Problem 12.2

(a) $m_A\mathbf{u}_A + m_B\mathbf{u}_B = m_C\mathbf{u}_C + m_D\mathbf{u}_D$; $\quad \mathbf{u}_i = \bar{\mathbf{u}}_i + \mathbf{v}$.

$m_A(\bar{\mathbf{u}}_A + \mathbf{v}) + m_B(\bar{\mathbf{u}}_B + \mathbf{v}) = m_C(\bar{\mathbf{u}}_C + \mathbf{v}) + m_D(\bar{\mathbf{u}}_D + \mathbf{v})$,

$m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B + (m_A + m_B)\mathbf{v} = m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D + (m_C + m_D)\mathbf{v}$.

Assuming *mass* is conserved, $(m_A + m_B) = (m_C + m_D)$, it follows that

$m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B = m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D$, so momentum is conserved in $\bar{\mathcal{S}}$.

(b) $\frac{1}{2}m_A u_A^2 + \frac{1}{2}m_B u_B^2 = \frac{1}{2}m_C u_C^2 + \frac{1}{2}m_D u_D^2 \Rightarrow$

$$\tfrac{1}{2}m_A(\bar{u}_A^2 + 2\bar{\mathbf{u}}_A\cdot\mathbf{v} + v^2) + \tfrac{1}{2}m_B(\bar{u}_B^2 + 2\bar{\mathbf{u}}_B\cdot\mathbf{v} + v^2) = \tfrac{1}{2}m_C(\bar{u}_C^2 + 2\bar{\mathbf{u}}_C\cdot\mathbf{v} + v^2) + \tfrac{1}{2}m_D(\bar{u}_D^2 + 2\bar{\mathbf{u}}_D\cdot\mathbf{v} + v^2)$$

$$\tfrac{1}{2}m_A\bar{u}_A^2 + \tfrac{1}{2}m_B\bar{u}_B^2 + 2\mathbf{v}\cdot(m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B) + \tfrac{1}{2}v^2(m_A + m_B)$$
$$= \tfrac{1}{2}m_C\bar{u}_C^2 + \tfrac{1}{2}m_D\bar{u}_D^2 + 2\mathbf{v}\cdot(m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D) + \tfrac{1}{2}v^2(m_C + m_D).$$

But the middle terms are equal by conservation of momentum, and the last terms are equal by conservation of mass, so $\frac{1}{2}m_A\bar{u}_A^2 + \frac{1}{2}m_B\bar{u}_B^2 = \frac{1}{2}m_C\bar{u}_C^2 + \frac{1}{2}m_D\bar{u}_D^2$. qed

Problem 12.13

Let brother's accident occur at origin, time zero, in both frames. In system $\mathcal{S}$ (Sophie's), the coordinates of Sophie's cry are $x = 5\times10^5$ m, $t = 0$. In system $\bar{\mathcal{S}}$ (scientist's), $\bar{t} = \gamma(t - \frac{v}{c^2}x) = -\gamma v x/c^2$. Since this is *negative*, **Sophie's cry occurred *before* the accident,** in $\bar{\mathcal{S}}$. $\gamma = \frac{1}{\sqrt{1-(12/13)^2}} = \frac{13}{\sqrt{169-144}} = \frac{13}{5}$. So $\bar{t} = -\left(\frac{13}{5}\right)\left(\frac{12}{13}c\right)(5\times10^5)/c^2 = -12\times10^5/3\times10^8 = -4\times10^{-3}$. $\boxed{4\times10^{-3}\text{ s earlier.}}$

Problem 12.14

(a) In $\mathcal{S}$ it moves a distance dy in time dt. In $\bar{\mathcal{S}}$, meanwhile, it moves a distance $d\bar{y} = dy$ in time $d\bar{t} = \gamma(dt - \frac{v}{c^2}dx)$.

$$\therefore \frac{d\bar{y}}{d\bar{t}} = \frac{dy}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{(dy/dt)}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)};\ \text{or}\ \boxed{\bar{u}_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)};\ \bar{u}_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}.}$$

(b) $\tan\bar{\theta} = -\dfrac{\bar{u}_y}{\bar{u}_x} = -\dfrac{u_y/\left[\gamma\left(1 - \frac{vu_x}{c^2}\right)\right]}{(u_x - v)/\left(1 - \frac{vu_x}{c^2}\right)} = \dfrac{1}{\gamma}\dfrac{(-u_y)}{(u_x - v)}$.

In this case $u_x = -c\cos\theta$; $u_y = c\sin\theta \Rightarrow \tan\bar{\theta} = \frac{1}{\gamma}\left(\frac{-c\sin\theta}{-c\cos\theta - v}\right)$.

$\boxed{\tan\bar{\theta} = \dfrac{1}{\gamma}\left(\dfrac{\sin\theta}{\cos\theta + v/c}\right).}$ [Compare $\tan\bar{\theta} = \gamma\frac{\sin\theta}{\cos\theta}$ in Prob. 12.10. The point is that *velocities* are sensitive not only to the transformation of *distances*, but also of *times*. That's why there is no *universal* rule for translating angles—you have to know whether it's an angle made by a *velocity* vector or a *position* vector.]

Problem 12.17

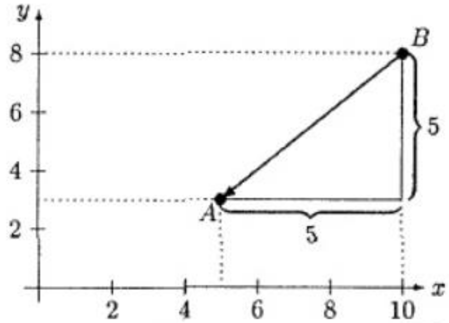
$$-\bar{a}^0\bar{b}^0 + \bar{a}^1\bar{b}^1 + \bar{a}^2\bar{b}^2 + \bar{a}^3\bar{b}^3 = -\gamma^2(a^0 - \beta a^1)(b^0 - \beta b^1) + \gamma^2(a^1 - \beta a^0)(b^1 - \beta a^0) + a^2b^2 + a^3b^3$$
$$= -\gamma^2(a^0b^0 - \beta a^0b^1 - \beta a^1b^0 + \beta^2a^1b^1 - a^1b^1 + \beta a^1b^0 + \beta a^0b^1 - \beta^2a^0b^0) + a^2b^2 + a^3b^3$$
$$= -\gamma^2a^0b^0(1-\beta^2) + \gamma^2a^1b^1(1-\beta^2) + a^2b^2 + a^3b^3$$
$$= -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3. \quad \text{qed} \quad [\textit{Note: } \gamma^2(1-\beta^2) = 1.]$$

Problem 12.20

(a) (i) $I = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -(5-15)^2 + (10-5)^2 + (8-3)^2 + (0-0)^2 = -100 + 25 + 25 = \boxed{-50.}$

(ii) $\boxed{\text{No.}}$ (In such a system $\Delta\bar{t} = 0$, so I would have to be *positive*, which it *isn't*.)

(iii) $\boxed{\text{Yes.}}$



$\bar{S}$ travels in the direction from B toward A, making the trip in time $10/c$.

$$\therefore \mathbf{v} = \frac{-5\hat{\mathbf{x}} - 5\hat{\mathbf{y}}}{10/c} = \boxed{-\frac{c}{2}\hat{\mathbf{x}} - \frac{c}{2}\hat{\mathbf{y}}.}$$

Note that $\frac{v^2}{c^2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $v = \frac{1}{\sqrt{2}}c$, safely less than c.

(b) (i) $I = -(3-1)^2 + (5-2)^2 + 0 + 0 = -4 + 9 = \boxed{5.}$

(ii) $\boxed{\text{Yes.}}$ By Lorentz transformation: $\Delta(c\bar{t}) = \gamma\left[\Delta(ct) - \beta(\Delta x)\right]$. We want $\Delta\bar{t} = 0$, so $\Delta(ct) = \beta(\Delta x)$; or $\frac{v}{c} = \frac{\Delta(ct)}{\Delta x} = \frac{(3-1)}{(5-2)} = \frac{2}{3}$. So $\boxed{v = \frac{2}{3}c,}$ in the $+x$ direction.

(iii) $\boxed{\text{No.}}$ (In such a system $\Delta x = \Delta y = \Delta z = 0$ so I would be negative, which it *isn't*.)

Problem 12.25

(a) $u_x = u_y = u\cos 45^\circ = \frac{1}{\sqrt{2}}\frac{2}{\sqrt{5}}c = \boxed{\sqrt{\frac{2}{5}}\,c.}$

(b) $\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-4/5}} = \frac{\sqrt{5}}{\sqrt{5-4}} = \sqrt{5}$; $\quad \eta = \frac{u}{\sqrt{1-u^2/c^2}} \Rightarrow \boxed{\eta_x = \eta_y = \sqrt{2}\,c.}$

(c) $\eta^0 = \gamma c = \boxed{\sqrt{5}\,c.}$

(d) Eq. 12.45$\Rightarrow \left\{ \begin{array}{l} \bar{u}_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{\sqrt{2/5}\,c - \sqrt{2/5}\,c}{1-\frac{2}{5}} = \boxed{0.} \\ \bar{u}_y = \frac{1}{\gamma}\left(\frac{u_y}{1-\frac{u_x v}{c^2}}\right) = \sqrt{1-\frac{2}{5}}\,\frac{\sqrt{2/5}\,c}{1-\frac{2}{5}} = \frac{\sqrt{2/5}}{\sqrt{3/5}}c = \boxed{\sqrt{\frac{2}{3}}\,c.} \end{array} \right.$

(e) $\bar{\eta}_x = \gamma(\eta_x - \beta\eta^0) = \sqrt{1-\frac{2}{5}}\left(\sqrt{2}\,c - \sqrt{\frac{2}{5}}\sqrt{5}\,c\right) = \boxed{0.}$ $\quad \boxed{\bar{\eta}_y = \eta_y = \sqrt{2}\,c.}$

(f) $\frac{1}{\sqrt{1-\bar{u}^2/c^2}} = \frac{1}{\sqrt{1-(2/3)}} = \sqrt{3}$; $\quad \bar{\eta} = \sqrt{3}\,\bar{\mathbf{u}} \Rightarrow \left\{ \begin{array}{l} \bar{\eta}_x = \sqrt{3}\,\bar{u}_x = 0. \checkmark \\ \bar{\eta}_y = \sqrt{3}\,\bar{u}_y = \sqrt{2}\,c. \checkmark \end{array} \right\}$