PHYSICS 161 Prof: Kim Griest
HOMEWORK 8 28 May 2014

1. An electron is a pointlike particle of zero radius, so it is natural to wonder whether an electron could be a black hole. However, a black hole of mass M cannot have an arbitrary angular momuntum J, or charge Q. These values must satisfy an inequality,

$$\left(\frac{GM}{c^2}\right)^2 \geq G\left(\frac{Q}{c^2}\right)^2 + \left(\frac{J}{Mc}\right)^2.$$

If this inequality were violated, the singularity would be found *outside* the event horizon violating the Cosmic Censorship hypothesis.
1(a) Check the units of the above inequality and show each term is in units of cm^2.
1(b) Use $\hbar/2$ for the angular momentum of the electron to determine whether or not an electron could be a black hole. (Note: you need to use cgs units throughout to make this problem work, that is, charge on electron is $4.8 \times 10^{-10} esu$, and all quantities in centimeters, grams, and seconds.)

2(a) Using the Kerr metric calculate the surface area of the horizon of a rotating black hole. You should get $A = 4\pi(r_+^2 + a^2)$. [Hint: if you have trouble, first try using the Schwarzschild metric to find the area of the horizon of a Schwarzschild black hole $4\pi r_S^2$. Use the same technique on the Kerr metric, but remember the horizon of a Kerr hole has $\Delta = 0$.]
2(b) Show that this can also be expressed as $A = 8\pi GM[GM + \sqrt{G^2M^2 - a^2}]$.
2(c) Venture a guess as to why is this not $4\pi r_S^2$ as it was for the Schwarzschild metric?

3. Using the Hawking Area theorem (area of horizons of black holes cannot decrease in classical processes), and the result of problem 2 find the maximum amount of energy that can be obtained from a Kerr black hole from the Penrose process (or any other classical process).
3(a) Energy extracted from a BH reduces its mass. But that will also reduce its surface area, unless its angular momentum, $a = J/M$, is also reduced by enough to compensate. So the question becomes what is the most you can reduce a to?
3(b) Suppose you extract energy from a spinning hole, reducing M to M' and a to its minimal value (part (a)). Set the horizon surface area of this new reduced hole equal to the surface area of the original hole (thus not reducing its area which would be against the Hawking area theorem.) Solve for the new mass M' in terms of the original mass and original a. This mass is called the irreducible mass.
3(c) Using this result find a formula for the maximal amount of energy that can be extracted classically from a spinning black hole.
3(d) Using this formula for a maximally spinning hole, show that around 29% of the original rest mass of a hole can be extracted by the Penrose process. In fact, this is a theoretical limit that can not be reached in practice. [Note, this is different than the 42% of rest mass of garbage that can be got from throwing garbage into the inmost stable orbit; here we are not throwing stuff in, but getting net energy out.]

4(a) Prof. Andrea Ghez at UCLA has observed stars are orbiting the $4 \times 10^7 M_\odot$ black hole at the center of our Milky Way Galaxy. Estimate the speed of these stars in km/sec. You may assume the stars are in face-on circular orbits of radius 0.1 arcsec. The center of the Milky Way is about 25000 lightyears away. Use just regular Newtonian mechanics.
[Hint: first convert the angle into km using the definition of a light year and the small angle approximation for triangles.]
4(b) How long will it take for these stars to make one complete orbit? Express your answer in years.