10.11

(a) As in Ex. 10.2, for $t < r/c$, $\mathbf{A} = 0$; for $t > r/c$,

$$\begin{aligned}\mathbf{A}(r,t) &= \left(\frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\right) 2 \int\limits_0^{\sqrt{(ct)^2-r^2}} \frac{k(t-\sqrt{r^2+z^2}/c)}{\sqrt{r^2+z^2}}\,dz = \frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}} \left\{ t \int\limits_0^{\sqrt{(ct)^2-r^2}} \frac{dz}{\sqrt{r^2+z^2}} - \frac{1}{c} \int\limits_0^{\sqrt{(ct)^2-r^2}} dz \right\} \\ &= \left(\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}}\right) \left[t \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) - \frac{1}{c}\sqrt{(ct)^2-r^2} \right].\end{aligned}$$

Accordingly,

$$\begin{aligned}\mathbf{E}(r,t) &= -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}} \left\{ \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + \right. \\ &\qquad \left. t\left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right)\left(\frac{1}{r}\right)\left(c + \frac{1}{2}\frac{2c^2 t}{\sqrt{(ct)^2-r^2}}\right) - \frac{1}{2c}\frac{2c^2 t}{\sqrt{(ct)^2-r^2}} \right\} \\ &= -\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}} \left\{ \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + \frac{ct}{\sqrt{(ct)^2-r^2}} - \frac{ct}{\sqrt{(ct)^2-r^2}} \right\} \\ &= \boxed{-\frac{\mu_0 k}{2\pi} \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right)\hat{\mathbf{z}}} \qquad \text{(or zero, for } t < r/c\text{).}\end{aligned}$$

$$\begin{aligned}\mathbf{B}(r,t) &= -\frac{\partial A_z}{\partial r}\,\hat{\boldsymbol{\phi}} \\ &= -\frac{\mu_0 k}{2\pi} \left\{ t\left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right) \frac{\left[r\frac{1}{2}\frac{(-2r)}{\sqrt{(ct)^2-r^2}} - ct - \sqrt{(ct)^2-r^2}\right]}{r^2} - \frac{1}{2c}\frac{(-2r)}{\sqrt{(ct)^2-r^2}} \right\}\hat{\boldsymbol{\phi}} \\ &= -\frac{\mu_0 k}{2\pi}\left\{ \frac{-ct^2}{r\sqrt{(ct)^2-r^2}} + \frac{r}{c\sqrt{(ct)^2-r^2}} \right\}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 k}{2\pi}\frac{(-c^2t^2+r^2)}{rc\sqrt{(ct)^2-r^2}}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0 k}{2\pi r c}\sqrt{(ct)^2-r^2}\,\hat{\boldsymbol{\phi}}.}\end{aligned}$$

(b) $\mathbf{A}(r,t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}} \int_{-\infty}^{\infty} \frac{q_0 \delta(t - \mathscr{r}/c)}{\mathscr{r}}\,dz$. But $\mathscr{r} = \sqrt{r^2+z^2}$, so the integrand is even in z:

$$\mathbf{A}(r,t) = \left(\frac{\mu_0 q_0}{4\pi}\,\hat{\mathbf{z}}\right) 2 \int_0^{\infty} \frac{\delta(t-\mathscr{r}/c)}{\mathscr{r}}\,dz.$$

Now $z = \sqrt{\mathscr{r}^2 - r^2} \Rightarrow dz = \frac{1}{2}\frac{2\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}} = \frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}}$, and $z = 0 \Rightarrow \mathscr{r} = r$, $z = \infty \Rightarrow \mathscr{r} = \infty$. So:

$$\mathbf{A}(r,t) = \frac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}} \int_r^{\infty} \frac{1}{\mathscr{r}}\delta\left(t - \frac{\mathscr{r}}{c}\right)\frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}}.$$

Now $\delta(t - \imath/c) = c\delta(\imath - ct)$ (Ex. 1.15); therefore $\mathbf{A} = \frac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}}\,c\int_r^\infty \frac{\delta(\imath - ct)}{\sqrt{\imath^2 - r^2}}\,d\imath$, so

$$\mathbf{A}(r,t) = \frac{\mu_0 q_0 c}{2\pi}\frac{1}{\sqrt{(ct)^2 - r^2}}\,\hat{\mathbf{z}} \quad \text{(or zero, if } ct < r\text{);}$$

$$\mathbf{E}(r,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{2c^2 t}{[(ct)^2 - r^2]^{3/2}}\,\hat{\mathbf{z}} = \boxed{\frac{\mu_0 q_0 c^3 t}{2\pi[(ct)^2 - r^2]^{3/2}}\,\hat{\mathbf{z}}} \quad \text{(or zero, for } t < r/c\text{);}$$

$$\mathbf{B}(r,t) = -\frac{\partial \mathbf{A}_z}{\partial r}\,\hat{\boldsymbol{\phi}} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{-2r}{[(ct)^2 - r^2]^{3/2}}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{-\mu_0 q_0 c r}{2\pi[(ct)^2 - r^2]^{3/2}}\,\hat{\boldsymbol{\phi}}} \quad \text{(or zero, for } t < r/c\text{).}$$

10.12

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}(t_r)}{\imath}\,dl = \frac{\mu_0 k}{4\pi}\int \frac{(t - \imath/c)}{\imath}\,d\mathbf{l} = \frac{\mu_0 k}{4\pi}\left\{t\int \frac{d\mathbf{l}}{\imath} - \frac{1}{c}\int d\mathbf{l}\right\}.$$

But for the complete loop, $\int d\mathbf{l} = 0$, so $\mathbf{A} = \frac{\mu_0 k t}{4\pi}\left\{\frac{1}{a}\int_1 d\mathbf{l} + \frac{1}{b}\int_2 d\mathbf{l} + 2\,\hat{\mathbf{x}}\int_a^b \frac{dx}{x}\right\}$. Here $\int_1 d\mathbf{l} = 2a\,\hat{\mathbf{x}}$ (inner circle), $\int_2 d\mathbf{l} = -2b\,\hat{\mathbf{x}}$ (outer circle), so

$$\mathbf{A} = \frac{\mu_0 k t}{4\pi}\left[\frac{1}{a}(2a) + \frac{1}{b}(-2b) + 2\ln(b/a)\right]\hat{\mathbf{x}} \Rightarrow \boxed{\mathbf{A} = \frac{\mu_0 k t}{2\pi}\ln(b/a)\,\hat{\mathbf{x}},} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \boxed{-\frac{\mu_0 k}{2\pi}\ln(b/a)\,\hat{\mathbf{x}}.}$$

The changing magnetic field induces the electric field. Since we only know $\mathbf{A}$ at *one point* (the center), we can't compute $\nabla \times \mathbf{A}$ to get $\mathbf{B}$.

10.15

At time t the charge is at $\mathbf{r}(t) = a[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$, so $\mathbf{v}(t) = \omega a[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}]$. Therefore $\boldsymbol{\imath} = z\,\hat{\mathbf{z}} - a[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}]$, and hence $\imath^2 = z^2 + a^2$ (of course), and $\imath = \sqrt{z^2 + a^2}$.

$$\hat{\boldsymbol{\imath}}\cdot\mathbf{v} = \frac{1}{\imath}(\boldsymbol{\imath}\cdot\mathbf{v}) = \frac{1}{\imath}\left\{-\omega a^2[-\sin(\omega t_r)\cos(\omega t_r) + \sin(\omega t_r)\cos(\omega t_r)]\right\} = 0, \text{ so } \left(1 - \frac{\hat{\boldsymbol{\imath}}\cdot\mathbf{v}}{c}\right) = 1.$$

Therefore

$$V(z,t) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2 + a^2}}}; \quad \mathbf{A}(z,t) = \boxed{\frac{q\omega a}{4\pi\epsilon_0 c^2\sqrt{z^2 + a^2}}[-\sin(\omega t_r)\,\hat{\mathbf{x}} + \cos(\omega t_r)\,\hat{\mathbf{y}}),} \text{ where } \boxed{t_r = t - \frac{\sqrt{z^2 + a^2}}{c}.}$$

10.16

Term under square root in (Eq. 9.98) is:

$$\begin{aligned} I &= c^4t^2 - 2c^2t(\mathbf{r}\cdot\mathbf{v}) + (\mathbf{r}\cdot\mathbf{v})^2 + c^2r^2 - c^4t^2 - v^2r^2 + v^2c^2t^2 \\ &= (\mathbf{r}\cdot\mathbf{v})^2 + (c^2-v^2)r^2 + c^2(vt)^2 - 2c^2(\mathbf{r}\cdot\mathbf{v}t). \quad \text{put in } \mathbf{v}t = \mathbf{r}-\mathbf{R}^2. \\ &= (\mathbf{r}\cdot\mathbf{v})^2 + (c^2-v^2)r^2 + c^2(r^2+R^2-2\mathbf{r}\cdot\mathbf{R}) - 2c^2(r^2-\mathbf{r}\cdot\mathbf{R}) = (\mathbf{r}\cdot\mathbf{v})^2 - r^2v^2 + c^2R^2. \end{aligned}$$

but

$$\begin{aligned} (\mathbf{r}\cdot\mathbf{v})^2 - r^2v^2 &= ((\mathbf{R}+\mathbf{v}t)\cdot\mathbf{v})^2 - (\mathbf{R}+\mathbf{v}t)^2v^2 \\ &= (\mathbf{R}\cdot\mathbf{v})^2 + v^4t^2 + 2(\mathbf{R}\cdot\mathbf{v})v^2t - R^2v^2 - 2(\mathbf{R}\cdot\mathbf{v})tv^2 - v^2t^2v^2 \\ &= (\mathbf{R}\cdot\mathbf{v})^2 - R^2v^2 = R^2v^2\cos^2\theta - R^2v^2 = -R^2v^2\left(1-\cos^2\theta\right) \\ &= -R^2v^2\sin^2\theta. \end{aligned}$$

Therefore

$$I = -R^2v^2\sin^2\theta + c^2R^2 = c^2R^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right).$$

Hence

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}. \quad \text{qed}$$

10.19

From Eq. 10.33, $c(t-t_r) = \eta \Rightarrow c^2(t-t_r)^2 = \eta^2 = \boldsymbol{\eta}\cdot\boldsymbol{\eta}$. Differentiate with respect to t:
$2c^2(t-t_r)\left(1-\frac{\partial t_r}{\partial t}\right) = 2\boldsymbol{\eta}\cdot\frac{\partial \boldsymbol{\eta}}{\partial t}$, or $c\eta\left(1-\frac{\partial t_r}{\partial t}\right) = \boldsymbol{\eta}\cdot\frac{\partial \boldsymbol{\eta}}{\partial t}$. Now $\boldsymbol{\eta} = \mathbf{r}-\mathbf{w}(t_r)$, so
$\frac{\partial \boldsymbol{\eta}}{\partial t} = -\frac{\partial \mathbf{w}}{\partial t} = -\frac{\partial \mathbf{w}}{\partial t_r}\frac{\partial t_r}{\partial t} = -\mathbf{v}\frac{\partial t_r}{\partial t}$; $c\eta\left(1-\frac{\partial t_r}{\partial t}\right) = -\boldsymbol{\eta}\cdot\mathbf{v}\frac{\partial t_r}{\partial t}$; $\quad c\eta = \frac{\partial t_r}{\partial t}(c\eta - \boldsymbol{\eta}\cdot\mathbf{v}) = \frac{\partial t_r}{\partial t}(\boldsymbol{\eta}\cdot\mathbf{u})$ (Eq. 10.64),
and hence $\frac{\partial t_r}{\partial t} = \frac{c\eta}{\boldsymbol{\eta}\cdot\mathbf{u}}$. qed

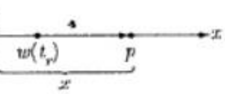
Now Eq. 10.40 says $\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$, so

$$\begin{aligned}
\frac{\partial \mathbf{A}}{\partial t} &= \frac{1}{c^2}\left(\frac{\partial \mathbf{v}}{\partial t}V + \mathbf{v}\frac{\partial V}{\partial t}\right) = \frac{1}{c^2}\left(\frac{\partial \mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t}V + \mathbf{v}\frac{\partial V}{\partial t}\right)\\
&= \frac{1}{c^2}\left[\mathbf{a}\frac{\partial t_r}{\partial t}\frac{1}{4\pi\epsilon_0}\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}} + \mathbf{v}\frac{1}{4\pi\epsilon_0}\frac{-qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\frac{\partial}{\partial t}(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})\right]\\
&= \frac{1}{c^2}\frac{qc}{4\pi\epsilon_0}\left[\frac{\mathbf{a}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\frac{\partial t_r}{\partial t} - \frac{\mathbf{v}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left(c\frac{\partial \mathscr{r}}{\partial t} - \frac{\partial \boldsymbol{\mathscr{r}}}{\partial t}\cdot\mathbf{v} - \boldsymbol{\mathscr{r}}\cdot\frac{\partial \mathbf{v}}{\partial t}\right)\right].
\end{aligned}$$

But $\mathscr{r} = c(t-t_r) \Rightarrow \frac{\partial \mathscr{r}}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right)$, $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \Rightarrow \frac{\partial \boldsymbol{\mathscr{r}}}{\partial t} = -\mathbf{v}\frac{\partial t_r}{\partial t}$ (as above), and $\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t} = \mathbf{a}\frac{\partial t_r}{\partial t}$.

$$\begin{aligned}
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{\mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\frac{\partial t_r}{\partial t} - \mathbf{v}\left[c^2\left(1-\frac{\partial t_r}{\partial t}\right) + v^2\frac{\partial t_r}{\partial t} - \boldsymbol{\mathscr{r}}\cdot\mathbf{a}\frac{\partial t_r}{\partial t}\right]\right\}\\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v} + \left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{\partial t_r}{\partial t}\right\}\\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v} + \left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\right\}\\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[-c^2\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}) + c\mathscr{r}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + c\mathscr{r}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\\
&= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})\left(-\mathbf{v} + \frac{\mathscr{r}}{c}\mathbf{a}\right) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]. \quad \text{qed}
\end{aligned}$$

10.20



$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$. Here $\mathbf{v} = v\,\hat{\mathbf{x}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$, and, for points to the *right*, $\hat{\boldsymbol{\mathscr{r}}} = \hat{\mathbf{x}}$. So $\mathbf{u} = (c-v)\,\hat{\mathbf{x}}$, $\mathbf{u}\times\mathbf{a} = 0$, and $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c-v)$.

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c-v)^3}(c^2 - v^2)(c-v)\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{(c+v)(c-v)^2}{(c-v)^3}\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}};\\
\mathbf{B} &= \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E} = 0. \quad \text{qed}
\end{aligned}$$

For field points to the *left*, $\hat{\boldsymbol{\mathscr{r}}} = -\hat{\mathbf{x}}$ and $\mathbf{u} = -(c+v)\,\hat{\mathbf{x}}$, so $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c+v)$, and

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c+v)^3}(c^2 - v^2)(c+v)\,\hat{\mathbf{x}} = \boxed{\frac{-q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}};\ \mathbf{B} = 0.}$$