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Chapter 1

Phenomenological Theories of Superconductivity

1.1 Basic Phenomenology of Superconductors

The superconducting state is a phase of matter, as is ferromagnetism, metallicity, *etc.* The phenomenon was discovered in the Spring of 1911 by the Dutch physicist H. Kamerlingh Onnes, who observed an abrupt vanishing of the resistivity of solid mercury at $T = 4.15\,\mathrm{K}$[1]. Under ambient pressure, there are 33 elemental superconductors[2], all of which have a metallic phase at higher temperatures, and hundreds of compounds and alloys which exhibit the phenomenon. A timeline of superconductors and their critical temperatures is provided in Fig. 1.1. The related phenomenon of superfluidity was first discovered in liquid helium below $T = 2.17\,\mathrm{K}$, at atmospheric pressure, independently in 1937 by P. Kapitza (Moscow) and by J. F. Allen and A. D. Misener (Cambridge). At some level, a superconductor may be considered as a charged superfluid – we will elaborate on this statement later on. Here we recite the basic phenomenology of superconductors:

- *Vanishing electrical resistance* : The DC electrical resistance at zero magnetic field vanishes in the superconducting state. This is established in some materials to better than one part in 10^{15} of the normal state resistance. Above the critical temperature T_c, the DC resistivity at $H = 0$ is finite. The AC resistivity remains zero up to a critical frequency, $\omega_c = 2\Delta/\hbar$, where Δ is the gap in the electronic excitation spectrum. The frequency threshold is 2Δ because the superconducting condensate is made up of electron *pairs*, so breaking a pair results in two *quasiparticles*, each with energy Δ or greater. For *weak coupling* superconductors, which are described by the famous BCS theory (1957), there is a relation between the gap energy and the superconducting transition temperature, $2\Delta_0 = 3.5\,k_B T_c$, which we derive when we study the BCS model. The gap $\Delta(T)$ is temperature-dependent and vanishes at T_c.

- *Flux expulsion* : In 1933 it was descovered by Meissner and Ochsenfeld that magnetic fields in superconducting tin and lead to not penetrate into the bulk of a superconductor, but rather are confined to a surface layer of thickness λ, called the *London penetration depth*. Typically λ in on the scale of tens to hundreds of nanometers.

  It is important to appreciate the difference between a superconductor and a perfect metal. If we set $\sigma = \infty$ then from $\boldsymbol{j} = \sigma\boldsymbol{E}$ we must have $\boldsymbol{E} = 0$, hence Faraday's law $\boldsymbol{\nabla} \times \boldsymbol{E} = -c^{-1}\partial_t \boldsymbol{B}$ yields $\partial_t \boldsymbol{B} = 0$, which

[1] Coincidentally, this just below the temperature at which helium liquefies under atmospheric pressure.
[2] An additional 23 elements are superconducting under high pressure.

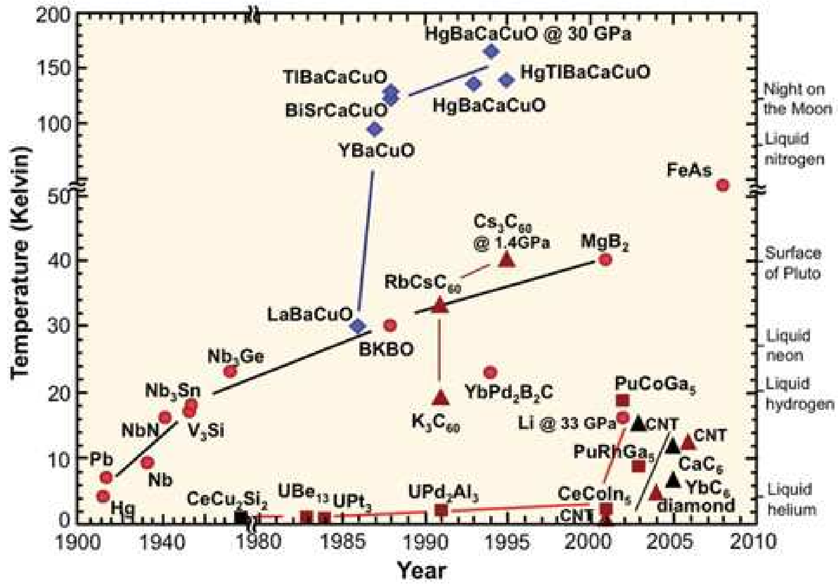

Figure 1.1: Timeline of superconductors and their transition temperatures (from Wikipedia).

says that $\boldsymbol{B}$ remains *constant* in a perfect metal. Yet Meissner and Ochsenfeld found that below T_c the flux was *expelled* from the bulk of the superconductor. If, however, the superconducting sample is not simply connected, *i.e.* if it has holes, such as in the case of a superconducting ring, then in the Meissner phase flux may be trapped in the holes. Such trapped flux is quantized in integer units of the superconducting fluxoid $\phi_L = hc/2e = 2.07 \times 10^{-7}\,\mathrm{G\,cm^2}$ (see Fig. 1.2).

- *Critical field(s)* : The Meissner state exists for $T < T_c$ only when the applied magnetic field H is smaller than the *critical field* $H_c(T)$, with

$$H_c(T) \simeq H_c(0)\left(1 - \frac{T^2}{T_c^2}\right) \quad . \tag{1.1}$$

In so-called type-I superconductors, the system goes normal[3] for $H > H_c(T)$. For most elemental type-I materials (*e.g.*, Hg, Pb, Nb, Sn) one has $H_c(0) \leq 1\,\mathrm{kG}$. In type-II materials, there are two critical fields, $H_{c1}(T)$ and $H_{c2}(T)$. For $H < H_{c1}$, we have flux expulsion, and the system is in the Meissner phase. For $H > H_{c2}$, we have uniform flux penetration and the system is normal. For $H_{c1} < H < H_{c2}$, the system in a *mixed state* in which quantized vortices of flux ϕ_L penetrate the system (see Fig. 1.3). There is a depletion of what we shall describe as the superconducting order parameter $\Psi(\boldsymbol{r})$ in the vortex cores over a length scale ξ, which is the *coherence length* of the superconductor. The upper critical field is set by the condition that the vortex cores start to overlap: $H_{c2} = \phi_L/2\pi\xi^2$. The vortex cores can be pinned by disorder. Vortices also interact with each other out to a distance λ, and at low temperatures in the absence of disorder the vortices order into a (typically triangular) *Abrikosov vortex lattice* (see Fig. 1.4). Typically one has $H_{c2} = \sqrt{2}\kappa\, H_{c1}$, where $\kappa = \lambda/\xi$ is a ratio of the two fundamental length scales. Type-II materials exist when $H_{c2} > H_{c1}$, *i.e.* when $\kappa > \frac{1}{\sqrt{2}}$. Type-II behavior tends to occur in superconducting alloys, such as Nb-Sn.

- *Persistent currents* : We have already mentioned that a metallic ring in the presence of an external magnetic field may enclosed a quantized trapped flux $n\phi_L$ when cooled below its superconducting transition temperature. If the field is now decreased to zero, the trapped flux remains, and is generated by a *persistent current* which flows around the ring. In thick rings, such currents have been demonstrated to exist undiminished for years, and may be stable for astronomically long times.

[3]Here and henceforth, "normal" is an abbreviation for "normal metal".

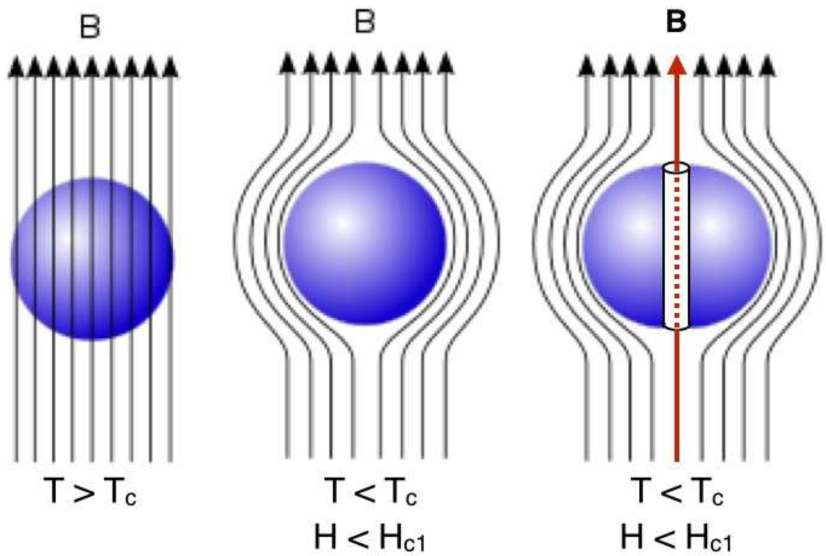

Figure 1.2: Flux expulsion from a superconductor in the Meissner state. In the right panel, quantized trapped flux penetrates a hole in the sample.

- *Specific heat jump* : The heat capacity of metals behaves as $c_V \equiv C_V/V = \frac{\pi^2}{3}\, k_B^2 T g(\varepsilon_F)$, where $g(\varepsilon_F)$ is the density of states at the Fermi level. In a superconductor, once one subtracts the low temperature phonon contribution $c_V^{\text{phonon}} = AT^3$, one is left for $T < T_c$ with an electronic contribution behaving as $c_V^{\text{elec}} \propto e^{-\Delta/k_B T}$. There is also a jump in the specific heat at $T = T_c$, the magnitude of which is generally about three times the normal specific heat just above T_c. This jump is consistent with a second order transition with critical exponent $\alpha = 0$.
- *Tunneling and Josephson effect* : The energy gap in superconductors can be measured by electron tunneling between a superconductor and a normal metal, or between two superconductors separated by an insulating layer. In the case of a weak link between two superconductors, current can flow at zero bias voltage, a situation known as the *Josephson effect*.

1.2 Thermodynamics of Superconductors

The differential free energy density of a magnetic material is given by

$$df = -s\,dT + \frac{1}{4\pi}\,\boldsymbol{H}\cdot d\boldsymbol{B} \quad , \tag{1.2}$$

which says that $f = f(T, \boldsymbol{B})$. Here s is the entropy density, and $\boldsymbol{B}$ the magnetic field. The quantity $\boldsymbol{H}$ is called the *magnetizing field* and is thermodynamically conjugate to $\boldsymbol{B}$:

$$s = -\left(\frac{\partial f}{\partial T}\right)_{\boldsymbol{B}} \quad , \quad \boldsymbol{H} = 4\pi \left(\frac{\partial f}{\partial \boldsymbol{B}}\right)_T \quad . \tag{1.3}$$

In the Ampère-Maxwell equation, $\nabla \times \boldsymbol{H} = 4\pi c^{-1} \boldsymbol{j}_{\text{ext}} + c^{-1}\partial_t \boldsymbol{D}$, the sources of $\boldsymbol{H}$ appear on the RHS[4]. Usually $c^{-1}\partial_t\boldsymbol{D}$ is negligible, in which case $\boldsymbol{H}$ is generated by external sources such as magnetic solenoids. The magnetic

[4]Throughout these notes, RHS/LHS will be used to abbreviate "right/left hand side".

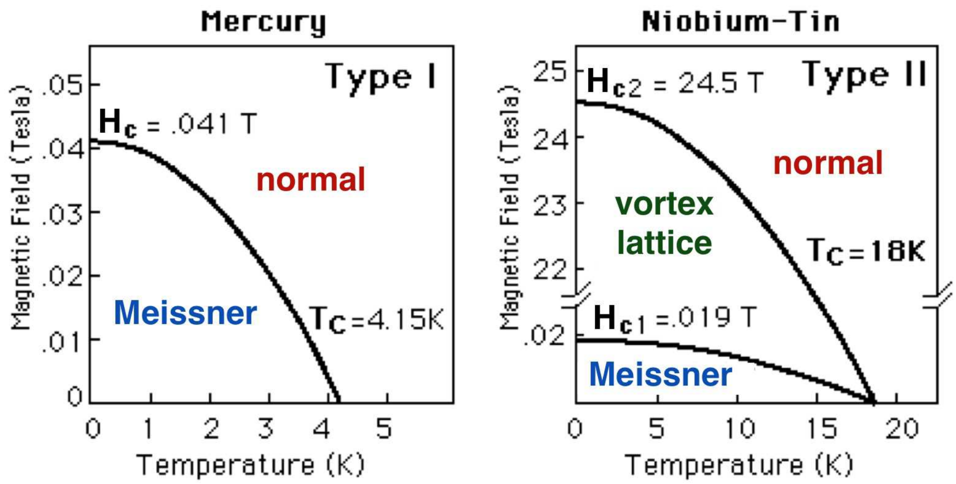

Figure 1.3: Phase diagrams for type I and type II superconductors in the (T, H) plane.

field $\boldsymbol{B}$ is given by $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M} \equiv \mu\boldsymbol{H}$, where $\boldsymbol{M}$ is the magnetization density. We therefore have no direct control over $\boldsymbol{B}$, and it is necessary to discuss the thermodynamics in terms of the Gibbs free energy density, $g(T, \boldsymbol{H})$:

$$
\begin{aligned}
g(T, \boldsymbol{H}) &= f(T, \boldsymbol{B}) - \frac{1}{4\pi}\boldsymbol{B}\cdot\boldsymbol{H} \\
dg &= -s\,dT - \frac{1}{4\pi}\,\boldsymbol{B}\cdot d\boldsymbol{H} \quad .
\end{aligned} \tag{1.4}
$$

Thus,

$$
s = -\left(\frac{\partial g}{\partial T}\right)_{\boldsymbol{H}} \quad , \quad \boldsymbol{B} = -4\pi\left(\frac{\partial g}{\partial \boldsymbol{H}}\right)_T \quad . \tag{1.5}
$$

Assuming a bulk sample which is isotropic, we then have

$$
g(T, H) = g(T, 0) - \frac{1}{4\pi}\int_0^H dH'\, B(H') \quad . \tag{1.6}
$$

In a normal metal, $\mu \approx 1$ (cgs units), which means $B \approx H$, which yields

$$
g_n(T, H) = g_n(T, 0) - \frac{H^2}{8\pi} \quad . \tag{1.7}
$$

In the Meissner phase of a superconductor, $B = 0$, so

$$
g_s(T, H) = g_s(T, 0) \quad . \tag{1.8}
$$

For a type-I material, the free energies cross at $H = H_c$, so

$$
g_s(T, 0) = g_n(T, 0) - \frac{H_c^2}{8\pi} \quad , \tag{1.9}
$$

which says that there is a negative *condensation energy density* $-\frac{H_c^2(T)}{8\pi}$ which stabilizes the superconducting phase. We call H_c the *thermodynamic critical field*. We may now write

$$
g_s(T, H) - g_n(T, H) = \frac{1}{8\pi}\Big(H^2 - H_c^2(T)\Big) \quad , \tag{1.10}
$$

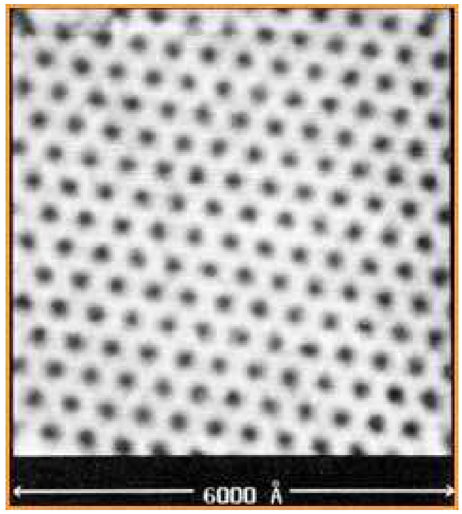

Figure 1.4: STM image of a vortex lattice in $NbSe_2$ at $H = 1\,$T and $T = 1.8\,$K. From H. F. Hess *et al.*, *Phys. Rev. Lett.* **62**, 214 (1989).

so the superconductor is the equilibrium state for $H < H_c$. Taking the derivative with respect to temperature, the entropy difference is given by

$$s_s(T,H) - s_n(T,H) = \frac{1}{4\pi}\,H_c(T)\,\frac{dH_c(T)}{dT} < 0 \quad , \tag{1.11}$$

since $H_c(T)$ is a decreasing function of temperature. Note that the entropy difference is independent of the external magnetizing field H. As we see from Fig. 1.3, the derivative $H_c'(T)$ changes discontinuously at $T = T_c$. The latent heat $\ell = T\,\Delta s$ vanishes because $H_c(T_c)$ itself vanishes, but the specific heat is discontinuous:

$$c_s(T_c, H=0) - c_n(T_c, H=0) = \frac{T_c}{4\pi}\left(\frac{dH_c(T)}{dT}\right)^2_{T_c} \quad , \tag{1.12}$$

and from the phenomenological relation of Eqn. 1.1, we have $H_c'(T_c) = -2H_c(0)/T_c$, hence

$$\Delta c \equiv c_s(T_c, H=0) - c_n(T_c, H=0) = \frac{H_c^2(0)}{\pi T_c} \quad . \tag{1.13}$$

We can appeal to Eqn. 1.11 to compute the difference $\Delta c(T,H)$ for general $T < T_c$:

$$\Delta c(T,H) = \frac{T}{8\pi}\,\frac{d^2}{dT^2}\,H_c^2(T) \quad . \tag{1.14}$$

With the approximation of Eqn. 1.1, we obtain

$$c_s(T,H) - c_n(T,H) \simeq \frac{T H_c^2(0)}{2\pi T_c^2}\left\{3\left(\frac{T}{T_c}\right)^2 - 1\right\} \quad . \tag{1.15}$$

In the limit $T \to 0$, we expect $c_s(T)$ to vanish exponentially as $e^{-\Delta/k_B T}$, hence we have $\Delta c(T \to 0) = -\gamma T$, where γ is the coefficient of the linear T term in the metallic specific heat. Thus, we expect $\gamma \simeq H_c^2(0)/2\pi T_c^2$. Note also

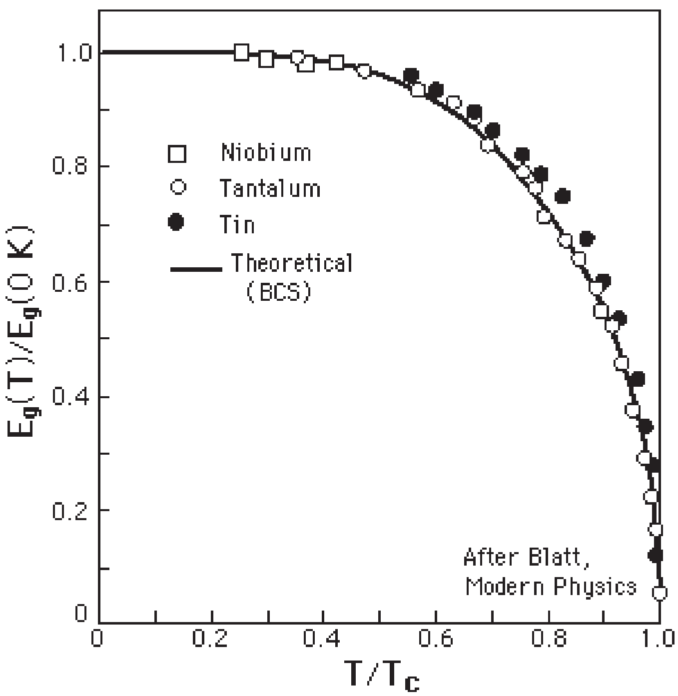

Figure 1.5: Dimensionless energy gap $\Delta(T)/\Delta_0$ in niobium, tantalum, and tin. The solid curve is the prediction from BCS theory, derived in chapter 3 below.

that this also predicts the ratio $\Delta c(T_c,0)/c_n(T_c,0) = 2$. In fact, within BCS theory, as we shall later show, this ratio is approximately 1.43. BCS also yields the low temperature form

$$H_c(T) = H_c(0)\left\{1 - \alpha\left(\frac{T}{T_c}\right)^2 + \mathcal{O}\big(e^{-\Delta/k_B T}\big)\right\} \tag{1.16}$$

with $\alpha \simeq 1.07$. Thus, $H_c^{\rm BCS}(0) = \big(2\pi\gamma T_c^2/\alpha\big)^{1/2}$.

1.3 London Theory

Fritz and Heinz London in 1935 proposed a two fluid model for the macroscopic behavior of superconductors. The two fluids are: (i) the normal fluid, with electron number density n_n, which has finite resistivity, and (ii) the superfluid, with electron number density n_s, and which moves with zero resistance. The associated velocities are v_n and v_s, respectively. Thus, the total number density and current density are

$$\begin{aligned} n &= n_n + n_s \\ \boldsymbol{j} &= \boldsymbol{j}_n + \boldsymbol{j}_s = -e\big(n_n \boldsymbol{v}_n + n_s \boldsymbol{v}_s\big) \quad . \end{aligned} \tag{1.17}$$

The normal fluid is dissipative, hence $\boldsymbol{j}_n = \sigma_n \boldsymbol{E}$, but the superfluid obeys $F = m\boldsymbol{a}$, *i.e.*

$$m\,\frac{d\boldsymbol{v}_s}{dt} = -e\boldsymbol{E} \quad \Rightarrow \quad \frac{d\boldsymbol{j}_s}{dt} = \frac{n_s e^2}{m}\,\boldsymbol{E} \quad . \tag{1.18}$$

In the presence of an external magnetic field, the superflow satisfies

$$\begin{aligned}\frac{d\boldsymbol{v}_s}{dt} &= -\frac{e}{m}\big(\boldsymbol{E} + c^{-1}\boldsymbol{v}_s \times \boldsymbol{B}\big) \\ &= \frac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{v}_s \cdot \boldsymbol{\nabla})\boldsymbol{v}_s = \frac{\partial \boldsymbol{v}_s}{\partial t} + \boldsymbol{\nabla}\big(\tfrac{1}{2}\boldsymbol{v}_s^2\big) - \boldsymbol{v}_s \times (\boldsymbol{\nabla} \times \boldsymbol{v}_s) \quad .\end{aligned} \tag{1.19}$$

We then have

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + \frac{e}{m}\,\boldsymbol{E} + \boldsymbol{\nabla}\big(\tfrac{1}{2}\boldsymbol{v}_s^2\big) = \boldsymbol{v}_s \times \left(\boldsymbol{\nabla} \times \boldsymbol{v}_s - \frac{e\boldsymbol{B}}{mc}\right) \quad . \tag{1.20}$$

Taking the curl, and invoking Faraday's law $\boldsymbol{\nabla} \times \boldsymbol{E} = -c^{-1}\partial_t \boldsymbol{B}$, we obtain

$$\frac{\partial}{\partial t}\left(\boldsymbol{\nabla} \times \boldsymbol{v}_s - \frac{e\boldsymbol{B}}{mc}\right) = \boldsymbol{\nabla} \times \left\{\boldsymbol{v}_s \times \left(\boldsymbol{\nabla} \times \boldsymbol{v}_s - \frac{e\boldsymbol{B}}{mc}\right)\right\} \quad , \tag{1.21}$$

which may be written as

$$\frac{\partial \boldsymbol{Q}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v}_s \times \boldsymbol{Q}) \quad , \tag{1.22}$$

where

$$\boldsymbol{Q} \equiv \boldsymbol{\nabla} \times \boldsymbol{v}_s - \frac{e\boldsymbol{B}}{mc} \quad . \tag{1.23}$$

Eqn. 1.22 says that if $\boldsymbol{Q} = 0$, it remains zero for all time. Assumption: the equilibrium state has $\boldsymbol{Q} = 0$. Thus,

$$\boldsymbol{\nabla} \times \boldsymbol{v}_s = \frac{e\boldsymbol{B}}{mc} \quad \Rightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{j}_s = -\frac{n_s e^2}{mc}\,\boldsymbol{B} \quad . \tag{1.24}$$

This equation implies the Meissner effect, for upon taking the curl of the last of Maxwell's equations (and assuming a steady state so $\dot{\boldsymbol{E}} = \dot{\boldsymbol{D}} = 0$),

$$-\nabla^2 \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \frac{4\pi}{c}\,\boldsymbol{\nabla} \times \boldsymbol{j} = -\frac{4\pi n_s e^2}{mc^2}\,\boldsymbol{B} \quad \Rightarrow \quad \nabla^2 \boldsymbol{B} = \lambda_L^{-2}\boldsymbol{B} \quad , \tag{1.25}$$

where $\lambda_L = \sqrt{mc^2/4\pi n_s e^2}$ is the *London penetration depth*. The magnetic field can only penetrate up to a distance on the order of λ_L inside the superconductor.

Note that

$$\boldsymbol{\nabla} \times \boldsymbol{j}_s = -\frac{c}{4\pi\lambda_L^2}\,\boldsymbol{B} \tag{1.26}$$

and the definition $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ licenses us to write

$$\boldsymbol{j}_s = -\frac{c}{4\pi\lambda_L^2}\,\boldsymbol{A} \quad , \tag{1.27}$$

provided an appropriate gauge choice for $\boldsymbol{A}$ is taken. Since $\boldsymbol{\nabla} \cdot \boldsymbol{j}_s = 0$ in steady state, we conclude $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ is the proper gauge. This is called the Coulomb gauge. Note, however, that this still allows for the little gauge transformation $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi$, provided $\nabla^2\chi = 0$. Consider now an isolated body which is simply connected, *i.e.* any closed loop drawn within the body is continuously contractable to a point. The normal component of the superfluid at the boundary, $\boldsymbol{J}_{s,\perp}$ must vanish, hence $\boldsymbol{A}_\perp = 0$ as well. Therefore $\nabla_\perp \chi$ must also vanish everywhere on the boundary, which says that χ is determined up to a global constant.

If the superconductor is multiply connected, though, the condition $\nabla_\perp \chi = 0$ allows for non-constant solutions for χ. The line integral of $\boldsymbol{A}$ around a closed loop surrounding a hole $\mathcal{D}$ in the superconductor is, by Stokes' theorem, the magnetic flux through the loop:

$$\oint\limits_{\partial\mathcal{D}} d\boldsymbol{l} \cdot \boldsymbol{A} = \int\limits_{\mathcal{D}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{B} = \Phi_{\mathcal{D}} \quad . \tag{1.28}$$

On the other hand, within the interior of the superconductor, since $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = 0$, we can write $\boldsymbol{A} = \boldsymbol{\nabla}\chi$, which says that the trapped flux $\Phi_{\mathcal{D}}$ is given by $\Phi_{\mathcal{D}} = \Delta\chi$, then change in the gauge function as one proceeds counterclockwise around the loop. F. London argued that if the gauge transformation $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi$ is associated with a quantum mechanical wavefunction associated with a charge e object, then the flux $\Phi_{\mathcal{D}}$ will be quantized in units of the Dirac quantum $\phi_0 = hc/e = 4.137 \times 10^{-7}\,\mathrm{G\,cm}^2$. The argument is simple. The transformation of the wavefunction $\Psi \to \Psi\, e^{-i\alpha}$ is cancelled by the replacement $\boldsymbol{A} \to \boldsymbol{A} + (\hbar c/e)\boldsymbol{\nabla}\alpha$. Thus, we have $\chi = \alpha\phi_0/2\pi$, and single-valuedness requires $\Delta\alpha = 2\pi n$ around a loop, hence $\Phi_{\mathcal{D}} = \Delta\chi = n\phi_0$.

The above argument is almost correct. The final piece was put in place by Lars Onsager in 1953. Onsager pointed out that if the particles described by the superconducting wavefunction Ψ were of charge $e^* = 2e$, then, *mutatis mutandis*, one would conclude the quantization condition is $\Phi_{\mathcal{D}} = n\phi_{\mathrm{L}}$, where $\phi_{\mathrm{L}} = hc/2e$ is the London flux quantum, which is half the size of the Dirac flux quantum. This suggestion was confirmed in subsequent experiments by Deaver and Fairbank, and by Doll and Näbauer, both in 1961.

De Gennes' derivation of London Theory

De Gennes writes the total free energy of the superconductor as

$$\begin{aligned} F &= \int\! d^3\!x\; f_0 + E_{\mathrm{kinetic}} + E_{\mathrm{field}} \\ E_{\mathrm{kinetic}} &= \int\! d^3\!x\; \tfrac{1}{2} m n_{\mathrm{s}} \boldsymbol{v}_{\mathrm{s}}^2(\boldsymbol{x}) = \int\! d^3\!x\; \frac{m}{2n_{\mathrm{s}}e^2}\, \boldsymbol{j}_{\mathrm{s}}^2(\boldsymbol{x}) \\ E_{\mathrm{field}} &= \int\! d^3\!x\; \frac{\boldsymbol{B}^2(\boldsymbol{x})}{8\pi} \quad . \end{aligned} \tag{1.29}$$

Here f_0 is the free energy density of the *metallic* state, in which no currents flow. What makes this a model of a superconductor is the assumption that a current $\boldsymbol{j}_{\mathrm{s}}$ flows in the presence of a magnetic field. Thus, under steady state conditions $\boldsymbol{\nabla} \times \boldsymbol{B} = 4\pi c^{-1} \boldsymbol{j}_{\mathrm{s}}$, so

$$F = \int\! d^3\!x \left\{ f_0 + \frac{\boldsymbol{B}^2}{8\pi} + \lambda_{\mathrm{L}}^2\, \frac{(\boldsymbol{\nabla} \times \boldsymbol{B})^2}{8\pi} \right\} \quad . \tag{1.30}$$

Taking the functional variation and setting it to zero,

$$4\pi\, \frac{\delta F}{\delta \boldsymbol{B}} = \boldsymbol{B} + \lambda_{\mathrm{L}}^2\, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{B} - \lambda_{\mathrm{L}}^2\, \nabla^2 \boldsymbol{B} = 0 \quad . \tag{1.31}$$

Pippard's nonlocal extension

The London equation $\boldsymbol{j}_{\mathrm{s}}(\boldsymbol{x}) = -c\boldsymbol{A}(\boldsymbol{x})/4\pi\lambda_{\mathrm{L}}^2$ says that the supercurrent is perfectly yoked to the vector potential, and on arbitrarily small length scales. This is unrealistic. A. B. Pippard undertook a phenomenological generalization of the (phenomenological) London equation, writing[5]

$$\begin{aligned} j_{\mathrm{s}}^{\alpha}(\boldsymbol{x}) &= -\frac{c}{4\pi\lambda_{\mathrm{L}}^2} \int\! d^3\!r\; K^{\alpha\beta}(\boldsymbol{r})\, A_\beta(\boldsymbol{x} + \boldsymbol{r}) \\ &= -\frac{c}{4\pi\lambda_{\mathrm{L}}^2} \cdot \frac{3}{4\pi\xi} \int\! d^3\!r\; \frac{e^{-r/\xi}}{r^2}\, \hat{r}^{\alpha}\, \hat{r}^{\beta}\, A_\beta(\boldsymbol{x} + \boldsymbol{r}) \quad . \end{aligned} \tag{1.32}$$

[5]See A. B. Pippard, *Proc. Roy. Soc. Lond.* **A216**, 547 (1953).

Note that the kernel $K^{\alpha\beta}(\boldsymbol{r}) = 3\, e^{-r/\xi}\, \hat{r}^\alpha \hat{r}^\beta / 4\pi\xi r^2$ is normalized so that

$$\int\! d^3\!r\; K^{\alpha\beta}(\boldsymbol{r}) = \frac{3}{4\pi\xi} \int\! d^3\!r\; \frac{e^{-r/\xi}}{r^2}\, \hat{r}^\alpha\, \hat{r}^\beta \;= \overbrace{\frac{1}{\xi}\int\limits_0^\infty \! dr\; e^{-r/\xi}}^{1} \cdot \overbrace{3\!\int\! \frac{d\hat{\boldsymbol{r}}}{4\pi}\, \hat{r}^\alpha\, \hat{r}^\beta}^{\delta^{\alpha\beta}} = \delta^{\alpha\beta} \quad . \tag{1.33}$$

The exponential factor means that $K^{\alpha\beta}(\boldsymbol{r})$ is negligible for $r \gg \xi$. If the vector potential is constant on the scale ξ, then we may pull $A_\beta(\boldsymbol{x})$ out of the integral in Eqn. 1.33, in which case we recover the original London equation. Invoking continuity in the steady state, $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{j} = 0$ requires

$$\frac{3}{4\pi\xi^2} \int\! d^3\!r\; \frac{e^{-r/\xi}}{r^2}\, \hat{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{x}+\boldsymbol{r}) = 0 \quad , \tag{1.34}$$

which is to be regarded as a gauge condition on the vector potential. One can show that this condition is equivalent to $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A} = 0$, the original Coulomb gauge.

In disordered superconductors, Pippard took

$$K^{\alpha\beta}(\boldsymbol{r}) = \frac{3}{4\pi\xi_0}\, \frac{e^{-r/\xi}}{r^2}\, \hat{r}^\alpha\, \hat{r}^\beta \quad , \tag{1.35}$$

with

$$\frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{a\,\ell} \quad , \tag{1.36}$$

where ℓ is the metallic elastic mean free path, and a is a dimensionless constant on the order of unity. Note that $\int\! d^3\!r\; K^{\alpha\beta}(\boldsymbol{r}) = (\xi/\xi_0)\, \delta^{\alpha\beta}$. Thus, for $\lambda_{\rm L} \gg \xi$, one obtains an effective penetration depth $\lambda = (\xi_0/\xi)^{1/2}\lambda_{\rm L}$, where $\lambda_{\rm L} = \sqrt{mc^2/4\pi n_{\rm s} e^2}$. In the opposite limit, where $\lambda_{\rm L} \ll \xi$, Pippard found $\lambda = (3/4\pi^2)^{1/6}\big(\xi_0 \lambda_{\rm L}^2\big)^{1/3}$. For strongly type-I superconductors, $\xi \gg \lambda_{\rm L}$. Since $\boldsymbol{j}_{\rm s}(\boldsymbol{x})$ is averaging the vector potential over a region of size $\xi \gg \lambda_{\rm L}$, the screening currents near the surface of the superconductor are weaker, which means the magnetic field penetrates deeper than $\lambda_{\rm L}$. The physical penetration depth is λ, where, according to Pippard, $\lambda/\lambda_{\rm L} \propto \big(\xi_0/\lambda_{\rm L}\big)^{1/3} \gg 1$.

1.4 Ginzburg-Landau Theory

The basic idea behind Ginzburg-Landau theory is to write the free energy as a simple functional of the *order parameter(s)* of a thermodynamic system and their derivatives. In ^{4}He, the order parameter $\Psi(\boldsymbol{x}) = \langle \psi(\boldsymbol{x}) \rangle$ is the quantum and thermal average of the field operator $\psi(\boldsymbol{x})$ which destroys a helium atom at position $\boldsymbol{x}$. When Ψ is nonzero, we have Bose condensation with condensate density $n_0 = |\Psi|^2$. Above the lambda transition, one has $n_0(T > T_\lambda) = 0$.

In an s-wave superconductor, the order parameter field is given by

$$\Psi(\boldsymbol{x}) \propto \big\langle \psi_\uparrow(\boldsymbol{x})\, \psi_\downarrow(\boldsymbol{x}) \big\rangle \quad , \tag{1.37}$$

where $\psi_\sigma(\boldsymbol{x})$ destroys a conduction band electron of spin σ at position $\boldsymbol{x}$. Owing to the anticommuting nature of the fermion operators, the fermion field $\psi_\sigma(\boldsymbol{x})$ itself cannot condense, and it is only the *pair field* $\Psi(\boldsymbol{x})$ (and other products involving an even number of fermion field operators) which can take a nonzero value.

1.4.1 Landau theory for superconductors

The superconducting order parameter $\Psi(\boldsymbol{x})$ is thus a complex scalar, as in a superfluid. As we shall see, the difference is that the superconductor is *charged*. In the absence of magnetic fields, the Landau free energy density is approximated as

$$f = a\,|\Psi|^2 + \tfrac{1}{2}b\,|\Psi|^4 \quad . \tag{1.38}$$

The coefficients a and b are real and temperature-dependent but otherwise constant in a spatially homogeneous system. The sign of a is negotiable, but $b > 0$ is necessary for thermodynamic stability. The free energy has an $\mathrm{O}(2)$ symmetry, *i.e.* it is invariant under the substitution $\Psi \to \Psi\, e^{i\alpha}$. For $a < 0$ the free energy is minimized by writing

$$\Psi = \sqrt{-\frac{a}{b}}\; e^{i\phi} \quad , \tag{1.39}$$

where ϕ, the phase of the superconductor, is a constant. The system spontaneously breaks the $\mathrm{O}(2)$ symmetry and chooses a direction in Ψ space in which to point.

In our formulation here, the free energy of the normal state, *i.e.* when $\Psi = 0$, is $f_\mathrm{n} = 0$ at all temperatures, and that of the superconducting state is $f_\mathrm{s} = -a^2/2b$. From thermodynamic considerations, therefore, we have

$$f_\mathrm{s}(T) - f_\mathrm{n}(T) = -\frac{H_\mathrm{c}^2(T)}{8\pi} \quad \Rightarrow \quad \frac{a^2(T)}{b(T)} = \frac{H_\mathrm{c}^2(T)}{4\pi} \quad . \tag{1.40}$$

Furthermore, from London theory we have that $\lambda_\mathrm{L}^2 = mc^2/4\pi n_\mathrm{s} e^2$, and if we normalize the order parameter according to

$$\big|\Psi\big|^2 = \frac{n_\mathrm{s}}{n} \quad , \tag{1.41}$$

where n_s is the number density of superconducting electrons and n the total number density of conduction band electrons, then

$$\frac{\lambda_\mathrm{L}^2(0)}{\lambda_\mathrm{L}^2(T)} = \big|\Psi(T)\big|^2 = -\frac{a(T)}{b(T)} \quad . \tag{1.42}$$

Here we have taken $n_\mathrm{s}(T=0) = n$, so $|\Psi(0)|^2 = 1$. Putting this all together, we find

$$a(T) = -\frac{H_\mathrm{c}^2(T)}{4\pi}\cdot\frac{\lambda_\mathrm{L}^2(T)}{\lambda_\mathrm{L}^2(0)} \qquad , \qquad b(T) = \frac{H_\mathrm{c}^2(T)}{4\pi}\cdot\frac{\lambda_\mathrm{L}^4(T)}{\lambda_\mathrm{L}^4(0)} \tag{1.43}$$

Close to the transition, $H_\mathrm{c}(T)$ vanishes in proportion to $\lambda_\mathrm{L}^{-2}(T)$, so $a(T_\mathrm{c}) = 0$ while $b(T_\mathrm{c}) > 0$ remains finite at T_c. Later on below, we shall relate the penetration depth λ_L to a stiffness parameter in the Ginzburg-Landau theory.

We may now compute the specific heat discontinuity from $c = -T\,\frac{\partial^2 f}{\partial T^2}$. It is left as an exercise to the reader to show

$$\Delta c = c_\mathrm{s}(T_\mathrm{c}) - c_\mathrm{n}(T_\mathrm{c}) = \frac{T_\mathrm{c}\,\big[a'(T_\mathrm{c})\big]^2}{b(T_\mathrm{c})} \quad , \tag{1.44}$$

where $a'(T) = da/dT$. Of course, $c_n(T)$ isn't zero! Rather, here we are accounting only for the specific heat due to that part of the free energy associated with the condensate. The Ginzburg-Landau description completely ignores the metal, and doesn't describe the physics of the normal state Fermi surface, which gives rise to $c_\mathrm{n} = \gamma T$. The discontinuity Δc is a mean field result. It works extremely well for superconductors, where, as we shall see, the Ginzburg criterion is satisfied down to extremely small temperature variations relative to T_c. In ^{4}He, one sees an cusp-like behavior with an apparent weak divergence at the lambda transition. Recall that in the language of critical phenomena, $c(T) \propto |T - T_\mathrm{c}|^{-\alpha}$. For the $\mathrm{O}(2)$ model in $d = 3$ dimensions, the exponent α is very close to zero, which is close to the mean field value $\alpha = 0$. The order parameter exponent is $\beta = \frac{1}{2}$ at the mean field level; the exact value is closer to $\frac{1}{3}$. One has, for $T < T_c$,

$$\big|\Psi(T < T_c)\big| = \sqrt{-\frac{a(T)}{b(T)}} = \sqrt{\frac{a'(T_c)}{b(T_c)}}\,(T_c - T)^{1/2} + \ldots \quad . \tag{1.45}$$

1.4.2 Ginzburg-Landau Theory

The Landau free energy is minimized by setting $|\Psi|^2 = -a/b$ for $a < 0$. The phase of Ψ is therefore free to vary, and indeed free to vary independently everywhere in space. Phase fluctuations should cost energy, so we posit an augmented free energy functional,

$$F\big[\Psi,\Psi^*\big] = \int\! d^d x \left\{ a\,\big|\Psi(\boldsymbol{x})\big|^2 + \tfrac{1}{2}\,b\,\big|\Psi(\boldsymbol{x})\big|^4 + K\,\big|\boldsymbol{\nabla}\Psi(\boldsymbol{x})\big|^2 + \ldots \right\} \quad . \tag{1.46}$$

Here K is a stiffness with respect to spatial variation of the order parameter $\Psi(\boldsymbol{x})$. From K and a, we can form a length scale, $\xi = \sqrt{K/|a|}$, known as the *coherence length*. This functional in fact is very useful in discussing properties of neutral superfluids, such as ^{4}He, but superconductors are *charged*, and we have instead

$$F\big[\Psi,\Psi^*,\boldsymbol{A}\big] = \int\! d^d x \left\{ a\,\big|\Psi(\boldsymbol{x})\big|^2 + \tfrac{1}{2}\,b\,\big|\Psi(\boldsymbol{x})\big|^4 + K\,\Big|\big(\boldsymbol{\nabla} + \tfrac{ie^*}{\hbar c}\,\boldsymbol{A}\big)\Psi(\boldsymbol{x})\Big|^2 + \tfrac{1}{8\pi}\,(\boldsymbol{\nabla}\times\boldsymbol{A})^2 + \ldots \right\} \quad . \tag{1.47}$$

Here $q = -e^* = -2e$ is the *charge* of the condensate. We assume $\boldsymbol{E} = 0$, so $\boldsymbol{A}$ is not time-dependent.

Under a local transformation $\Psi(\boldsymbol{x}) \to \Psi(\boldsymbol{x})\,e^{i\alpha(\boldsymbol{x})}$, we have

$$\big(\boldsymbol{\nabla} + \tfrac{ie^*}{\hbar c}\,\boldsymbol{A}\big)\big(\Psi\,e^{i\alpha}\big) = e^{i\alpha}\big(\boldsymbol{\nabla} + i\boldsymbol{\nabla}\alpha + \tfrac{ie^*}{\hbar c}\boldsymbol{A}\big)\,\Psi \quad , \tag{1.48}$$

which, upon making the gauge transformation $\boldsymbol{A} \to \boldsymbol{A} - \frac{\hbar c}{e^*}\,\boldsymbol{\nabla}\alpha$, reverts to its original form. Thus, the free energy is unchanged upon replacing $\Psi \to \Psi e^{i\alpha}$ and $\boldsymbol{A} \to \boldsymbol{A} - \frac{\hbar c}{e^*}\,\boldsymbol{\nabla}\alpha$. Since gauge transformations result in no physical consequences, we conclude that the *longitudinal* phase fluctuations of a charged order parameter do not really exist. More on this later when we discuss the Anderson-Higgs mechanism.

1.4.3 Equations of motion

Varying the free energy in Eqn. 1.47 with respect to Ψ^* and $\boldsymbol{A}$, respectively, yields

$$\begin{aligned} 0 &= \frac{\delta F}{\delta \Psi^*} = a\,\Psi + b\,|\Psi|^2\,\Psi - K\,\big(\boldsymbol{\nabla} + \tfrac{ie^*}{\hbar c}\,\boldsymbol{A}\big)^2\,\Psi \\ 0 &= \frac{\delta F}{\delta \boldsymbol{A}} = \frac{2Ke^*}{\hbar c}\left[\frac{1}{2i}\big(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\big) + \frac{e^*}{\hbar c}\,|\Psi|^2\boldsymbol{A}\right] + \frac{1}{4\pi}\,\boldsymbol{\nabla}\times\boldsymbol{B} \quad . \end{aligned} \tag{1.49}$$

The second of these equations is the Ampère-Maxwell law, $\boldsymbol{\nabla}\times\boldsymbol{B} = 4\pi c^{-1}\boldsymbol{j}$, with

$$\boldsymbol{j} = -\frac{2Ke^*}{\hbar^2}\left[\frac{\hbar}{2i}\big(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\big) + \frac{e^*}{c}\,|\Psi|^2\boldsymbol{A}\right] \quad . \tag{1.50}$$

If we set Ψ to be constant, we obtain $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \lambda_L^{-2}\boldsymbol{B} = 0$, with

$$\lambda_L^{-2} = 8\pi K\left(\frac{e^*}{\hbar c}\right)^2 |\Psi|^2 \quad . \tag{1.51}$$

Thus we recover the relation $\lambda_{\rm L}^{-2} \propto |\Psi|^2$. Note that $|\Psi|^2 = |a|/b$ in the ordered phase, hence

$$\lambda_{\rm L}^{-1} = \left[\frac{8\pi a^2}{b} \cdot \frac{K}{|a|}\right]^{1/2} \frac{e^*}{\hbar c} = \frac{\sqrt{2}\, e^*}{\hbar c} H_{\rm c}\, \xi \quad , \tag{1.52}$$

which says

$$H_{\rm c} = \frac{\phi_{\rm L}}{\sqrt{8}\, \pi\, \xi \lambda_{\rm L}} \quad . \tag{1.53}$$

At a superconductor-vacuum interface, we should have

$$\hat{\boldsymbol{n}} \cdot \left(\frac{\hbar}{i} \boldsymbol{\nabla} + \frac{e^*}{c} \boldsymbol{A}\right) \Psi\big|_{\partial\Omega} = 0 \quad , \tag{1.54}$$

where Ω denotes the superconducting region and $\hat{\boldsymbol{n}}$ the surface normal. This guarantees $\hat{\boldsymbol{n}} \cdot \boldsymbol{j}\big|_{\partial\Omega} = 0$, since

$$\boldsymbol{j} = -\frac{2Ke^*}{\hbar^2} \,\mathrm{Re}\left(\frac{\hbar}{i}\, \Psi^* \boldsymbol{\nabla} \Psi + \frac{e^*}{c}\, |\Psi|^2 \boldsymbol{A}\right) \quad . \tag{1.55}$$

Note that $\hat{\boldsymbol{n}} \cdot \boldsymbol{j} = 0$ also holds if

$$\hat{\boldsymbol{n}} \cdot \left(\frac{\hbar}{i} \boldsymbol{\nabla} + \frac{e^*}{c} \boldsymbol{A}\right) \Psi\big|_{\partial\Omega} = ir\Psi \quad , \tag{1.56}$$

with r a real constant. This boundary condition is appropriate at a junction with a normal metal.

1.4.4 Critical current

Consider the case where $\Psi = \Psi_0$. The free energy density is

$$f = a\, |\Psi_0|^2 + \tfrac{1}{2}\, b\, |\Psi_0|^4 + K \left(\frac{e^*}{\hbar c}\right)^2 \boldsymbol{A}^2\, |\Psi_0|^2 \quad . \tag{1.57}$$

If $a > 0$ then f is minimized for $\Psi_0 = 0$. What happens for $a < 0$, *i.e.* when $T < T_{\rm c}$. Minimizing with respect to $|\Psi_0|$, we find

$$|\Psi_0|^2 = \frac{|a| - K(e^*/\hbar c)^2 \boldsymbol{A}^2}{b} \quad . \tag{1.58}$$

The current density is then

$$\boldsymbol{j} = -2cK \left(\frac{e^*}{\hbar c}\right)^2 \left(\frac{|a| - K(e^*/\hbar c)^2 \boldsymbol{A}^2}{b}\right) \boldsymbol{A} \quad . \tag{1.59}$$

Taking the magnitude and extremizing with respect to $A = |\boldsymbol{A}|$, we obtain the *critical current density* $j_{\rm c}$:

$$A^2 = \frac{|a|}{3K(e^*/\hbar c)^2} \quad \Rightarrow \quad j_{\rm c} = \frac{4}{3\sqrt{3}} \, \frac{c\, K^{1/2}\, |a|^{3/2}}{b} \quad . \tag{1.60}$$

Physically, what is happening is this. When the kinetic energy density in the superflow exceeds the condensation energy density $H_{\rm c}^2/8\pi = a^2/2b$, the system goes normal. Note that $j_{\rm c}(T) \propto (T_{\rm c} - T)^{3/2}$.

Should we feel bad about using a gauge-covariant variable like $\boldsymbol{A}$ in the above analysis? Not really, because when we write $\boldsymbol{A}$, what we really mean is the gauge-*invariant* combination $\boldsymbol{A} + \frac{\hbar c}{e^*} \boldsymbol{\nabla}\varphi$, where $\varphi = \mathrm{arg}(\Psi)$ is the phase of the order parameter.

London limit

In the so-called *London limit*, we write $\Psi = \sqrt{n_0}\, e^{i\varphi}$, with n_0 constant. Then

$$\boldsymbol{j} = -\frac{2Ke^* n_0}{\hbar}\Big(\boldsymbol{\nabla}\varphi + \frac{e^*}{\hbar c}\boldsymbol{A}\Big) = -\frac{c}{4\pi\lambda_{\rm L}^2}\Big(\frac{\phi_{\rm L}}{2\pi}\,\boldsymbol{\nabla}\varphi + \boldsymbol{A}\Big) \quad . \tag{1.61}$$

Thus,

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{j} &= \frac{c}{4\pi}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) \\ &= -\frac{c}{4\pi\lambda_{\rm L}^2}\,\boldsymbol{B} - \frac{c}{4\pi\lambda_{\rm L}^2}\,\frac{\phi_{\rm L}}{2\pi}\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi \quad , \end{aligned} \tag{1.62}$$

which says

$$\lambda_{\rm L}^2\,\nabla^2\boldsymbol{B} = \boldsymbol{B} + \frac{\phi_{\rm L}}{2\pi}\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi \quad . \tag{1.63}$$

If we assume $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and the phase field φ has singular vortex lines of topological index $n_i \in \mathbb{Z}$ located at position $\boldsymbol{\rho}_i$ in the (x,y) plane, we have

$$\lambda_{\rm L}^2\,\nabla^2 B = B + \phi_{\rm L}\sum_i n_i\,\delta\big(\boldsymbol{\rho}-\boldsymbol{\rho}_i\big) \quad . \tag{1.64}$$

Taking the Fourier transform, we solve for $\hat{B}(\boldsymbol{q})$, where $\boldsymbol{k} = (\boldsymbol{q}, k_z)$:

$$\hat{B}(\boldsymbol{q}) = -\frac{\phi_{\rm L}}{1+\boldsymbol{q}^2\lambda_{\rm L}^2}\sum_i n_i\, e^{-i\boldsymbol{q}\cdot\boldsymbol{\rho}_i} \quad , \tag{1.65}$$

whence

$$B(\boldsymbol{\rho}) = -\frac{\phi_{\rm L}}{2\pi\lambda_{\rm L}^2}\sum_i n_i\, K_0\bigg(\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_i|}{\lambda_{\rm L}}\bigg) \quad , \tag{1.66}$$

where $K_0(z)$ is the MacDonald function, whose asymptotic behaviors are given by

$$K_0(z) \sim \begin{cases} -{\rm C} - \ln(z/2) & (z\to 0) \\ (\pi/2z)^{1/2}\exp(-z) & (z\to\infty) \end{cases} \quad , \tag{1.67}$$

where ${\rm C} = 0.57721566\ldots$ is the Euler-Mascheroni constant. The logarithmic divergence as $\rho\to 0$ is an artifact of the London limit. Physically, the divergence should be cut off when $|\boldsymbol{\rho}-\boldsymbol{\rho}_i| \sim \xi$. The current density for a single vortex at the origin is

$$\boldsymbol{j}(\boldsymbol{r}) = \frac{nc}{4\pi}\,\boldsymbol{\nabla}\times\boldsymbol{B} = -\frac{c}{4\pi\lambda_{\rm L}}\cdot\frac{\phi_{\rm L}}{2\pi\lambda_{\rm L}^2}\,K_1\big(\rho/\lambda_{\rm L}\big)\,\hat{\boldsymbol{\varphi}} \quad , \tag{1.68}$$

where $n\in\mathbb{Z}$ is the vorticity, and $K_1(z) = -K_0'(z)$ behaves as z^{-1} as $z\to 0$ and $\exp(-z)/\sqrt{2\pi z}$ as $z\to\infty$. Note the $i^{\rm th}$ vortex carries magnetic flux $n_i\,\phi_{\rm L}$.

1.4.5 Ginzburg criterion

Consider fluctuations in $\Psi(\boldsymbol{x})$ above $T_{\rm c}$. If $|\Psi| \ll 1$, we may neglect quartic terms and write

$$F = \int\! d^d\!x\,\Big(a\,|\Psi|^2 + K\,|\boldsymbol{\nabla}\Psi|^2\Big) = \sum_{\boldsymbol{k}}\big(a + K\boldsymbol{k}^2\big)|\hat{\Psi}(\boldsymbol{k})|^2 \quad , \tag{1.69}$$

where we have expanded

$$\Psi(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} \hat{\Psi}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad . \tag{1.70}$$

The Helmholtz free energy $A(T)$ is given by

$$e^{-A/k_{\rm B}T} = \int\!D[\Psi,\Psi^*]\; e^{-F/T} = \prod_{\boldsymbol{k}} \left(\frac{\pi k_{\rm B} T}{a + K\boldsymbol{k}^2}\right) \quad , \tag{1.71}$$

which is to say

$$A(T) = k_{\rm B} T \sum_{\boldsymbol{k}} \ln\!\left(\frac{\pi k_{\rm B} T}{a + K\boldsymbol{k}^2}\right) \quad . \tag{1.72}$$

We write $a(T) = \alpha t$ with $t = (T - T_{\rm c})/T_{\rm c}$ the reduced temperature. We now compute the singular contribution to the specific heat $C_V = -TA''(T)$, which only requires we differentiate with respect to T as it appears in $a(T)$. Dividing by $N_{\rm s} k_{\rm B}$, where $N_{\rm s} = V/\mathsf{a}^d$ is the number of lattice sites, we obtain the dimensionless heat capacity per unit cell,

$$c = \frac{\alpha^2 \mathsf{a}^d}{K^2} \int^{\Lambda\xi} \!\frac{d^d\!k}{(2\pi)^d}\, \frac{1}{(\xi^{-2} + \boldsymbol{k}^2)^2} \quad , \tag{1.73}$$

where $\Lambda \sim \mathsf{a}^{-1}$ is an ultraviolet cutoff on the order of the inverse lattice spacing, and $\xi = (K/a)^{1/2} \propto |t|^{-1/2}$. We define $R_* \equiv (K/\alpha)^{1/2}$, in which case $\xi = R_*\, |t|^{-1/2}$, and

$$c = R_*^{-4}\, \mathsf{a}^d\, \xi^{4-d} \!\int^{\Lambda\xi}\! \frac{d^d\bar{q}}{(2\pi)^d}\, \frac{1}{(1 + \bar{q}^2)^2} \;, \tag{1.74}$$

where $\bar{\boldsymbol{q}} \equiv \boldsymbol{q}\xi$. Thus,

$$c(t) \sim \begin{cases} \text{const.} & \text{if } d > 4 \\ -\ln t & \text{if } d = 4 \\ t^{\frac{d}{2}-2} & \text{if } d < 4 \;. \end{cases} \tag{1.75}$$

For $d > 4$, mean field theory is qualitatively accurate, with finite corrections. In dimensions $d \le 4$, the mean field result is overwhelmed by fluctuation contributions as $t \to 0^+$ (*i.e.* as $T \to T_{\rm c}^+$). We see that the Ginzburg-Landau mean field theory is sensible provided the fluctuation contributions are small, *i.e.* provided

$$R_*^{-4}\, \mathsf{a}^d\, \xi^{4-d} \ll 1 \;, \tag{1.76}$$

which entails $t \gg t_{\rm G}$, where

$$t_{\rm G} = \left(\frac{\mathsf{a}}{R_*}\right)^{\frac{2d}{4-d}} \tag{1.77}$$

is the *Ginzburg reduced temperature*. The criterion for the sufficiency of mean field theory, namely $t \gg t_{\rm G}$, is known as the *Ginzburg criterion*. The region $|t| < t_{\rm G}$ is known as the *critical region*.

In a lattice ferromagnet, as we have seen, $R_* \sim \mathsf{a}$ is on the scale of the lattice spacing itself, hence $t_{\rm G} \sim 1$ and the critical regime is very large. Mean field theory then fails quickly as $T \to T_{\rm c}$. In a (conventional) three-dimensional superconductor, R_* is on the order of the Cooper pair size, and $R_*/\mathsf{a} \sim 10^2 - 10^3$, hence $t_{\rm G} = (a/R_*)^6 \sim 10^{-18} - 10^{-12}$ is negligibly narrow. The mean field theory of the superconducting transition – BCS theory – is then valid essentially all the way to $T = T_{\rm c}$.

Another way to think about it is as follows. In dimensions $d > 2$, for $|\boldsymbol{r}|$ fixed and $\xi \to \infty$, one has[6]

$$\big\langle \Psi^*(\boldsymbol{r})\Psi(0)\big\rangle \simeq \frac{C_d}{k_{\rm B}T\,R_*^2}\,\frac{e^{-r/\xi}}{r^{d-2}} \quad , \tag{1.78}$$

where C_d is a dimensionless constant. If we compute the ratio of fluctuations to the mean value over a patch of linear dimension ξ, we have

$$\begin{aligned}\frac{\text{fluctuations}}{\text{mean}} &= \frac{\int^{\xi} d^d r\, \langle \Psi^*(\boldsymbol{r})\,\Psi(0)\rangle}{\int^{\xi} d^d r\, \langle |\Psi(\boldsymbol{r})|^2\rangle} \\ &\propto \frac{1}{R_*^2\,\xi^d\,|\Psi|^2}\int^{\xi} d^d r\,\frac{e^{-r/\xi}}{r^{d-2}} \propto \frac{1}{R_*^2\,\xi^{d-2}\,|\Psi|^2} \quad .\end{aligned} \tag{1.79}$$

Close to the critical point we have $\xi \propto R_*\,|t|^{-\nu}$ and $|\Psi| \propto |t|^{\beta}$, with $\nu = \frac{1}{2}$ and $\beta = \frac{1}{2}$ within mean field theory. Setting the ratio of fluctuations to mean to be small, we recover the Ginzburg criterion.

1.4.6 Domain wall solution

Consider first the simple case of the neutral superfluid. The additional parameter K provides us with a new length scale, $\xi = \sqrt{K/|a|}$, which is called the coherence length. Varying the free energy with respect to $\Psi^*(\boldsymbol{x})$, one obtains

$$\frac{\delta F}{\delta \Psi^*(\boldsymbol{x})} = a\,\Psi(\boldsymbol{x}) + b\,\big|\Psi(\boldsymbol{x})\big|^2\,\Psi(\boldsymbol{x}) - K\nabla^2\Psi(\boldsymbol{x}) \quad . \tag{1.80}$$

Rescaling, we write $\Psi \equiv \big(|a|/b\big)^{1/2}\psi$, and setting the above functional variation to zero, we obtain

$$-\xi^2\nabla^2\psi + \,\mathrm{sgn}\,(T-T_{\rm c})\,\psi + |\psi|^2\psi = 0 \quad . \tag{1.81}$$

Consider the case of a domain wall when $T < T_{\rm c}$. We assume all spatial variation occurs in the x-direction, and we set $\psi(x=0)=0$ and $\psi(x=\infty)=1$. Furthermore, we take $\psi(x) = f(x)\,e^{i\alpha}$ where α is a constant[7]. We then have $-\xi^2 f''(x) - f + f^3 = 0$, which may be recast as

$$\xi^2\,\frac{d^2 f}{dx^2} = \frac{\partial}{\partial f}\Big[\tfrac{1}{4}\big(1-f^2\big)^2\Big] \quad . \tag{1.82}$$

This looks just like $F = ma$ if we regard f as the coordinate, x as time, and $-V(f) = \frac{1}{4}\big(1-f^2\big)^2$. Thus, the potential describes an *inverted* double well with symmetric minima at $f = \pm 1$. The solution to the equations of motion is then that the 'particle' rolls starts at 'time' $x = -\infty$ at 'position' $f = +1$ and 'rolls' down, eventually passing the position $f = 0$ exactly at time $x = 0$. Multiplying the above equation by $f'(x)$ and integrating once, we have

$$\xi^2\bigg(\frac{df}{dx}\bigg)^2 = \tfrac{1}{2}\big(1-f^2\big)^2 + C \quad , \tag{1.83}$$

where C is a constant, which is fixed by setting $f(x\to\infty) = +1$, which says $f'(\infty) = 0$, hence $C = 0$. Integrating once more,

$$f(x) = \tanh\bigg(\frac{x-x_0}{\sqrt{2}\,\xi}\bigg) \quad , \tag{1.84}$$

[6] Exactly at $T = T_{\rm c}$, the correlations behave as $\big\langle \Psi^*(\boldsymbol{r})\,\Psi(0)\big\rangle \propto r^{-(d-2+\eta)}$, where η is a critical exponent.

[7] Remember that for a superconductor, phase fluctuations of the order parameter are nonphysical since they are eliiminable by a gauge transformation.

where x_0 is the second constant of integration. This, too, may be set to zero upon invoking the boundary condition $f(0) = 0$. Thus, the width of the domain wall is $\xi(T)$. This solution is valid provided that the local magnetic field averaged over scales small compared to ξ, *i.e.* $\boldsymbol{b} = \langle \boldsymbol{\nabla} \times \boldsymbol{A} \rangle$, is negligible.

The energy per unit area of the domain wall is given by $\tilde{\sigma}$, where

$$\begin{aligned}\tilde{\sigma} &= \int_0^\infty \! dx \left\{ K \left| \frac{d\Psi}{dx} \right|^2 + a\, |\Psi|^2 + \tfrac{1}{2}\, b\, |\Psi|^4 \right\} \\ &= \frac{a^2}{b} \int_0^\infty \! dx \left\{ \xi^2 \left(\frac{df}{dx} \right)^2 - f^2 + \tfrac{1}{2}\, f^4 \right\} \quad .\end{aligned} \tag{1.85}$$

Now we ask: is domain wall formation energetically favorable in the superconductor? To answer, we compute the difference in surface energy between the domain wall state and the uniform superconducting state. We call the resulting difference σ, the true domainwall energy relative to the superconducting state:

$$\begin{aligned}\sigma &= \tilde{\sigma} - \int_0^\infty \! dx \left(-\frac{H_c^2}{8\pi} \right) \\ &= \frac{a^2}{b} \int_0^\infty \! dx \left\{ \xi^2 \left(\frac{df}{dx} \right)^2 + \tfrac{1}{2} \left(1 - f^2 \right)^2 \right\} \equiv \frac{H_c^2}{8\pi}\, \delta \quad ,\end{aligned} \tag{1.86}$$

where we have used $H_c^2 = 4\pi a^2/b$. Invoking the previous result $f' = (1 - f^2)/\sqrt{2}\,\xi$, the parameter δ is given by

$$\delta = 2 \int_0^\infty \! dx \, \left(1 - f^2 \right)^2 = 2 \int_0^1 \! df \, \frac{\left(1 - f^2 \right)^2}{f'} = \frac{4\sqrt{2}}{3}\, \xi(T) \quad . \tag{1.87}$$

Had we permitted a field to penetrate over a distance $\lambda_L(T)$ in the domain wall state, we'd have obtained

$$\delta(T) = \frac{4\sqrt{2}}{3}\, \xi(T) - \lambda_L(T) \quad . \tag{1.88}$$

Detailed calculations show

$$\delta = \begin{cases} \frac{4\sqrt{2}}{3}\, \xi \approx 1.89\, \xi & \text{if } \xi \gg \lambda_L \\ 0 & \text{if } \xi = \sqrt{2}\, \lambda_L \\ -\frac{8(\sqrt{2}-1)}{3}\, \lambda_L \approx -1.10\, \lambda_L & \text{if } \lambda_L \gg \xi \quad . \end{cases} \tag{1.89}$$

Accordingly, we define the Ginzburg-Landau parameter $\kappa \equiv \lambda_L/\xi$, which is temperature-dependent near $T = T_c$, as we'll soon show.

So the story is as follows. In type-I materials, the positive ($\delta > 0$) N-S surface energy keeps the sample spatially homogeneous for all $H < H_c$. In type-II materials, the negative surface energy causes the system to break into domains, which are vortex structures, as soon as H exceeds the lower critical field H_{c1}. This is known as the *mixed state*.

1.4.7 Scaled Ginzburg-Landau equations

For $T < T_c$, we write

$$\Psi = \sqrt{\frac{|a|}{b}}\, \psi \quad , \quad \boldsymbol{x} = \lambda_L\, \boldsymbol{r} \quad , \quad \boldsymbol{A} = \sqrt{2}\, \lambda_L\, H_c\, \boldsymbol{a} \tag{1.90}$$

as well as the GL parameter,

$$\kappa = \frac{\lambda_{\rm L}}{\xi} = \frac{\sqrt{2}\, e^*}{\hbar c}\, H_{\rm c}\, \lambda_{\rm L}^2 \quad . \tag{1.91}$$

The Gibbs free energy is then

$$G = \frac{H_{\rm c}^2 \lambda_{\rm L}^3}{4\pi} \int d^3\!r \, \bigg\{ -|\psi|^2 + \tfrac{1}{2}\,|\psi|^2 + \big|(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a})\,\psi\big|^2 + (\boldsymbol{\nabla}\times\boldsymbol{a})^2 - 2\boldsymbol{h}\cdot\boldsymbol{\nabla}\times\boldsymbol{a}\bigg\} \quad . \tag{1.92}$$

Setting $\delta G = 0$, we obtain

$$\begin{aligned} (\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a})^2\,\psi + \psi - |\psi|^2\psi = 0 \\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a} - \boldsymbol{h}) + |\psi|^2\boldsymbol{a} - \frac{i}{2\kappa}\big(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\big) = 0 \quad . \end{aligned} \tag{1.93}$$

The condition that no current flow through the boundary is

$$\hat{\boldsymbol{n}}\cdot\big(\boldsymbol{\nabla} + i\kappa\boldsymbol{a}\big)\psi\Big|_{\partial\Omega} = 0 \quad . \tag{1.94}$$

1.5 Applications of Ginzburg-Landau Theory

The applications of GL theory are numerous. Here we run through some examples.

1.5.1 Domain wall energy

Consider a domain wall interpolating between a normal metal at $x \to -\infty$ and a superconductor at $x \to +\infty$. The difference between the Gibbs free energies is

$$\begin{aligned} \Delta G = G_{\rm s} - G_{\rm n} &= \int d^3\!x \, \bigg\{ a\,|\Psi|^2 + \tfrac{1}{2} b\,|\Psi|^4 + K\big|(\boldsymbol{\nabla} + \tfrac{ie^*}{\hbar c}\,\boldsymbol{A})\Psi\big|^2 + \frac{(B-H)^2}{8\pi}\bigg\} \\ &= \frac{H_{\rm c}^2\,\lambda_{\rm L}^3}{4\pi} \int d^3\!r \, \Big[-|\psi|^2 + \tfrac{1}{2}|\psi|^4 + \big|(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a})\,\psi\big|^2 + (b-h)^2 \Big] \quad , \end{aligned} \tag{1.95}$$

with $b = B/\sqrt{2}\,H_{\rm c}$ and $h = H/\sqrt{2}\,H_{\rm c}$. We define

$$\Delta G(T, H_{\rm c}) \equiv \frac{H_{\rm c}^2}{8\pi} \cdot A\,\lambda_{\rm L} \cdot \delta \quad , \tag{1.96}$$

as we did above in Eqn. 1.86, except here δ is rendered dimensionless by scaling it by $\lambda_{\rm L}$. Here A is the cross-sectional area, so δ is a dimensionless domain wall energy per unit area. Integrating by parts and appealing to the Euler-Lagrange equations, we have

$$\int d^3\!r \, \Big[-|\psi|^2 + |\psi|^4 + \big|(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a})\,\psi\big|^2 \Big] = \int d^3\!r \; \psi^* \Big[-\psi + |\psi|^2\psi - (\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a})^2\,\psi \Big] = 0 \quad , \tag{1.97}$$

and therefore

$$\delta = \int\limits_{-\infty}^{\infty} dx \, \Big[-|\psi|^4 + 2\,(b-h)^2 \Big] \quad . \tag{1.98}$$

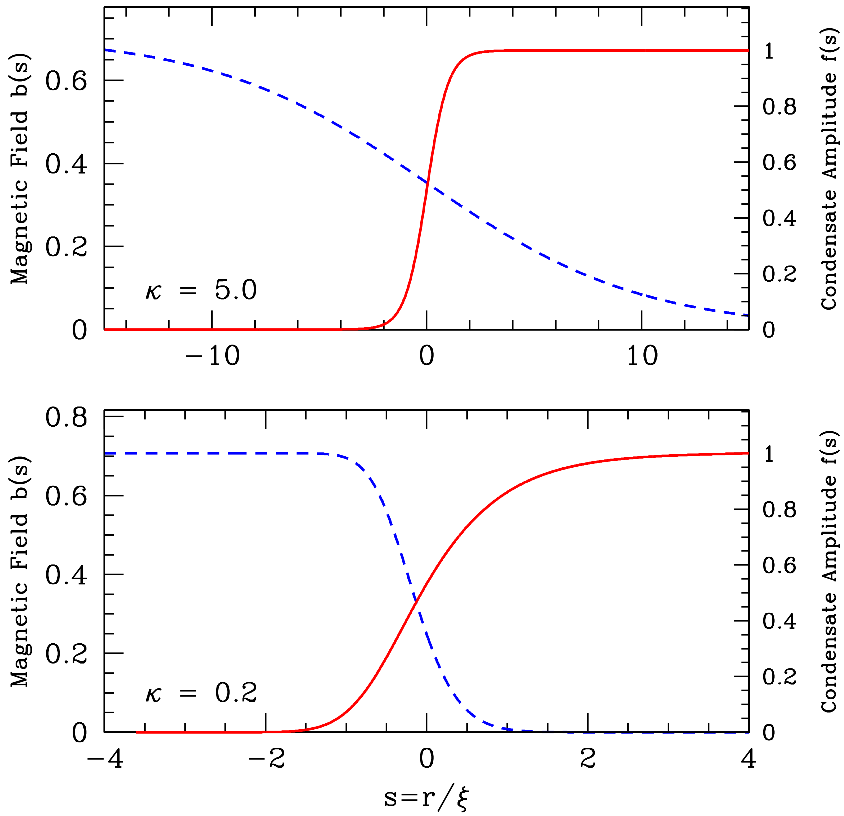

Figure 1.6: Numerical solution to a Ginzburg-Landau domain wall interpolating between normal metal ($x \to -\infty$) and superconducting ($x \to +\infty$) phases, for $H = H_{c2}$. Upper panel corresponds to $\kappa = 5$, and lower panel to $\kappa = 0.2$. Condensate amplitude $f(s)$ is shown in red, and dimensionless magnetic field $b(s) = B(s)/\sqrt{2}\,H_c$ in dashed blue.

Deep in the metal, as $x \to -\infty$, we expect $\psi \to 0$ and $b \to h$. Deep in the superconductor, as $x \to +\infty$, we expect $|\psi| \to 1$ and $b \to 0$. The bulk energy contribution then vanishes for $h = h_c = \frac{1}{\sqrt{2}}$, which means δ is finite, corresponding to the domain wall free energy per unit area.

We take $\psi = f \in \mathbb{R}$, $\boldsymbol{a} = a(x)\,\hat{\boldsymbol{y}}$, so $\boldsymbol{b} = b(x)\,\hat{\boldsymbol{z}}$ with $b(x) = a'(x)$. Thus, $\boldsymbol{\nabla} \times \boldsymbol{b} = -a''(x)\,\hat{\boldsymbol{y}}$, and the Euler-Lagrange equations are

$$\begin{aligned} \frac{1}{\kappa^2}\,\frac{d^2 f}{dx^2} &= \big(a^2 - 1\big) f + f^3 \\ \frac{d^2 a}{dx^2} &= a f^2 \quad . \end{aligned} \tag{1.99}$$

These equations must be solved simultaneously to obtain the full solution. They are equivalent to a nonlinear dynamical system of dimension $N = 4$, where the phase space coordinates are (f, f', a, a'), *i.e.*

$$\frac{d}{dx}\begin{pmatrix} f \\ f' \\ a \\ a' \end{pmatrix} = \begin{pmatrix} f' \\ \kappa^2 (a^2 - 1) f + \kappa^2 f^3 \\ a' \\ a f^2 \end{pmatrix} \quad . \tag{1.100}$$

Four boundary conditions must be provided, which we can take to be

$$f(-\infty) = 0 \quad , \quad a'(-\infty) = \frac{1}{\sqrt{2}} \quad , \quad f(+\infty) = 1 \quad , \quad a'(+\infty) = 0 \quad . \tag{1.101}$$

Usually with dynamical systems, we specify N boundary conditions at some initial value $x = x_0$ and then integrate to the final value, using a Runge-Kutta method. Here we specify $\frac{1}{2}N$ boundary conditions at each of the two ends, which requires we use something such as the *shooting method* to solve the coupled ODEs, which effectively converts the boundary value problem to an initial value problem. In Fig. 1.6, we present such a numerical solution to the above system, for $\kappa = 0.2$ (type-I) and for $\kappa = 5$ (type-II).

Vortex solution

To describe a vortex line of strength $n \in \mathbb{Z}$, we choose cylindrical coordinates (ρ, φ, z), and assume no variation in the vertical (z) direction. We write $\psi(\boldsymbol{r}) = f(\rho)\, e^{in\varphi}$ and $\boldsymbol{a}(\boldsymbol{r}) = a(\rho)\,\hat{\boldsymbol{\varphi}}$. which says $\boldsymbol{b}(\boldsymbol{r}) = b(\rho)\,\hat{\boldsymbol{z}}$ with $b(\rho) = \frac{\partial a}{\partial \rho} + \frac{a}{\rho}$. We then obtain

$$\begin{aligned} \frac{1}{\kappa^2}\left(\frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho}\right) &= \left(\frac{n}{\kappa\rho} + a\right)^2 f - f + f^3 \\ \frac{d^2 a}{d\rho^2} + \frac{1}{\rho}\frac{da}{d\rho} &= \frac{a}{\rho^2} + \left(\frac{n}{\kappa\rho} + a\right) f^2 \quad . \end{aligned} \tag{1.102}$$

As in the case of the domain wall, this also corresponds to an $N = 4$ dynamical system boundary value problem, which may be solved numerically using the shooting method.

1.5.2 Thin type-I films : critical field strength

Consider a thin extreme type-I (*i.e.* $\kappa \ll 1$) film. Let the finite dimension of the film be along $\hat{\boldsymbol{x}}$, and write $f = f(x)$, $\boldsymbol{a} = a(x)\,\hat{\boldsymbol{y}}$, so $\boldsymbol{\nabla}\times\boldsymbol{a} = b(x)\,\hat{\boldsymbol{z}} = \frac{\partial a}{\partial x}\,\hat{\boldsymbol{z}}$. We assue $f(x) \in \mathbb{R}$. Now $\boldsymbol{\nabla}\times\boldsymbol{b} = -\frac{\partial^2 a}{\partial x^2}\,\hat{\boldsymbol{y}}$, so we have from the second of Eqs. 1.93 that

$$\frac{d^2 f}{dx^2} = a f^2 \quad , \tag{1.103}$$

while the first of Eqs. 1.93 yields

$$\frac{1}{\kappa^2}\frac{d^2 f}{dx^2} + (1 - a^2) f - f^3 = 0 \quad . \tag{1.104}$$

We require $f'(x) = 0$ on the boundaries, which we take to lie at $x = \pm\frac{1}{2}d$. For $\kappa \ll 1$, we have, to a first approximation, $f''(x) = 0$ with $f'(\pm\frac{1}{2}d) = 0$. This yields $f = f_0$, a constant, in which case $a''(x) = f_0^2 a(x)$, yielding

$$a(x) = \frac{h_0 \sinh(f_0 x)}{f_0 \cosh(\frac{1}{2} f_0 d)} \qquad , \qquad b(x) = \frac{h_0 \cosh(f_0 x)}{\cosh(\frac{1}{2} f_0 d)} \quad , \tag{1.105}$$

with $h_0 = H_0/\sqrt{2}\,H_c$ the scaled field outside the superconductor. Note $b(\pm\frac{1}{2}d) = h_0$. To determine the constant f_0, we set $f = f_0 + f_1$ and solve for f_1:

$$-\frac{d^2 f_1}{dx^2} = \kappa^2\Big[\big(1 - a^2(x)\big)\, f_0 - f_0^3\Big] \quad . \tag{1.106}$$

In order for a solution to exist, the RHS must be orthogonal to the zeroth order solution[8], *i.e.* we demand

$$\int_{-d/2}^{d/2} dx \left[1 - a^2(x) - f_0^2\right] \equiv 0 \quad , \tag{1.107}$$

which requires

$$h_0^2 = \frac{2f_0^2\,(1-f_0^2)\cosh^2(\frac{1}{2}f_0 d)}{\big[\sinh(f_0 d)/f_0 d\big] - 1} \quad , \tag{1.108}$$

which should be considered an implicit relation for $f_0(h_0)$. The magnetization is

$$m = \frac{1}{4\pi d}\int_{-d/2}^{d/2} dx\, b(x) - \frac{h_0}{4\pi} = \frac{h_0}{4\pi}\left[\frac{\tanh(\frac{1}{2}f_0 d)}{\frac{1}{2}f_0 d} - 1\right] \quad . \tag{1.109}$$

Note that for $f_0 d \gg 1$, we recover the complete Meissner effect, $h_0 = -4\pi m$. In the opposite limit $f_0 d \ll 1$, we find

$$m \simeq -\frac{f_0^2 d^2 h_0}{48\pi} \quad , \quad h_0^2 \simeq \frac{12(1-f_0^2)}{d^2} \quad \Rightarrow \quad m \simeq -\frac{h_0 d^2}{8\pi}\left(1 - \frac{h_0^2 d^2}{12}\right) \quad . \tag{1.110}$$

Next, consider the free energy difference,

$$\begin{aligned} G_s - G_n &= \frac{H_c^2 \lambda_L^3}{4\pi}\int_{-d/2}^{d/2} dx \left[-f^2 + \tfrac{1}{2}f^4 + (b-h_0)^2 + \left|(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a})\,f\right|^2\right] \\ &= \frac{H_c^2\lambda_L^3 d}{4\pi}\left[\left(1 - \frac{\tanh(\frac{1}{2}f_0 d)}{\frac{1}{2}f_0 d}\right)h_0^2 - f_0^2 + \tfrac{1}{2}f_0^4\right] \quad . \end{aligned} \tag{1.111}$$

The critical field $h_0 = h_c$ occurs when $G_s = G_n$, hence

$$h_c^2 = \frac{f_0^2\,(1-\frac{1}{2}f_0^2)}{\left[1 - \frac{\tanh(\frac{1}{2}f_0 d)}{\frac{1}{2}f_0 d}\right]} = \frac{2\,f_0^2\,(1-f_0^2)\cosh^2(\frac{1}{2}f_0 d)}{\big[\sinh(f_0 d)/f_0 d\big] - 1} \quad . \tag{1.112}$$

We must eliminate f_0 to determine $h_c(d)$.

When the film is thick we can write $f_0 = 1 - \varepsilon$ with $\varepsilon \ll 1$. Then $df_0 = d(1-\varepsilon) \gg 1$ and we have $h_c^2 \simeq 2d\varepsilon$ and $\varepsilon = h_c^2/2d \ll 1$. We also have

$$h_c^2 \approx \frac{\frac{1}{2}}{1-\frac{2}{d}} \approx \tfrac{1}{2}\left(1 + \frac{2}{d}\right) \quad , \tag{1.113}$$

which says

$$h_c(d) = \tfrac{1}{\sqrt{2}}\big(1 + d^{-1}\big) \quad \Rightarrow \quad H_c(d) = H_c(\infty)\left(1 + \frac{\lambda_L}{d}\right) \quad , \tag{1.114}$$

where in the very last equation we restore dimensionful units for d.

For a thin film, we have $f_0 \approx 0$, in which case

$$h_c = \frac{2\sqrt{3}}{d}\sqrt{1-f_0^2} \quad , \tag{1.115}$$

[8] If $\hat{L}f_1 = R$, then $\langle\, f_0 \,|\, R\,\rangle = \langle\, f_0 \,|\, \hat{L} \,|\, f_1\,\rangle = \langle\, \hat{L}^\dagger f_0 \,|\, f_1\,\rangle$. Assuming $\hat{L}$ is self-adjoint, and that $\hat{L}f_0 = 0$, we obtain $\langle\, f_0 \,|\, R\,\rangle = 0$. In our case, the operator $\hat{L}$ is given by $\hat{L} = -d^2/dx^2$.

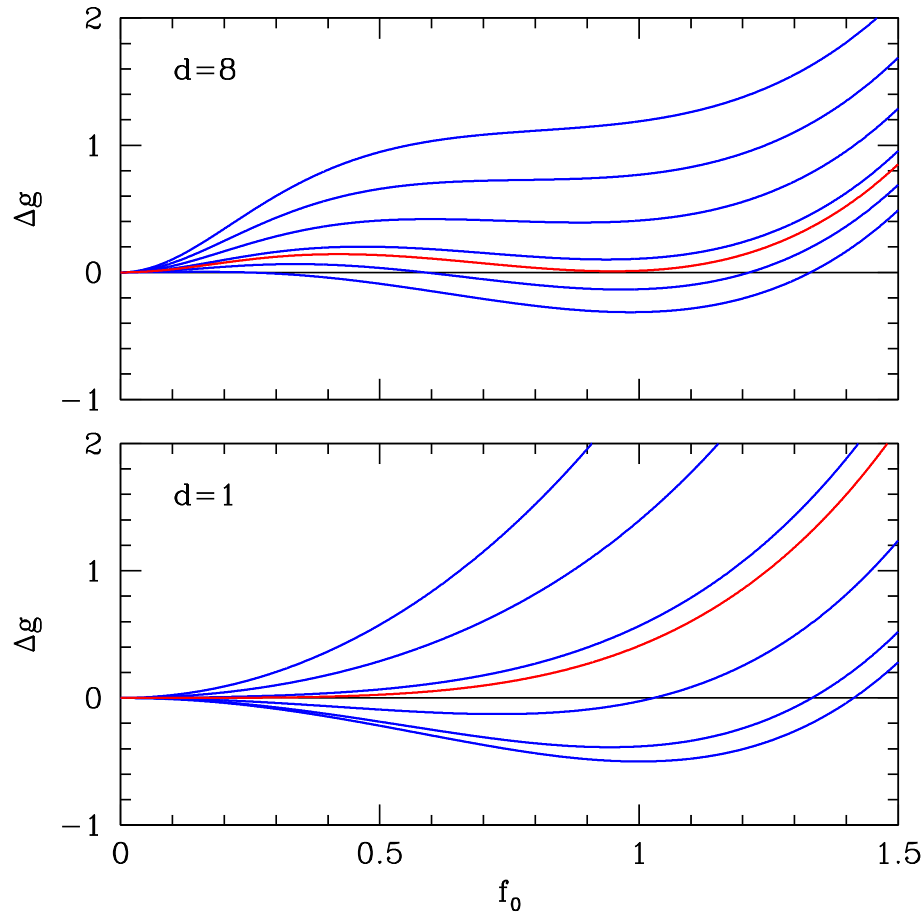

Figure 1.7: Difference in dimensionless free energy density Δg between superconducting and normal state for a thin extreme type-I film of thickness $d\lambda_{\rm L}$. Free energy curves are shown as a function of the amplitude f_0 for several values of the applied field $h_0 = H/\sqrt{2}\,H_{\rm c}(\infty)$ (upper curves correspond to larger h_0 values). Top panel: $d = 8$ curves, with the critical field (in red) at $h_{\rm c} \approx 0.827$ and a first order transition. Lower panel: $d = 1$ curves, with $h_{\rm c} = \sqrt{12} \approx 3.46$ (in red) and a second order transition. The critical thickness is $d_{\rm c} = \sqrt{5}$.

and expanding the hyperbolic tangent, we find

$$h_{\rm c}^2 = \frac{12}{d^2}\big(1 - \tfrac{1}{2}f_0^2\big) \quad . \tag{1.116}$$

This gives

$$f_0 \approx 0 \quad , \quad h_{\rm c} \approx \frac{2\sqrt{3}}{d} \quad \Rightarrow \quad H_{\rm c}(d) = 2\sqrt{6}\,H_{\rm c}(\infty)\,\frac{\lambda_{\rm L}}{d} \quad . \tag{1.117}$$

Note for d large we have $f_0 \approx 1$ at the transition (first order), while for d small we have $f_0 \approx 0$ at the transition (second order). We can see this crossover from first to second order by plotting

$$g = \frac{4\pi}{d\lambda_{\rm L}^3 H_{\rm c}^3}\big(G_{\rm s} - G_{\rm n}\big) = \left(1 - \frac{\tanh(\frac{1}{2}f_0 d)}{\frac{1}{2}f_0 d}\right)h_0^2 - f_0^2 + \tfrac{1}{2}f_0^4 \tag{1.118}$$

as a function of f_0 for various values of h_0 and d. Setting $dg/df_0 = 0$ and $d^2g/df_0^2 = 0$ and $f_0 = 0$, we obtain $d_{\rm c} = \sqrt{5}$. See Fig. 1.7. For consistency, we must have $d \ll \kappa^{-1}$.

1.5.3 Critical current of a wire

Consider a wire of radius R and let the total current carried be I. The magnetizing field $\boldsymbol{H}$ is azimuthal, and integrating around the surface of the wire, we obtain

$$2\pi R H_0 = \oint_{r=R} d\boldsymbol{l} \cdot \boldsymbol{H} = \int d\boldsymbol{S} \cdot \boldsymbol{\nabla} \times \boldsymbol{H} = \frac{4\pi}{c} \int d\boldsymbol{S} \cdot \boldsymbol{j} = \frac{4\pi I}{c} \quad . \tag{1.119}$$

Thus,

$$H_0 = H(R) = \frac{2I}{cR} \quad . \tag{1.120}$$

We work in cylindrical coordinates (ρ, φ, z), taking $\boldsymbol{a} = a(\rho)\,\hat{\boldsymbol{z}}$ and $f = f(\rho)$. The scaled GL equations give

$$\big(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a}\big)^2 f + f - f^3 = 0 \tag{1.121}$$

with[9]

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\,\frac{\partial}{\partial\rho} + \frac{\hat{\boldsymbol{\varphi}}}{\rho}\,\frac{\partial}{\partial\varphi} + \hat{\boldsymbol{z}}\,\frac{\partial}{\partial z} \quad . \tag{1.122}$$

Thus,

$$\frac{1}{\kappa^2}\,\frac{\partial^2 f}{\partial\rho^2} + \big(1 - a^2\big) f - f^3 = 0 \quad , \tag{1.123}$$

with $f'(R) = 0$. From $\boldsymbol{\nabla} \times \boldsymbol{b} = -\big(\kappa^{-1}\boldsymbol{\nabla}\theta + \boldsymbol{a}\big)\,|\psi|^2$, where $\arg(\psi) = \theta$, we have $\psi = f \in \mathbb{R}$ hence $\theta = 0$, and therefore

$$\frac{\partial^2 a}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial a}{\partial\rho} = a f^2 \quad . \tag{1.124}$$

The magnetic field is

$$\boldsymbol{b} = \boldsymbol{\nabla} \times a(\rho)\,\hat{\boldsymbol{z}} = -\frac{\partial a}{\partial\rho}\,\hat{\boldsymbol{\varphi}} \quad , \tag{1.125}$$

hence $b(\rho) = -\frac{\partial a}{\partial\rho}$, with

$$b(R) = \frac{H(R)}{\sqrt{2}\,H_{\mathrm{c}}} = \frac{\sqrt{2}\,I}{cRH_{\mathrm{c}}} \quad . \tag{1.126}$$

Again, we assume $\kappa \ll 1$, hence $f = f_0$ is the leading order solution to Eqn. 1.123. The vector potential and magnetic field, accounting for boundary conditions, are then given by

$$a(\rho) = -\frac{b(R)\,I_0(f_0\,\rho)}{f_0\,I_1(f_0 R)} \qquad , \qquad b(\rho) = \frac{b(R)\,I_1(f_0\,\rho)}{I_1(f_0 R)} \quad , \tag{1.127}$$

where $I_n(z)$ is a modified Bessel function. As in §1.5.2, we determine f_0 by writing $f = f_0 + f_1$ and demanding that f_1 be orthogonal to the uniform solution. This yields the condition

$$\int_0^R d\rho\,\rho\,\Big(1 - f_0^2 - a^2(\rho)\Big) = 0 \quad , \tag{1.128}$$

which gives

$$b^2(R) = \frac{f_0^2\,(1 - f_0^2)\,I_1^2(f_0 R)}{I_0^2(f_0 R) - I_1^2(f_0 R)} \quad . \tag{1.129}$$

[9]Though we don't need to invoke these results, it is good to recall $\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\varphi} = \hat{\boldsymbol{\varphi}}$ and $\frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\varphi} = -\hat{\boldsymbol{\rho}}$.

Thin wire : $R \ll 1$

When $R \ll 1$, we expand the Bessel functions, using

$$I_n(z) = \left(\tfrac{1}{2}z\right)^n \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\,(k+n)!} \quad . \tag{1.130}$$

Thus

$$\begin{aligned} I_0(z) &= 1 + \tfrac{1}{4}z^2 + \ldots \\ I_1(z) &= \tfrac{1}{2}z + \tfrac{1}{16}z^3 + \ldots \quad , \end{aligned} \tag{1.131}$$

and therefore

$$b^2(R) = \tfrac{1}{4} f_0^4 \big(1 - f_0^2\big) R^2 + \mathcal{O}(R^4) \quad . \tag{1.132}$$

To determine the critical current, we demand that the maximum value of $b(\rho)$ take place at $\rho = R$, yielding

$$\frac{\partial(b^2)}{\partial f_0} = \big(f_0^3 - \tfrac{3}{2} f_0^5\big) R^2 \equiv 0 \quad \Rightarrow \quad f_{0,\text{max}} = \sqrt{\tfrac{2}{3}} \quad . \tag{1.133}$$

From $f_{0,\text{max}}^2 = \frac{2}{3}$, we then obtain

$$b(R) = \frac{R}{3\sqrt{3}} = \frac{\sqrt{2}\, I_{\text{c}}}{cRH_{\text{c}}} \quad \Rightarrow \quad I_{\text{c}} = \frac{cR^2 H_{\text{c}}}{3\sqrt{6}} \quad . \tag{1.134}$$

The critical current density is then

$$j_{\text{c}} = \frac{I_{\text{c}}}{\pi R^2} = \frac{c\, H_{\text{c}}}{3\sqrt{6}\,\pi\,\lambda_{\text{L}}} \quad , \tag{1.135}$$

where we have restored physical units.

Thick wire : $1 \ll R \ll \kappa^{-1}$

For a thick wire, we use the asymptotic behavior of $I_n(z)$ for large argument:

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} (-1)^k\, \frac{a_k(\nu)}{z^k} \quad , \tag{1.136}$$

which is known as Hankel's expansion. The expansion coefficients are given by[10]

$$a_k(\nu) = \frac{\big(4\nu^2 - 1^2\big)\big(4\nu^2 - 3^2\big) \cdots \big(4\nu^2 - (2k-1)^2\big)}{8^k\, k!} \quad , \tag{1.137}$$

and we then obtain

$$b^2(R) = f_0^3 (1 - f_0^2) R + \mathcal{O}(R^0) \quad . \tag{1.138}$$

Extremizing with respect to f_0, we obtain $f_{0,\text{max}} = \sqrt{\frac{3}{5}}$ and

$$b_{\text{c}}(R) = \left(\frac{4 \cdot 3^3}{5^5}\right)^{1/4} R^{1/2} \quad . \tag{1.139}$$

[10] See *e.g.* the *NIST Handbook of Mathematical Functions*, §10.40.1 and §10.17.1.

Restoring units, the critical current of a thick wire is

$$I_c = \frac{3^{3/4}}{5^{5/4}}\, c\, H_c\, R^{3/2}\, \lambda_L^{-1/2} \quad . \tag{1.140}$$

To be consistent, we must have $R \ll \kappa^{-1}$, which explains why our result here does not coincide with the bulk critical current density obtained in Eqn. 1.60.

1.5.4 Magnetic properties of type-II superconductors

Consider an incipient type-II superconductor, when the order parameter is just beginning to form. In this case we can neglect the nonlinear terms in ψ in the Ginzburg-Landau equations 1.93. The first of these equations then yields

$$-\big(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a}\big)^2 \psi = \psi + \overbrace{\mathcal{O}\big(|\psi|^2\psi\big)}^{\approx 0} \quad . \tag{1.141}$$

We neglect the second term on the RHS. This is an eigenvalue equation, with the eigenvalue fixed at 1. In fact, this is to be regarded as an equation for $\boldsymbol{a}$, or, more precisely, for the gauge-invariant content of $\boldsymbol{a}$, which is $\boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{a}$. The second of the GL equations says $\boldsymbol{\nabla}\times(\boldsymbol{b}-\boldsymbol{h}) = \mathcal{O}\big(|\psi|^2\big)$, from which we conclude $\boldsymbol{b} = \boldsymbol{h} + \boldsymbol{\nabla}\zeta$, but inspection of the free energy itself tells us $\boldsymbol{\nabla}\zeta = 0$.

We assume $\boldsymbol{b} = h\hat{\boldsymbol{z}}$ and choose a gauge for $\boldsymbol{a}$:

$$\boldsymbol{a} = -\tfrac{1}{2} b\, y\, \hat{\boldsymbol{x}} + \tfrac{1}{2} b\, x\, \hat{\boldsymbol{y}} \quad , \tag{1.142}$$

with $b = h$. We define the operators

$$\pi_x = \frac{1}{i\kappa}\frac{\partial}{\partial x} - \tfrac{1}{2} b\, y \qquad , \qquad \pi_y = \frac{1}{i\kappa}\frac{\partial}{\partial y} + \tfrac{1}{2} b\, x \quad . \tag{1.143}$$

Note that $\big[\pi_x, \pi_y\big] = b/i\kappa$, and that

$$-\big(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a}\big)^2 = -\frac{1}{\kappa^2}\frac{\partial^2}{\partial z^2} + \pi_x^2 + \pi_y^2 \quad . \tag{1.144}$$

We now define the ladder operators

$$\begin{aligned} \gamma &= \sqrt{\frac{\kappa}{2b}}\,\big(\pi_x - i\pi_y\big) \\ \gamma^\dagger &= \sqrt{\frac{\kappa}{2b}}\,\big(\pi_x + i\pi_y\big) \quad , \end{aligned} \tag{1.145}$$

which satisfy $\big[\gamma, \gamma^\dagger\big] = 1$. Then

$$\hat{L} \equiv -\big(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a}\big)^2 = -\frac{1}{\kappa^2}\frac{\partial^2}{\partial z^2} + \frac{2b}{\kappa}\big(\gamma^\dagger\gamma + \tfrac{1}{2}\big) \quad . \tag{1.146}$$

The eigenvalues of the operator $\hat{L}$ are therefore

$$\varepsilon_n(k_z) = \frac{k_z^2}{\kappa^2} + \big(n + \tfrac{1}{2}\big)\cdot\frac{2b}{\kappa} \quad . \tag{1.147}$$

The lowest eigenvalue is therefore b/κ. This crosses the threshold value of 1 when $b = \kappa$, *i.e.* when

$$H = \sqrt{2}\,\kappa\, H_c \equiv H_{c2} \quad . \tag{1.148}$$

So, what have we shown? When $b = h < \frac{1}{\sqrt{2}}$, so $H_{c2} < H_c$ (we call H_c the *thermodynamic critical field*), a complete Meissner effect occurs when H is decreased below H_c. The order parameter ψ jumps discontinuously, and the transition across H_c is first order. If $\kappa > \frac{1}{\sqrt{2}}$, then $H_{c2} > H_c$, and for H just below H_{c2} the system wants $\psi \neq 0$. However, a complete Meissner effect cannot occur for $H > H_c$, so for $H_c < H < H_{c2}$ the system is in the so-called *mixed phase*. Recall that $H_c = \phi_L / \sqrt{8}\,\pi\,\xi\lambda_L$, hence

$$H_{c2} = \sqrt{2}\,\kappa\,H_c = \frac{\phi_L}{2\pi\xi^2} \quad . \tag{1.149}$$

Thus, H_{c2} is the field at which neighboring vortex lines, each of which carry flux ϕ_L, are separated by a distance on the order of ξ.

1.5.5 Lower critical field

We now compute the energy of a perfectly straight vortex line, and ask at what field H_{c1} vortex lines first penetrate. Let's consider the regime $\rho > \xi$, where $\psi \simeq e^{i\varphi}$, *i.e.* $|\psi| \simeq 1$. Then the second of the Ginzburg-Landau equations gives

$$\boldsymbol{\nabla} \times \boldsymbol{b} = -\big(\kappa^{-1}\boldsymbol{\nabla}\varphi + \boldsymbol{a}\big) \quad . \tag{1.150}$$

Therefore the Gibbs free energy is

$$G_V = \frac{H_c^2\,\lambda_L^3}{4\pi}\int\! d^3r\,\Big\{-\tfrac{1}{2} + \boldsymbol{b}^2 + (\boldsymbol{\nabla}\times\boldsymbol{b})^2 - 2\boldsymbol{h}\cdot\boldsymbol{b}\Big\} \quad . \tag{1.151}$$

The first term in the brackets is the condensation energy density $-H_c^2/8\pi$. The second term is the electromagnetic field energy density $\boldsymbol{B}^2/8\pi$. The third term is $\lambda_L^2\,(\boldsymbol{\nabla}\times\boldsymbol{B})^2/8\pi$, and accounts for the kinetic energy density in the superflow.

The energy penalty for a vortex is proportional to its length. We have

$$\begin{aligned}\frac{G_V - G_0}{L} &= \frac{H_c^2\,\lambda_L^2}{4\pi}\int\! d^2\rho\,\Big\{\boldsymbol{b}^2 + (\boldsymbol{\nabla}\times\boldsymbol{b})^2 - 2\,\boldsymbol{h}\cdot\boldsymbol{b}\Big\}\\ &= \frac{H_c^2\,\lambda_L^2}{4\pi}\int\! d^2\rho\,\Big\{\boldsymbol{b}\cdot\big[\boldsymbol{b} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\big] - 2\boldsymbol{h}\cdot\boldsymbol{b}\Big\} \quad .\end{aligned} \tag{1.152}$$

The total flux is

$$\int\! d^2\rho\;\boldsymbol{b}(\boldsymbol{\rho}) = -2\pi n\kappa^{-1}\,\hat{\boldsymbol{z}} \quad , \tag{1.153}$$

in units of $\sqrt{2}\,H_c\,\lambda_L^2$. We also have $b(\boldsymbol{\rho}) = -n\kappa^{-1}\,K_0(\rho)$ and, taking the curl of Eqn. 1.150, we have $\boldsymbol{b}+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) = -2\pi n\kappa^{-1}\,\delta(\boldsymbol{\rho})\,\hat{\boldsymbol{z}}$. As mentioned earlier above, the logarithmic divergence of $b(\rho \to 0)$ is an artifact of the London limit, where the vortices have no core structure. The core can crudely be accounted for by simply replacing $B(0)$ by $B(\xi)$, *i.e.* replacing $b(0)$ by $b(\xi/\lambda_L) = b(\kappa^{-1})$. Then, for $\kappa \gg 1$, after invoking Eqn. 1.67,

$$\frac{G_V - G_0}{L} = \frac{H_c^2\,\lambda_L^2}{4\pi}\Big\{2\pi n^2\kappa^{-2}\ln\big(2\,e^{-C}\kappa\big) + 4\pi n h\kappa^{-1}\Big\} \quad . \tag{1.154}$$

For vortices with vorticity $n = -1$, this first turns negative at a field

$$h_{c1} = \tfrac{1}{2}\kappa^{-1}\ln\big(2\,e^{-C}\kappa\big) \quad . \tag{1.155}$$

With $2\,e^{-C} \simeq 1.23$, we have, restoring units,

$$H_{c1} = \frac{H_c}{\sqrt{2}\,\kappa}\,\ln\big(2\,e^{-C}\kappa\big) = \frac{\phi_L}{4\pi\lambda_L^2}\,\ln(1.23\,\kappa) \quad . \tag{1.156}$$

So we have

$$\begin{aligned} H_{c1} &= \frac{\ln(1.23\,\kappa)}{\sqrt{2}\,\kappa}\,H_c \quad (\kappa \gg 1) \\ H_{c2} &= \sqrt{2}\,\kappa\,H_c \quad , \end{aligned} \tag{1.157}$$

where H_c is the thermodynamic critical field. Note in general that if E_v is the energy of a single vortex, then the lower critical field is given by the relation $H_{c1}\phi_L = 4\pi E_v$, *i.e.*

$$H_{c1} = \frac{4\pi E_v}{\phi_L} \quad . \tag{1.158}$$

1.5.6 Abrikosov vortex lattice

Consider again the linearized GL equation $-\big(\kappa^{-1}\boldsymbol{\nabla} + i\boldsymbol{a}\big)^2\psi = \psi$ with $\boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{a} = b\,\hat{\boldsymbol{z}}$, with $b = \kappa$, *i.e.* $B = H_{c2}$. We chose the gauge $\boldsymbol{a} = \frac{1}{2}b\,(-y, x, 0)$. We showed that $\psi(\boldsymbol{\rho})$ with no z-dependence is an eigenfunction with unit eigenvalue. Recall also that $\gamma\,\psi(\boldsymbol{\rho}) = 0$, where

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{2}}\left(\frac{1}{i\kappa}\frac{\partial}{\partial x} - \frac{\kappa}{2}\,y - \frac{1}{\kappa}\frac{\partial}{\partial y} - \frac{i\kappa}{2}\,x\right) \\ &= \frac{\sqrt{2}}{i\kappa}\left(\frac{\partial}{\partial w} + \tfrac{1}{4}\kappa^2\bar{w}\right) \quad , \end{aligned} \tag{1.159}$$

where $w = x + iy$ and $\bar{w} = x - iy$ are complex. To find general solutions of $\gamma\,\psi = 0$, note that

$$\gamma = \frac{\sqrt{2}}{i\kappa}\,e^{-\kappa^2\bar{w}w/4}\,\frac{\partial}{\partial w}\,e^{+\kappa^2\bar{w}w/4} \quad . \tag{1.160}$$

Thus, $\gamma\,\psi(x,y)$ is satisfied by any function of the form

$$\psi(x,y) = f(\bar{w})\,e^{-\kappa^2\bar{w}w/4} \quad . \tag{1.161}$$

where $f(\bar{w})$ is analytic in the complex coordinate $\bar{w}$. This set of functions is known as the *lowest Landau level*.

The most general such function[11] is of the form

$$f(\bar{w}) = C\prod_i(\bar{w} - \bar{w}_i) \quad , \tag{1.162}$$

where each $\bar{w}_i$ is a zero of $f(\bar{w})$. Any analytic function on the plane is, up to a constant, uniquely specified by the positions of its zeros. Note that

$$\big|\psi(x,y)\big|^2 = |C|^2\,e^{-\kappa^2\bar{w}w/2}\prod_i\big|w - w_i\big|^2 \equiv |C|^2\,e^{-\Phi(\boldsymbol{\rho})} \quad , \tag{1.163}$$

where

$$\Phi(\boldsymbol{\rho}) = \tfrac{1}{2}\kappa^2\boldsymbol{\rho}^2 - 2\sum_i \ln\big|\boldsymbol{\rho} - \boldsymbol{\rho}_i\big| \quad . \tag{1.164}$$

[11] We assume that ψ is square-integrable, which excludes poles in $f(\bar{w})$.

$\Phi(\boldsymbol{\rho})$ may be interpreted as the electrostatic potential of a set of point charges located at $\boldsymbol{\rho}_i$, in the presence of a uniform neutralizing background. To see this, recall that $\nabla^2 \ln\rho = 2\pi\,\delta(\boldsymbol{\rho})$, so

$$\nabla^2\Phi(\boldsymbol{\rho}) = 2\kappa^2 - 4\pi\sum_i \delta\big(\boldsymbol{\rho}-\boldsymbol{\rho}_i\big) \quad . \tag{1.165}$$

Therefore if we are to describe a state where the local density $|\psi|^2$ is uniform on average, we must impose $\langle\nabla^2\Phi\rangle = 0$, which says

$$\Big\langle \sum_i \delta(\boldsymbol{\rho}-\boldsymbol{\rho}_i)\Big\rangle = \frac{\kappa^2}{2\pi} \quad . \tag{1.166}$$

The zeroes $\boldsymbol{\rho}_i$ are of course the positions of (anti)vortices, hence the uniform state has vortex density $n_{\rm v} = \kappa^2/2\pi$. Recall that in these units each vortex carries $2\pi/\kappa$ London flux quanta, which upon restoring units is

$$\frac{2\pi}{\kappa}\cdot\sqrt{2}\,H_{\rm c}\,\lambda_{\rm L}^2 = 2\pi\cdot\sqrt{2}\,H_{\rm c}\,\lambda_{\rm L}\,\xi = \frac{hc}{e^*} = \phi_{\rm L} \quad . \tag{1.167}$$

Multiplying the vortex density $n_{\rm v}$ by the vorticity $2\pi/\kappa$, we obtain the magnetic field strength,

$$b = h = \frac{\kappa^2}{2\pi}\times\frac{2\pi}{\kappa} = \kappa \quad . \tag{1.168}$$

In other words, $H = H_{\rm c2}$.

Just below the upper critical field

Next, we consider the case where H is just below the upper critical field $H_{\rm c2}$. We write $\psi = \psi_0 + \delta\psi$, and $b = \kappa + \delta b$, with $\delta b < 0$. We apply the method of successive approximation, and solve for b using the second GL equation. This yields

$$b = h - \frac{|\psi_0|^2}{2\kappa} \qquad , \qquad \delta b = h - \kappa - \frac{|\psi_0|^2}{2\kappa} \tag{1.169}$$

where $\psi_0(\boldsymbol{\rho})$ is our initial solution for $\delta b = 0$. To see this, note that the second GL equation may be written

$$\boldsymbol{\nabla}\times(\boldsymbol{h}-\boldsymbol{b}) = \tfrac{1}{2}\Big(\psi^*\,\boldsymbol{\pi}\,\psi + \psi\,\boldsymbol{\pi}^*\,\psi^*\Big) = \mathsf{Re}\,\Big(\psi^*\,\boldsymbol{\pi}\,\psi\Big) \quad , \tag{1.170}$$

where $\boldsymbol{\pi} = -i\kappa^{-1}\boldsymbol{\nabla} + \boldsymbol{a}$. On the RHS we now replace ψ by ψ_0 and b by κ, corresponding to our lowest order solution. This means we write $\boldsymbol{\pi} = \boldsymbol{\pi}_0 + \delta\boldsymbol{a}$, with $\boldsymbol{\pi}_0 = -i\kappa^{-1}\boldsymbol{\nabla} + \boldsymbol{a}_0$, $\boldsymbol{a}_0 = \frac{1}{2}\kappa\,\hat{\boldsymbol{z}}\times\boldsymbol{\rho}$, and $\boldsymbol{\nabla}\times\delta\boldsymbol{a} = \delta b\,\hat{\boldsymbol{z}}$. Assuming $h - b = |\psi_0|^2/2\kappa$, we have

$$\begin{aligned}\boldsymbol{\nabla}\times\bigg(\frac{|\psi_0|^2}{2\kappa}\hat{\boldsymbol{z}}\bigg) &= \frac{1}{2\kappa}\bigg[\frac{\partial}{\partial y}\,(\psi_0^*\,\psi_0)\,\hat{\boldsymbol{x}} - \frac{\partial}{\partial x}\,(\psi_0^*\,\psi_0)\,\hat{\boldsymbol{y}}\bigg] \\ &= \frac{1}{\kappa}\,\mathsf{Re}\,\Big[\psi_0^*\,\partial_y\,\psi_0\,\hat{\boldsymbol{x}} - \psi_0^*\,\partial_x\,\psi_0\,\hat{\boldsymbol{y}}\Big] \\ &= \mathsf{Re}\,\Big[\psi_0^*\,i\pi_{0y}\,\psi_0\,\hat{\boldsymbol{x}} - \psi_0^*\,i\pi_{0x}\,\psi_0\,\hat{\boldsymbol{y}}\Big] = \mathsf{Re}\,\Big[\psi_0^*\,\boldsymbol{\pi}_0\,\psi_0\Big] \quad ,\end{aligned} \tag{1.171}$$

since $i\pi_{0y} = \kappa^{-1}\partial_y + ia_{0y}$ and $\mathsf{Re}\,\big[i\psi_0^*\,\psi_0\,a_{0y}\big] = 0$. Note also that since $\gamma\,\psi_0 = 0$ and $\gamma = \frac{1}{\sqrt{2}}\big(\pi_{0x} - i\pi_{0y}\big) = \frac{1}{\sqrt{2}}\pi_0^\dagger$, we have $\pi_{0y}\psi_0 = -i\pi_{0x}\psi_0$ and, equivalently, $\pi_{0x}\psi_0 = i\pi_{0y}\psi_0$.

Inserting this result into the first GL equation yields an inhomogeneous equation for $\delta\psi$. The original equation is

$$\Big(\boldsymbol{\pi}^2 - 1\Big)\psi = -|\psi|^2\psi \quad . \tag{1.172}$$

With $\boldsymbol{\pi} = \boldsymbol{\pi}_0 + \delta\boldsymbol{a}$, we then have

$$\Big(\boldsymbol{\pi}_0^2 - 1\Big)\delta\psi = -\delta\boldsymbol{a}\cdot\boldsymbol{\pi}_0\,\psi_0 - \boldsymbol{\pi}_0\cdot\delta\boldsymbol{a}\,\psi_0 - |\psi_0|^2\psi_0 \quad . \tag{1.173}$$

The RHS of the above equation must be orthogonal to ψ_0, since $\big(\boldsymbol{\pi}_0^2 - 1\big)\psi_0 = 0$. That is to say,

$$\int\! d^2r\;\psi_0^*\,\Big[\delta\boldsymbol{a}\cdot\boldsymbol{\pi}_0 + \boldsymbol{\pi}_0\cdot\delta\boldsymbol{a} + |\psi_0|^2\Big]\psi_0 = 0 \quad . \tag{1.174}$$

Note that

$$\delta\boldsymbol{a}\cdot\boldsymbol{\pi}_0 + \boldsymbol{\pi}_0\cdot\delta\boldsymbol{a} = \tfrac{1}{2}\,\delta a\,\pi_0^\dagger + \tfrac{1}{2}\,\pi_0^\dagger\,\delta a + \tfrac{1}{2}\,\delta\bar{a}\,\pi_0 + \tfrac{1}{2}\,\pi_0\,\delta\bar{a} \quad , \tag{1.175}$$

where

$$\pi_0 = \pi_{0x} + i\pi_{0y} \quad , \quad \pi_0^\dagger = \pi_{0x} - i\pi_{0y} \quad , \quad \delta a = \delta a_x + i\delta a_y \quad , \quad \delta\bar{a} = \delta a_x - i\delta a_y \quad . \tag{1.176}$$

We also have, from Eqn. 1.143,

$$\pi_0 = -2i\kappa^{-1}\big(\partial_{\bar{w}} - \tfrac{1}{4}\kappa^2 w\big) \qquad , \qquad \pi_0^\dagger = -2i\kappa^{-1}\big(\partial_w + \tfrac{1}{4}\kappa^2\bar{w}\big) \quad . \tag{1.177}$$

Note that

$$\begin{aligned} \pi_0^\dagger\,\delta a &= \big[\pi_0^\dagger\,,\,\delta a\big] + \delta a\,\pi_0^\dagger = -2i\kappa^{-1}\,\partial_w\,\delta a + \delta a\,\pi_0^\dagger \\ \delta\bar{a}\,\pi_0 &= \big[\delta\bar{a}\,,\,\pi_0\big] + \pi_0\,\delta\bar{a} = +2i\kappa^{-1}\,\partial_{\bar{w}}\,\delta\bar{a} + \pi_0\,\delta\bar{a} \end{aligned} \tag{1.178}$$

Therefore,

$$\int\! d^2r\;\psi_0^*\,\Big[\delta a\,\pi_0^\dagger + \pi_0\,\delta\bar{a} - i\kappa^{-1}\,\partial_w\,\delta a + i\kappa^{-1}\,\partial_{\bar{w}}\,\delta\bar{a} + |\psi_0|^2\Big]\psi_0 = 0 \quad . \tag{1.179}$$

We now use the fact that $\pi_0^\dagger\,\psi_0 = 0$ and $\psi_0^*\,\pi_0 = 0$ (integrating by parts) to kill off the first two terms inside the square brackets. The third and fourth term combine to give

$$-i\,\partial_w\,\delta a + i\,\partial_{\bar{w}}\,\delta\bar{a} = \partial_x\,\delta a_y - \partial_y\,\delta a_x = \delta b \quad . \tag{1.180}$$

Plugging in our expression for δb, we finally have our prize:

$$\int\! d^2r\,\Bigg[\bigg(\frac{h}{\kappa} - 1\bigg)\,|\psi_0|^2 + \bigg(1 - \frac{1}{2\kappa^2}\bigg)\,|\psi_0|^4\Bigg] = 0 \quad . \tag{1.181}$$

We may write this as

$$\bigg(1 - \frac{h}{\kappa}\bigg)\big\langle|\psi_0|^2\big\rangle = \bigg(1 - \frac{1}{2\kappa^2}\bigg)\big\langle|\psi_0|^4\big\rangle \quad , \tag{1.182}$$

where

$$\big\langle F(\boldsymbol{\rho})\big\rangle = \frac{1}{A}\int\! d^2\rho\;F(\boldsymbol{\rho}) \tag{1.183}$$

denotes the global spatial average of $F(\boldsymbol{\rho})$. It is customary to define the ratio

$$\beta_{\rm A} \equiv \frac{\big\langle|\psi_0|^4\big\rangle}{\big\langle|\psi_0|^2\big\rangle^2} \quad , \tag{1.184}$$

which depends on the distribution of the zeros $\{\boldsymbol{\rho}_i\}$. Note that

$$\big\langle|\psi_0|^2\big\rangle = \frac{1}{\beta_{\rm A}}\cdot\frac{\big\langle|\psi_0|^4\big\rangle}{\big\langle|\psi_0|^2\big\rangle} = \frac{2\kappa(\kappa - h)}{(2\kappa^2 - 1)\beta_{\rm A}} \quad . \tag{1.185}$$

Now let's compute the Gibbs free energy density. We have

$$\begin{aligned} g_s - g_n &= -\langle|\psi_0|^4\rangle + 2\,\langle(b-h)^2\rangle \\ &= -\left(1 - \frac{1}{2\kappa^2}\right)\langle|\psi_0|^4\rangle = -\left(1 - \frac{h}{\kappa}\right)\langle|\psi_0|^2\rangle = -\frac{2\,(\kappa-h)^2}{(2\kappa^2-1)\beta_A} \quad . \end{aligned} \tag{1.186}$$

Since $g_n = -2\,h^2$, we have, restoring physical units

$$g_s = -\frac{1}{8\pi}\left[H^2 + \frac{(H_{c2}-H)^2}{(2\kappa^2-1)\beta_A}\right] \quad . \tag{1.187}$$

The average magnetic field is then

$$\bar{B} = -4\pi\,\frac{\partial g_s}{\partial H} = H - \frac{H_{c2}-H}{(2\kappa^2-1)\beta_A} \quad , \tag{1.188}$$

hence

$$M = \frac{B-H}{4\pi} = \frac{H-H_{c2}}{4\pi\,(2\kappa^2-1)\,\beta_A} \quad \Rightarrow \quad \chi = \frac{\partial M}{\partial H} = \frac{1}{4\pi\,(2\kappa^2-1)\,\beta_A} \quad . \tag{1.189}$$

Clearly g_s is minimized by making β_A as small as possible, which is achieved by a regular lattice structure. Since $\beta_A^{\text{square}} = 1.18$ and $\beta_A^{\text{triangular}} = 1.16$, the triangular lattice just barely wins.

Just above the lower critical field

When H is just slightly above H_{c1}, vortex lines penetrate the superconductor, but their density is very low. To see this, we once again invoke the result of Eqn. 1.152, extending that result to the case of many vortices:

$$\frac{G_{VL} - G_0}{L} = \frac{H_c^2\,\lambda_L^2}{4\pi}\int\! d^2\!\rho\,\Big\{\boldsymbol{b}\cdot\big[\boldsymbol{b} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\big] - 2\boldsymbol{h}\cdot\boldsymbol{b}\Big\} \quad . \tag{1.190}$$

Here we have

$$\begin{aligned} \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{b} &= -\frac{2\pi}{\kappa}\sum_i n_i\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_i) \\ \boldsymbol{b} &= -\frac{1}{\kappa}\sum_i n_i\,K_0\big(|\boldsymbol{\rho}-\boldsymbol{\rho}_i|\big) \quad . \end{aligned} \tag{1.191}$$

Thus, again replacing $K_0(0)$ by $K_0(\kappa^{-1})$ and invoking Eqn. 1.67 for $\kappa \gg 1$,

$$\frac{G_{VL} - G_0}{L} = \frac{H_c^2\,\lambda_L^2}{\kappa^2}\left\{\tfrac{1}{2}\ln(1.23\,\kappa)\sum_i n_i^2 + \sum_{i<j} n_i\,n_j\,K_0\big(|\boldsymbol{\rho}_i-\boldsymbol{\rho}_j|\big) + \kappa h\sum_i n_i\right\} \quad . \tag{1.192}$$

The first term on the RHS is the self-interaction, cut off at a length scale κ^{-1} (ξ in physical units). The second term is the interaction between different vortex lines. We've assumed a perfectly straight set of vortex lines – no wiggling! The third term arises from $\boldsymbol{B}\cdot\boldsymbol{H}$ in the Gibbs free energy. If we assume a finite density of vortex lines, we may calculate the magnetization. For $H - H_{c1} \ll H_{c1}$, the spacing between the vortices is huge, and since $K_0(r) \simeq (\pi/2r)^{1/2}\exp(-r)$ for large $|r|$, we may safely neglect all but nearest neighbor interaction terms. We assume $n_i = -1$ for all i. Let the vortex lines form a regular lattice of coordination number z and nearest neighbor separation d. Then

$$\frac{G_{VL} - G_0}{L} = \frac{N H_c^2\,\lambda_L^2}{\kappa^2}\left\{\tfrac{1}{2}\ln(1.23\,\kappa) + \tfrac{1}{2}zK_0(d) - \kappa h\right\} \quad , \tag{1.193}$$

where N is the total number of vortex lines, given by $N = A/\Omega$ for a lattice with unit cell area Ω. Assuming a triangular lattice, $\Omega = \frac{\sqrt{3}}{2}\,d^2$ and $z = 6$. Then

$$\frac{G_{\rm VL} - G_0}{L} = \frac{H_{\rm c}^2\,\lambda_{\rm L}^2}{\sqrt{3}\,\kappa^2}\,\Big\{\big[\ln(1.23\,\kappa) - 2\kappa h\big]\,d^{-2} + 6d^{-2}K_0(d)\Big\} \quad . \tag{1.194}$$

Provided $h > h_{\rm c1} = \ln(1.23\,\kappa)/2\kappa$, this is minimized at a finite value of d.