $\hat{r}_\epsilon'(\phi)$. The mapping might look like this:



Since M_ϵ is area-preserving, these curves must intersect at an even number of distinct points. These intersections correspond to the fixed points of M_ϵ^q, thus replacing the circle of fixed points at $\hat{r}$ for the unperturbed map.

For $r > \hat{r}_\epsilon$, points go CCW under M_ϵ^q, and for $r < \hat{r}_\epsilon$, points go CW. We therefore obtain the following picture of alternating elliptic and hyperbolic points:



Since there is an even number of alternating fixed points,

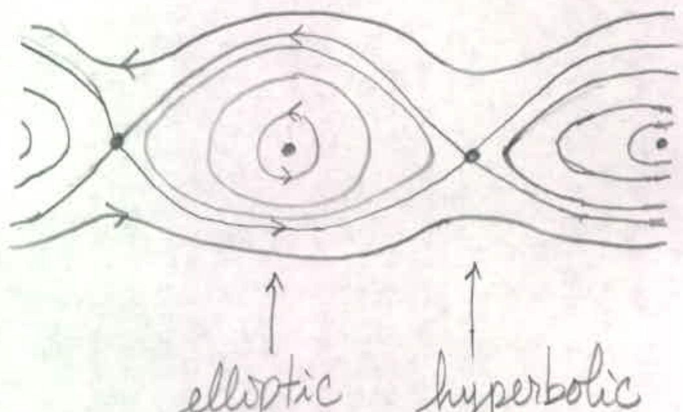
$$\#\text{ elliptic pts.} = \#\text{ hyperbolic pts.}$$

This is the <u>Poincaré–Birkhoff theorem</u>. It states, further, that since fixed points of M_ϵ^q are on period q orbits of M_ϵ, there must be (a multiple of) q fixed points of each type.

Very near an elliptic fixed point, the map again takes the form of M_ϵ. Thus, surrounding an elliptic point are encircling curves between which are destroyed resonant curves that have themselves been replaced by elliptic and hyperbolic curves, ad infinitum.

Hyperbolic orbits, on the other hand, typically result in heteroclinic intersections. That is, orbits passing through the hyperbolic fixed points travel from one hyperbolic fixed point to the next.

All told, we get a picture like this one:



Numerical studies show how this picture descends into chaos as the perturbation is strengthened.

WHY IS THIS OF INTEREST TO OUR DISCUSSION OF CHAOS?

Ott's account illustrates the destruction of resonant tori from the map perspective. Poincare-Birkhoff in particular makes it precise how the islands form for small perturbations. As the perturbation grows, this picture reproduces what we observed from a calculational approach in class.

b) TABLE OF CORRESPONDENCES

CLASS	OTT
• Calculational approach relying on Hamilton's eqtns, etc.	• Analysis using maps, ultimately geometric.
• Remove resonance by averaging over fast variable (to wit, $r\theta_1 - s\theta_2$ is slow, θ_2 is fast).	• Treat resonance using its q-periodicity under map M.
• $\hat{J}_2 = I_2 + \frac{s}{r} I_1$ is an adiabatic invariant of the averaged Hamiltonian.	• Intersections of $\hat{r}_\epsilon$ and $\vec{r}_\epsilon'$ are fixed points under M_ϵ^q.
• We solve for the fixed pts. of $\langle H \rangle = \frac{G}{2}(\hat{J}_1 - \hat{J}_{10})^2 - F\cos\phi$, yielding $\phi_1 = 0$ stable, $\phi_2 = \pm\pi$ unstable.	• We obtain Poincaré–Birkhoff theorem, telling us how the fixed points are arranged.

C) WHAT DO WE LEARN FROM OTT?

We gain geometric insight into the destruction of resonant tori. This is encapsulated in the Poincaré-Birkhoff theorem, which tells us that

- There are (a multiple of) q fixed points of each type, so that #elliptic = #hyperbolic.
- The fixed points alternate between elliptic and hyperbolic.

We also see how elliptic points give rise to fractal behavior.

WHAT DO WE LEARN FROM CLASS?

Nothing.

Just kidding. We obtain the equations behind the phase-space pictures, and we see explicitly that the perturbed system with $N=2$ is isomorphic to the pendulum. Moreover, we get the island width

$$(\Delta J)_{max} \cong 2\left(-2\epsilon H_{r,-s} \Big/ \left.\frac{\partial^2 H_0}{\partial \hat{J}_1^2}\right|_{\hat{J}_{1,0}}\right)^{1/2},$$

which leads directly into Chirikov.

4.) $dx/dt = v$ / $\frac{dv}{dt} = \frac{q}{M} E$

$E = -\frac{d\phi}{dx}$

$\phi = \sum_n \phi_{kn} \cos(k_n x - w(k_n)t)$

a.) Monochromatic $\Rightarrow$ $\phi = \phi_K \cos(Kx - wt)$

$E = \phi_K K \sin(Kx - wt)$

discretize

$\frac{X_{n+1} - x_n}{dt} = v_n$

$\frac{v_{n+1} - v_n}{dt} = \frac{q}{m} \phi_K K \sin(kx_n - wt)$

$$\begin{cases} x_{n+1} = v_n dt + x_n \\ v_{n+1} = \frac{q}{m} \phi_K K \sin(Kx_n - wt) dt + v_n \end{cases}$$

Fixed points if $v_n = 0$

$\sin(kx_n - wt) = 0 \Rightarrow x_n = \frac{wt}{K} + \frac{\pi n}{k}$

"Resonance" $\rightarrow$ $\boxed{x = w/k\, t + \frac{\pi n}{k} \text{ particle moves with wave fronts}}$ $\frac{w}{k} = v_{phase}$

There are a discrete # of fixed points. Poincare Birkhoff theorem says this will lead to Chaos

$\boxed{\begin{array}{l}\text{Perturbative integrability breaks down at} \\ \text{fixed points } v_{particle} = v_{ph} \text{ because} \\ x_{n+1} = x_n,\ v_{n+1} = v_n \text{ gets us nowhere}\end{array}}$

b) $H = \frac{p^2}{2m} + q\Phi(x)$

Expand around fixed point $x = \omega/k\, t$

$H_{eff} = \frac{p^2}{2m} + q\left[\Phi(\omega/k\, t) + \frac{1}{2}\Phi''(\omega/k\, t)(x - \omega/k\, t)^2\right]$

$H_{eff} \approx \frac{p^2}{2m} + q\Phi_0\left[1 - \frac{k^2}{2}(x - \omega/k\, t)^2\right]\cos(0)$

$$\boxed{\tilde{H}_{eff} = \frac{\tilde{p}^2}{2m} + q\Phi_0 - \frac{1}{2}q\Phi_0 k^2 (x - \omega/k\, t)^2}$$

like a

$H(t=0) = q\Phi_0 + \frac{\tilde{p}^2}{2m} - \frac{1}{2}q\Phi_0 k^2 x^2$ Harmonic oscillator

$\cancel{x_{max} = 1/k \approx \frac{1}{2}(\text{island width})} \quad \cancel{k = \frac{2\pi n}{L}}$

$$\omega_1^2 = \frac{q\Phi_0 k^2}{m}$$

Island width

$$\boxed{\frac{\omega}{k} = \Delta v = \sqrt{\frac{q\Phi_0}{m}}}$$

d) Liouville equation $f = f(t, x, v)$

$\frac{df}{dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \dot{v}\frac{\partial f}{\partial v} = 0 \qquad \dot{v} = \frac{qE}{m}$

$$\boxed{\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial v} = 0}$$

e) $\boxed{\text{Average Liouville over velocities}}$ $\langle\ \rangle = \int \frac{dv}{(v_f - v_i)}$

$$\left\langle \frac{\partial f}{\partial t} \right\rangle = -\left\langle \cancel{v \frac{\partial f}{\partial x}}^{\text{odd}} \right\rangle - \left\langle \frac{qE}{m} \frac{\partial f}{\partial v} \right\rangle$$

Let $f = \langle f \rangle + \delta f$

(const compared to δf)

$$\left\langle \frac{\partial \langle f \rangle}{\partial t} \right\rangle + \left\langle \cancel{\frac{\partial}{\partial t} \delta f} \right\rangle = -\left\langle \frac{qE}{m} \left[\cancel{\frac{\partial}{\partial v} \langle f \rangle} + \frac{\partial}{\partial v} \delta f \right] \right\rangle$$

no time dependence

$$\langle\langle f \rangle\rangle = \langle f \rangle$$

$$\boxed{\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v} \left\langle \frac{qE}{m} \delta f \right\rangle}$$

Linear response truncation

$$\delta f = \tilde{\delta f} + \langle \delta f \rangle$$

Use ansatz $\tilde{\delta f} = -\pi \delta(\Omega - m\omega)\, m\tilde{H} \frac{\partial}{\partial v} \langle f \rangle$

$$\Omega = \omega_n \qquad m\omega = k_n v \qquad m\tilde{H} = \frac{E_n q}{m}$$

$$\tilde{\delta f} = -\pi \frac{q}{m} \left(\sum_n E_n \delta(\omega_n - k_n v) \right) \frac{\partial}{\partial v} \langle f \rangle$$

Plug in linear response

$$\boxed{\frac{\partial}{\partial t} \langle f \rangle = \frac{\partial}{\partial v} \underbrace{\left\langle \left(\frac{q}{m}\right)^2 \pi \left(\sum E_n^2 \delta(\omega_n - kv) \right) \right.}_{D} \frac{\partial}{\partial v} \langle F \rangle}$$

"diffusion

Kinetic energy ~ $\langle v^2 \rangle$

$$\langle v^2 \rangle = \int dv\, v^2 f$$

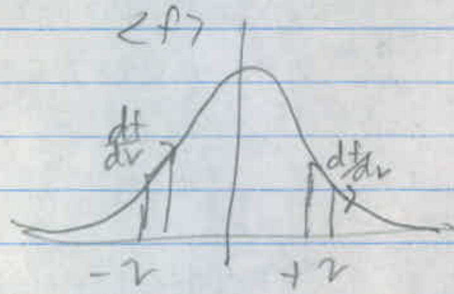
$$\frac{d\langle v^2\rangle}{dt} \sim \int dv\, v^2 \frac{\partial \langle f\rangle}{\partial t}$$

$$= \int dv\, v^2 \left[\frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}\right]$$

Integrate by parts. Surface terms $\to 0$

$$\frac{d\langle v^2\rangle}{dt} = -2\int dv\, D\, v \frac{\partial \langle f\rangle}{\partial v}$$

Assume $\langle f \rangle$ is centered around $v=0$



$$v \frac{\partial \langle f \rangle}{\partial v} < 0$$
for all v

$$\therefore \boxed{\frac{d\langle v^2\rangle}{dt} > 0}$$