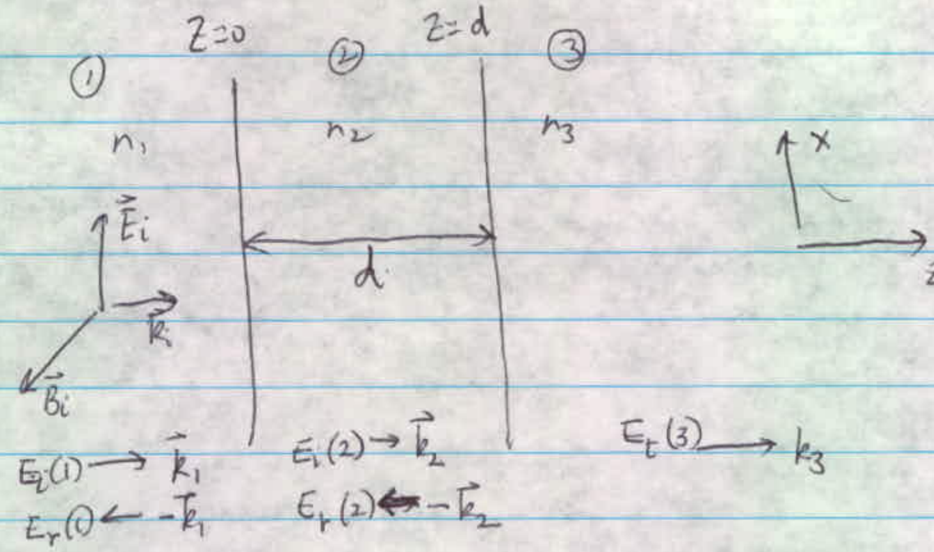
7.2



Electric field is given by

Region 1 $\quad \left(E_i(1)\, e^{ik_1 z} + E_r(1)\, e^{-ik_1 z}\right) \hat{x} \times e^{-i\omega t}$

Region 2 $\quad \left(E_i(2)\, e^{ik_2 z} + E_r(2)\, e^{-ik_2 z}\right) \hat{x} \times e^{-i\omega t}$

Region 3 $\quad E_t(3)\, e^{ik_3 z}\ \hat{x} \times e^{-i\omega t}$

Assume all μ's are $\simeq \mu_0$. $\quad n_1 = \sqrt{\frac{\epsilon_1}{\epsilon_0}},\ n_2 = \sqrt{\frac{\epsilon_2}{\epsilon_0}},\ n_3 = \sqrt{\frac{\epsilon_3}{\epsilon_0}}$

$$k_1 = n_1 k_0,\ k_2 = n_2 k_0,\ k_3 = n_3 k_0 \qquad k_0 = \omega/c$$

Boundary conditions at $z=0$:

$$E_i(1) + E_r(1) = E_i(2) + E_r(2) \qquad \text{(from tangential E field)}$$

$$n_1 E_i(1) - n_1 E_r(1) = n_2 E_i(2) - n_2 E_r(2) \qquad \text{(From tangential H field)}$$

$$\text{or} \quad \begin{pmatrix} E_i(2) \\ E_r(2) \end{pmatrix} = \underset{\sim}{t_1} \begin{pmatrix} E_i(1) \\ E_r(1) \end{pmatrix} ; \quad \underset{\sim}{t_1} \equiv \begin{pmatrix} \frac{n_1+n_2}{2n_2} & \frac{n_2-n_1}{2n_2} \\ \frac{n_2-n_1}{2n_2} & \frac{n_1+n_2}{2n_2} \end{pmatrix}$$

B. C. at $z=d$

$$E_i(2)\, e^{ik_2 d} + E_r(2)\, e^{-ik_2 d} = E_t(3)\, e^{ik_3 d}$$

$$n_2\left(E_i(2) e^{ik_2 d}\right) - n_2 E_r(2)\, e^{-ik_2 d} = n_3 E_t(3)\, e^{ik_3 d}$$

$$\text{or} \quad \begin{pmatrix} E_t(3) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{n_2+n_3}{2n_3} e^{i(k_2-k_3)d} & \frac{n_3-n_2}{2n_3} e^{-i(k_2+k_3)d} \\ \frac{n_3-n_2}{2n_3} e^{i(k_2-k_3)d} & \frac{n_3+n_2}{2n_3} e^{-i(k_2+k_3)d} \end{pmatrix} \begin{pmatrix} E_i(2) \\ E_r(2) \end{pmatrix}$$

Define as $\underset{\sim}{t_2}$.

so $$\begin{pmatrix}E_t(3)\\ 0\end{pmatrix} = \overset{\underset{\sim}{t_2}\underset{\sim}{t_1}}{\underset{\sim}{T}}\begin{pmatrix}E_i(1)\\ E_r(1)\end{pmatrix} = \begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}\begin{pmatrix}E_i(1)\\ E_r(1)\end{pmatrix} \qquad (1)$$

$T = \underset{\sim}{t_2}\underset{\sim}{t_1}$ has the following matrix elements:

$$T_{11} = \left(\frac{n_2+n_3}{2n_3}\right)\left(\frac{n_1+n_2}{2n_2}\right)e^{i(n_2-n_3)k_0 d} + \frac{(n_3-n_2)}{2n_3}\frac{(n_2-n_1)}{2n_2}e^{-i(n_2+n_3)k_0 d}$$

$$T_{12} = \left(\frac{n_2+n_3}{2n_3}\right)\left(\frac{n_2-n_1}{2n_2}\right)e^{i(n_2-n_3)k_0 d} + \left(\frac{n_3-n_2}{2n_3}\right)\left(\frac{n_1+n_2}{2n_2}\right)e^{-i(n_2+n_3)k_0 d}$$

$$T_{21} = \left(\frac{n_3-n_2}{2n_3}\right)\left(\frac{n_1+n_2}{2n_2}\right)e^{i(n_2-n_3)k_0 d} + \left(\frac{n_2+n_3}{2n_3}\right)\left(\frac{n_2-n_1}{2n_2}\right)e^{-i(n_2+n_3)k_0 d}$$

$$T_{22} = \left(\frac{n_3-n_2}{2n_3}\right)\left(\frac{n_2-n_1}{2n_2}\right)e^{i(n_2-n_3)k_0 d} + \left(\frac{n_2+n_3}{2n_3}\right)\left(\frac{n_1+n_2}{2n_2}\right)e^{-i(n_2+n_3)k_0 d}$$

From Eq. (1) above,

$$\frac{E_r(1)}{E_i(1)} = -\frac{T_{21}}{T_{22}} = r$$

$$\frac{E_t(3)}{E_i(1)} = \left[T_{11} - \frac{T_{12}T_{21}}{T_{22}}\right] = t$$

From Poynting Vector, intensities are:

$$I_i(1) = \frac{1}{2}\sqrt{\frac{\epsilon_1}{\mu_1}}\,|E_i(1)|^2$$

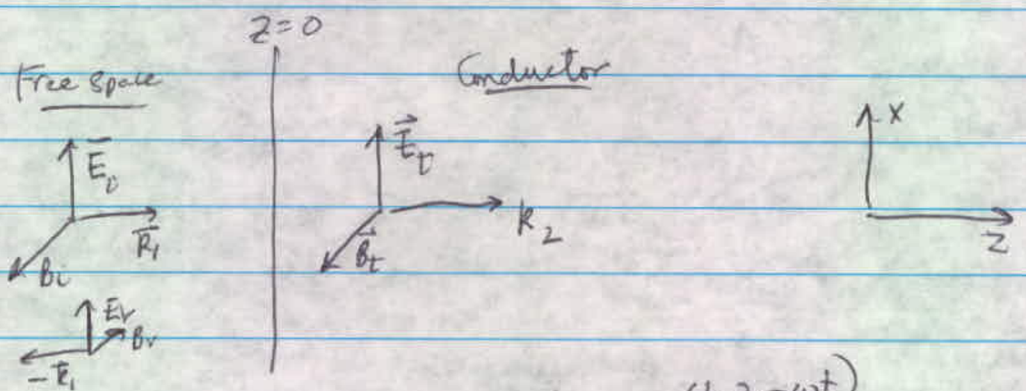
$$I_r(1) = \frac{1}{2}\sqrt{\frac{\epsilon_1}{\mu_1}}\,|E_r(1)|^2$$

$$I_t(3) = \frac{1}{2}\sqrt{\frac{\epsilon_3}{\mu_3}}\,|E_t(3)|^2$$

Reflection Coefficient $R = \dfrac{|E_r(1)|^2}{|E_t(1)|^2} = r^2$

Transmission Coefficient $\cancel{T} = \dfrac{n_3}{n_1}\left|\dfrac{E_t(3)}{E_i(1)}\right|^2 = \dfrac{n_3}{n_1}|t|^2$

7.4. (a) Write waves as



Write waves as

$$E_i(z,t) = E_{0i}\, e^{i(k_1 z - \omega t)}\,\hat{x}$$
$$B_i(z,t) = \frac{1}{c} E_{0i}\, e^{i(k_1 z - \omega t)}\,\hat{y}$$

$B = \frac{k}{\omega} E$

If free space $\frac{k}{\omega} = \frac{1}{c}$

In conductor, $\tilde{k}_2$ is complex

$$E_r(z,t) = E_{0r}\, e^{i(-k_1 z - \omega t)}\,\hat{x}$$
$$B_r(z,t) = -\frac{1}{c} E_{0r}\, e^{i(-k_1 z - \omega t)}\,\hat{y}$$
$$E_t(z,t) = E_{0t}\, e^{i(\tilde{k}_2 z - \omega t)}\,\hat{x}$$
$$B_t(z,t) = \frac{\tilde{k}_2}{\omega} E_{0t}\, e^{i(\tilde{k}_2 z - \omega t)}\,\hat{y}$$

$\tilde{k}_2$ = complex wave number in conductor $= k_2 + i\kappa_2$

$$\tilde{k}_2^2 = \mu_0\left(\epsilon_2\omega^2 + i\sigma_2\omega\right) \rightarrow \quad k_2 = \sqrt{\frac{\mu_0\epsilon_2}{2}}\,\omega\left[\sqrt{1+\left(\frac{\sigma_2}{\epsilon_2\omega}\right)^2} + 1\right]^{\frac{1}{2}}$$

$$\kappa_2 = \sqrt{\frac{\mu_0\epsilon_2}{2}}\,\omega\left[\sqrt{1+\left(\frac{\sigma_2}{\epsilon_2\omega}\right)^2} - 1\right]^{\frac{1}{2}}$$

$\kappa_2 = \frac{1}{\text{skin depth}}$

Boundary conditions that can be used are

$$E_{0i} + E_{0r} = E_{0t}$$
$$\frac{1}{c}(E_{0i} - E_{0r}) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{0t}$$

gives $\frac{E_{0r}}{E_{0i}} = \frac{1-\beta}{1+\beta} = r \;\Big|\; \beta = \frac{c}{\omega}\tilde{k}_2$ (complex)

~~Amplitude~~

$$r = \frac{E_{0r}}{E_{0i}} = A e^{i\phi} = \frac{\left(1 - \frac{c}{\omega}k_2\right) - i\frac{c}{\omega}K_2}{\left(1 + \frac{c}{\omega}k_2\right) + i\frac{c}{\omega}K_2}$$

$$A = \left[|r|^2\right]^{\frac{1}{2}} = \left\{\frac{\left(1 - \frac{c}{\omega}k_2\right)^2 + \frac{c^2K_2^2}{\omega^2}}{\left(1 + \frac{c}{\omega}k_2\right)^2 + \frac{c^2}{\omega^2}K_2^2}\right\}^{\frac{1}{2}}$$

$$\phi = -\,\mathrm{Tan}^{-1}\frac{\frac{c}{\omega}K_2}{1 - \frac{c}{\omega}k_2}$$

(b) Consider limit $\sigma_2 \to 0$ then $k_2 \to \omega\sqrt{\mu_0\epsilon_2}$ $K_2 \to 0$

$\beta \to \frac{c}{\omega}k_2 = \sqrt{\frac{\epsilon_2}{\epsilon_0}} = n_2$ = Refractive index of conductor

$r \to \frac{1-n_2}{1+n_2}$ (real) $R = |r|^2 = \left|\frac{1-n_2}{1+n_2}\right|^2$

If $\sigma_2 \gg \epsilon_2\omega$, $k_2^2 = \mu_0\omega(\epsilon_2\omega + i\sigma) \simeq i\sigma\omega\mu_0$

$$\beta = k_2\frac{c}{\omega} \simeq \frac{c}{\omega}(i)^{\frac{1}{2}}(\mu_0\sigma\omega)^{\frac{1}{2}} = (k_2 + iK_2)\frac{c}{\omega}$$

Now $i^{\frac{1}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = \frac{1}{2^{\frac{1}{2}}}(1+i)$

$$k_2 = \left(\frac{\mu_0\sigma\omega}{2}\right)^{\frac{1}{2}} = \frac{1}{\delta} \qquad K_2 = \left(\frac{\mu\sigma\omega}{2}\right)^{\frac{1}{2}} = \frac{1}{\delta}$$

δ = skin depth (defined as $\frac{1}{K_2}$)

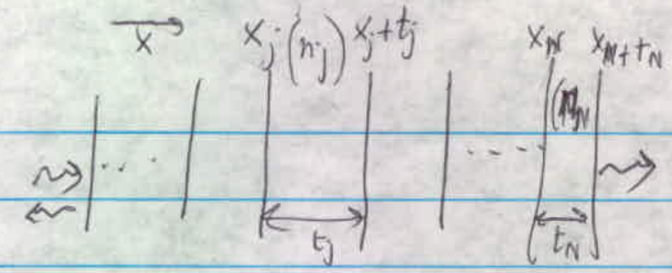
Then $r = \frac{1-\beta}{1+\beta} = \frac{\left(1 - \frac{c}{\omega\delta}\right) - i\frac{c}{\omega\delta}}{\left(1 + \frac{c}{\omega\delta}\right) + i\frac{c}{\omega\delta}}$

Then

$$R = |r|^2 = \frac{\left(1 - \frac{c}{\omega\delta}\right)^2 + \frac{c^2}{\omega^2\delta^2}}{\left(1 + \frac{c}{\omega\delta}\right)^2 + \frac{c^2}{\omega^2\delta^2}} = \frac{1 + \frac{2c}{\omega\delta} + \frac{c^2}{\omega^2\delta^2}}{1 + \frac{2c}{\omega\delta} + \frac{c^2}{\omega^2\delta^2}} - \frac{\frac{4c}{\omega\delta}}{1 + \frac{2c}{\omega\delta} + \frac{c^2}{\omega^2\delta^2}}$$

$$\simeq 1 - 2\frac{\omega}{c}\delta \quad \text{if } \omega\delta \ll c$$

7.8



In j^{th} layer $\quad E = E_+ e^{ik_j x} + E_- e^{-ik_j x}$

$$k_j = n_j \frac{\omega}{c}$$

E field on left side of j^{th} layer

$$E = E_+^l(j) e^{ik_j x_j} + E_-^l(j) e^{-ik_j x_j}$$

E field on right side of j^{th} layer

$$E = E_+^r(j) e^{ik_x x_j} + E_-^r(j) e^{-ik_j x_j}$$

where
$$\begin{pmatrix} E_+^r(j) \\ E_-^r(j) \end{pmatrix} = \underbrace{\begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix}}_{\underset{\sim}{T}_{layer}(j)} \begin{pmatrix} E_+^l(j) \\ E_-^l(j) \end{pmatrix}$$

Boundary conditions at interface give (see Problem 7.2 solution)

$$\begin{pmatrix} E_+^l(j+1) \\ E_-^l(j+1) \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} 1+\frac{n_1}{n_2} & -\left(\frac{n_1}{n_2}-1\right) \\ -\left(\frac{n_1}{n_2}-1\right) & \left(1+\frac{n_1}{n_2}\right) \end{pmatrix}}_{\underset{\sim}{T}_{interface}(j,j+1)} \begin{pmatrix} E_+^r(j) \\ E_-^r(j) \end{pmatrix}$$

so
$$\begin{pmatrix} E_+^l(j+1) \\ E_-^l(j+1) \end{pmatrix} = \underset{\sim}{T}_{interface}(j,j+1)\, \underset{\sim}{T}_{layer}(j) \begin{pmatrix} E_+^l(j) \\ E_-^l(j) \end{pmatrix}$$

$$= \left[\underset{\sim}{T}_{interface}(j,j+1)\, \underset{\sim}{T}_{layer}(j)\, \underset{\sim}{T}_{interface}(j-1,j)\, \underset{\sim}{T}_{layer}(j-1) \cdots\cdots \right] \begin{pmatrix} E_i \\ E_r \end{pmatrix}$$

at end
$$\begin{pmatrix} E_t \\ 0 \end{pmatrix} = \underbrace{\underset{\sim}{T}_{interface}(N, N+1)\, \underset{\sim}{T}_{layer}(N) \cdots\cdots \underset{\sim}{T}_{interface}(1,0)}_{\underset{\sim}{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}} \begin{pmatrix} E_i \\ E_r \end{pmatrix}$$

so $$E_t = t_{11} E_i + t_{12} E_r$$
$$0 = t_{21} E_i + t_{22} E_r$$

$$\longrightarrow \frac{E_r}{E_i} = -\frac{t_{21}}{t_{22}} \qquad \frac{E_t}{E_i} = t_{11} - \frac{t_{12} t_{21}}{t_{22}} = \frac{t_{11} t_{22} - t_{12} t_{21}}{t_{22}}$$
$$= \frac{\mathrm{Det}\,|t|}{t_{22}}$$

7-12 Assume $\rho(x,t) = \rho(x,\omega)\, e^{-i\omega t}$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}(x,t)$$

Taking time Fourier transforms $-i\omega\rho(x,\omega) = -\nabla\cdot\vec{J}(x,\omega)$

But $\vec{J}(x,\omega) = \sigma(\omega)\,\vec{E}(\omega)$

$\Rightarrow\; -i\omega\rho(x,\omega) = -\sigma(\omega)\left(\nabla\cdot\vec{E}(\omega)\right)$

But $\nabla\cdot\vec{E}(\omega) = \dfrac{\rho(\cancel{x},\omega)}{\epsilon_0}$ (Maxwell-Gauss)

$\Rightarrow\; -i\omega\rho(\vec{x},\omega) = -\dfrac{\sigma(\omega)}{\epsilon_0}\rho(\vec{x},\omega)$

or $\left[\sigma(\omega) - i\omega\epsilon_0\right]\rho(\vec{x},\omega) = 0$

This is a secular equation for oscillations of $\rho(\vec{x},\omega)$

Solve :- $\sigma(\omega) = i\omega\epsilon_0$

$$\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau} = i\omega\epsilon_0$$

yields quadratic equation $\omega^2\tau - i\omega - \omega_p^2\tau = 0$

$$\longrightarrow\quad \omega = \frac{i}{2\tau} \pm \frac{1}{2\tau}\sqrt{-1+4\omega_p^2\tau^2}$$

$\Rightarrow$ if $\omega_p\tau \gg 1$, $\quad\omega = \pm\omega_p + \frac{i}{2\tau}$

$\Rightarrow$ electron gas oscillates at ω_p & decays as $-\lambda t$

$\lambda = \frac{1}{2\tau}$

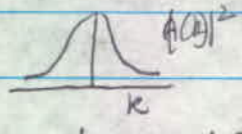
(8)

7.19. $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$

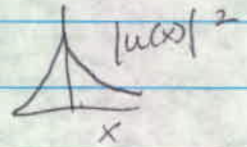
(a) $A(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha|x|}{2}} e^{-ikx} dx = \frac{2N}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{\alpha x}{2}} \cos kx \, dx$

$= \frac{2N}{\sqrt{2\pi}} \frac{\left(\frac{\alpha}{2}\right)}{\left(\frac{\alpha}{2}\right)^2 + k^2} = \frac{\alpha N}{\sqrt{2\pi}} \frac{1}{\frac{\alpha^2}{4} + k^2}$

$|A(k)|^2 = \frac{(\alpha N)^2}{2\pi} \frac{1}{\left(\frac{\alpha^2}{4}+k^2\right)^2} \longrightarrow$



$|f(x)|^2 = N^2 e^{-\alpha|x|} \longrightarrow$



$(\Delta k)^2 = \frac{\int_0^{\infty} \frac{k^2}{\left(\frac{\alpha^2}{4}+k^2\right)^2} dk}{\int_0^{\infty} \frac{1}{\left(\frac{\alpha^2}{4}+k^2\right)^2} dk} = \frac{\frac{2}{\alpha}}{\frac{2\pi}{\alpha^3}} = \frac{\alpha^2}{\pi} \qquad \Delta k = \frac{\alpha}{\sqrt{\pi}}$

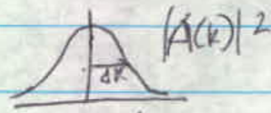
$(\Delta x)^2 = \frac{\int_0^{\infty} e^{-\alpha x} x^2 dx}{\int_0^{\infty} e^{-\alpha x} dx} = \frac{\left(\frac{2}{\alpha^3}\right)}{\left(\frac{1}{\alpha}\right)} = \frac{2}{\alpha^2} \qquad \Delta x = \frac{\sqrt{2}}{\alpha}$

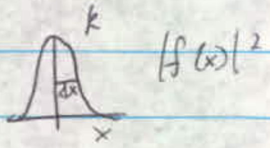
(b) $A(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2 x^2}{4}} e^{-ikx} dx = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2 x^2}{4}} \cos kx \, dx$

$= \frac{N}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\alpha} 2 e^{-\frac{k^2}{\alpha^2}} = \frac{\sqrt{2} N}{\alpha} e^{-k^2/\alpha^2}$

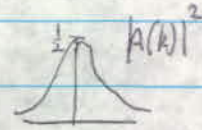
$|A(k)|^2 = \frac{2N^2}{\alpha^2} e^{-\frac{2k^2}{\alpha^2}}$

$(\Delta k)^2 = \frac{\int_0^{\infty} e^{-\frac{2k^2}{\alpha^2}} k^2 dk}{\int_0^{\infty} e^{-\frac{2k^2}{\alpha^2}} dk} = \frac{\alpha^2}{4} \qquad \Delta k = \frac{\alpha}{2}$

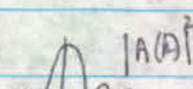




$(\Delta x)^2 = \frac{\int_0^{\infty} e^{-\frac{x^2 \alpha^2}{2}} x^2 dx}{\int_0^{\infty} e^{-\frac{x^2 \alpha^2}{4}} dx} = \frac{1}{\alpha^2} \qquad (\Delta x) = \frac{1}{\alpha}.$



(c) $A(k) = \frac{2N}{\sqrt{2\pi}} \int_0^{\frac{1}{\alpha}} \left[1 - \alpha x\right] \cos kx \, dx = \frac{\alpha}{k^2}\left(1 - \cos\left(\frac{k}{\alpha}\right)\right)$



(d) $A(k) = \frac{2N}{\sqrt{2\pi}} \int_0^{a} \cos kx = \frac{2N}{\sqrt{2\pi}} a \left(\frac{\sin ka}{ka}\right)$