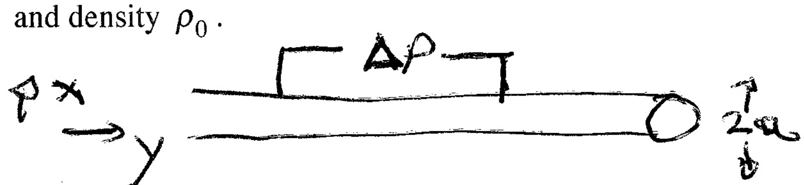
Problem Set II: Due Thursday, June 5, 2014 by 5:00 PM – NO LATE HOMEWORK

1.) *Play it again, Prandtl*

A pipe of radius a, with smooth inner surface, has fluid flowing steadily thru it with pressure drop per unit length Δp. Assume the fluid has kinematic viscosity ν and density ρ_0.



a.) Give the scaling of a 'typical' flow speed in both the high and low Reynolds number limit.

b.) Keeping in mind the no-slip boundary condition, state qualitatively what the mechanism for drag on the pipe flow is, in the limit of $Re >> 1$. Estimate the flux of momentum to the wall.

c.) Calculate the mean flow profile $v_y(x)$ very close to the wall. Over what distance is ν relevant?

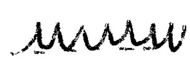
d.) Now, try to model the structure of the mean flow profile $v_y(x)$ for $x >$ viscous layer. Do this by balancing the momentum flux – related to Δp - with diffusive transport of the Fickian form:

$$\Pi_{y,x} = -\nu_T \, \partial V_y / \partial x$$

Here v_T is a 'turbulent viscosity' or 'eddy viscosity'. Note here the ubiquitous idea of modeling turbulent mixing by ideas from kinetic theory. To model v_T, you need a velocity and a length. For the velocity, use the 'typical velocity' from part a.). For the length, use the distance from the wall, as Prandtl did. What is $v_y(x)$? Discuss your result.

e.) Make a table which compares and contrasts this pipe flow problem with K41 theory for homogeneous turbulence.

f.) Now assume the pipe's interior surface is *rough*, with roughness scale ℓ_0

i.e. _____________ $\rightarrow$  ℓ_0

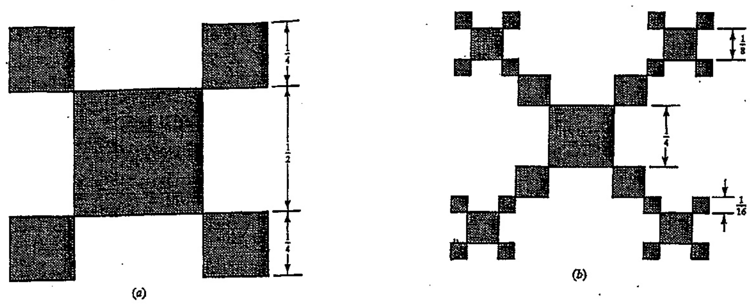
What size must ℓ_0 be in order to change the story of parts a.) $\rightarrow$ e.)? What will be the impact of ℓ_0?

2.) *Time*

Estimate the scaling of the *time* required for energy in 3D turbulence for cascade from the input scale to the dissipation scale. Use the K41 model. What is the implication of your result?

3.) *Fractal Fest*

a.) What is the box counting dimension of the "cake-cutting" fractal, sketched below?



b.) How does the length of the Koch curve depend upon the resolution scale ℓ_d?

c.) Computer simulations often employ a "hyper-viscosity" μ to increase the effective size of the inertial range. A hyper-viscosity works by the substitution

$$\nu \nabla^2 \to -\mu (\nabla^2)^2$$

Estimate the dissipation scale for hyper-viscous incompressible fluid turbulence. How does the number of degrees-of-freedom scale with "Reynolds number" for hyper-viscous fluids? [N.B. Here, note that the definition of Reynolds number changes for hyper-viscosity.]

4.) *Burgulence*

Burgulence refers to 1D turbulence, described by Burgers equation

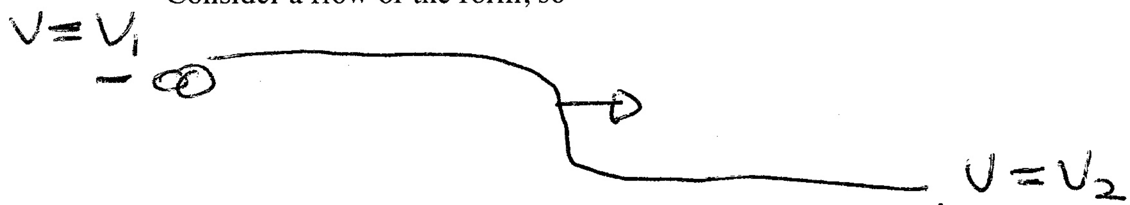
$$\partial V/\partial t + V\partial V/\partial x = \nu\, \partial^2 V/\partial x^2 + f_{ext}.$$

Here f is an external force.

→ Take $f_{ext} = 0$, $\nu = 0$ so

$$\partial_t V + V \partial_x V = 0$$

Consider a flow of the form, so



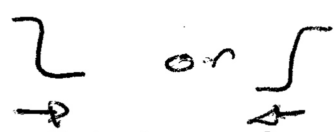
that $v = f(x - ct + x_0)$. This profile is that of a shock.

a.) Why are shock solutions the ones of relevance? Argue your answer without resorting to the full crank.

b.) Use the behavior at ($\pm\infty$) to determine the shock speed C.

c.) Now, restore viscosity ($\nu \pm 0$), and obtain an *exact* solution. See Barenblatt ("Scaling") or Whitham ("Linear and Nonlinear Waves") for guidance. Estimate the width of the shock. [N.B. This part requires a calculation.]

d.) Use a.) – c.) and any other analysis you can employ to argue that the long time state of Burgulence for a given initial condition will be one of a mix of

ramps  and shocks 

Where will shocks originate from? Where does dissipation occur?

e.) Given d.), what is the fractal dimension of the dissipative structure for Burgulence? What does the beta model predict for the spectrum of such "Burgulence"? Is K41 thinking relevant here?

f.) Show that the spectrum predicted by the beta model can be obtained by assuming Burgulence is in essence, a superposition of uncorrelated shocks. [Hint: This is a 3-liner. Start by writing the velocity gradient as a sum of uncorrelated delta functions.]

5.) Short Problems

a.) Estimate the extent of the turbulence wake of a super-tanker. Your answer should include:

1.) what assumptions you made about the ship.

2.) the length, width and fluid velocity in the wake.

b.) The velocity of fluid flow in a porous medium is related to the pressure drop-per-length $\Delta p/\ell$ by

$$V = k \Delta P / \eta \ell$$

Here η is the viscosity and k is the permeability constant. Calculate the scaling of the mass flow rate through a cross-section S of a porous medium. How does the scaling of this flow rate compare to the scaling of that for viscous flow thru a pipe of cross section area S?

6.) Propose a problem related to the material in this mini-course.