Problem (1): (20 points)

(a) Using the formula derived in the book (Eqn. 9.161) for the dielectric constant ($\varepsilon/\varepsilon_0$) of a material with N molecules/unit volume, where the oscillator strengths satisfy the sum rule $\sum_j f_j = Z$ where Z is the total no. of electrons/molecule, show that in the *X-ray regime*, ($\omega >>$ all ω_j, γ_j) the dielectric constant can be written as

$$\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$$ where the plasma frequency ω_p is given by

$$\omega_p^2 = \frac{NZq^2}{\varepsilon_0 m}$$ (3 points)

(b) Use Snell's Law of refraction to show that for EM waves incident from vacuum on a solid medium at angle of incidence θ, there exists a critical angle of incidence θ_c at which the refracted ray just grazes the surface of the medium if its refractive index $n < 1$. For incident angles greater than θ_c the EM wave is totally reflected. Assuming $\omega >> \omega_p$ use the result in part (a) to show that θ_c is very close to $\pi/2$, and that $\theta'_c = (\pi/2 - \theta_c)$ (the so-called "grazing angle of incidence") is well approximated by

$$\theta_c' \cong \omega_p / \omega$$ (10 points)

(c) Show that the z-component of the propagation wave vector k_{Tz} of the transmitted wave in the medium for incident angle θ is given by

$k_{Tz} = \frac{\omega}{c}\sqrt{n^2 - \sin^2\theta}$ for all angles of incidence. Discuss the general form of the electric field in the medium when $\theta > \theta_c$.
(7 points)

Solution:

(a) By Eq. (9.161) of book,

$$\frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Nq^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

since $\omega >> \omega_j$, γ_j we can write

$$\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{Nq^2}{m\varepsilon_0\omega^2}\sum_j f_j = 1 - \frac{Nq^2}{m\varepsilon_0\omega^2} Z = 1 - \frac{\omega_p^2}{\omega^2}$$

(b) Refractive index of medium $n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \cong 1 - \frac{\omega_p^2}{2\omega^2}$

since $\omega >> \omega_p$; Refractive index of incident medium =1

By Snell's Law, $\frac{\sin\theta_t}{\sin\theta_i} = \frac{1}{n}$ i.e. $\sin\theta_i = n\sin\theta_t$

so critical angle is when $\theta_t = \pi/2$ or

$$\cos\theta_c = \sqrt{1-n^2} \cong \frac{\omega_p}{\omega} = \sin\theta'_c$$

since $\theta'_c = \pi/2 - \theta_c$

Thus since $\omega_p/\omega << 1$,

$\theta'_c = \omega_p/\omega$

(c) In 1st medium magnitude of wave vector $k = \omega/c$

In 2nd medium magnitude of wave vector $k_T = nk$

so $(k_{Tz}^2 + k_{II}^2) = n^2 k^2$

but the parallel component of **k** is the same on both sides of the interface and = $k\sin\theta$, so

Problem (2): (20 points)

A resonant cavity is made of a metal which is a perfect conductor and consists of a section of rectangular

waveguide of length c extending along the z-direction with sides parallel to the x- and y-axes (of lengths a and b respectively), but with metal closing both ends.

(a) Show that the electric fields represented in their complex notation as

$$\tilde{E}_x = A_1 \cos(k_x x)\sin(k_y y)\sin(k_z z)e^{-i\omega t}$$

$$\tilde{E}_y = A_2 \sin(k_x x)\cos(k_y y)\sin(k_z z)e^{-i\omega t}$$

$$\tilde{E}_3 = A_3 \sin(k_x x)\sin(k_y y)\cos(k_z z)e^{-i\omega t}$$

satisfy the wave equation for **E** inside the cavity and all the required boundary conditions at the walls, if k_x, k_y, k_z satisfy certain conditions. Write down these conditions. (6 points)

(b) Derive a linear relation between A_1, A_2, A_3 if Maxwell's Equations are to be satisfied. (3 points)

(c) Derive expressions for $\tilde{B}_x, \tilde{B}_y, \tilde{B}_z$ and show that they also satisfy the correct boundary conditions at the walls. (6 points)

(d) If c > b > a, derive an expression for the smallest resonant frequency ω of this cavity. (5 points)

Solution:

(a) Partially differentiating the first equation with respect to x,y and z and adding we get

$$\nabla^2 \tilde{E}_x = -(k_x^2 + k_y^2 + k_z^2) A_1 \cos(k_x x) \sin(k_y y) \sin(k_z z) e^{-i\omega t}$$
$$= -(k_x^2 + k_y^2 + k_z^2) \tilde{E}_x$$

Differentiating with respect to time, we get

$$\frac{1}{c^2} \frac{\partial^2 \tilde{E}_x}{\partial t^2} = -\frac{\omega^2}{c^2} \tilde{E}_x$$

Thus $\nabla^2 \tilde{E}_x = \frac{1}{c^2} \frac{\partial^2 \tilde{E}_x}{\partial t^2}$

if $\omega^2 = c^2((k_x^2 + k_y^2 + k_z^2)$ (1)

Same can be shown for $\tilde{E}_y, \tilde{E}_z$.

Boundary conditions are tangential electric field is zero at all metal surfaces, i.e.

$E_x = 0$ whenever y=0 ; y=b; z=0 and z=C
i.e. $k_y = m\pi/b$ and $k_z = n\pi/c$ (m,n =0,1,2,……)

Ey=0 whenever x=0 or x=a; z=0 and z=C
requires also $k_x = l\pi/a$ (l=0,1,2,….)

Then E_z=0 whenever x=0 and x=a; y=0 and y=b is automatically satisfied.
So conditions are $k_x=l\pi/a$; $k_y= m\pi/b$ and $k_z =n\pi/c$

(l,m,n = 0,1,2, ……. BUT NOTE that two of them cannot be 0 at the same time as all fields would vanish!))

(b) Only Maxwell's Equation involving **E** alone is
Div **E** = 0

Using Equations for components of E and differentiating, we obtain

$k_x A_1 + k_y A_2 + k_z A_3 = 0$

(c) Using Maxwell's Equation $i\omega\tilde{\mathbf{B}} = \nabla \times \tilde{\mathbf{E}}$ (see Eqs.

(9.179) i, ii, iii of book)

$$\tilde{B}_x = -\frac{i}{\omega}[\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}]$$

or $\tilde{B}_x = -\frac{i}{\omega}[A_3 k_y - A_2 k_z]\sin(k_x x)\cos(k_y y)\cos(k_z z)e^{-i\omega t}$

similarly, we obtain

$$\tilde{B}_y = -\frac{i}{\omega}[A_1 k_z - A_3 k_x]\cos(k_x x)\sin(k_y y)\cos(k_z z)e^{-i\omega t}$$

$$\tilde{B}_z = -\frac{i}{\omega}[A_2 k_x - A_1 k_y]\cos(k_x x)\cos(k_y y)\sin(k_z z)e^{-i\omega t}$$

Boundary conditions for **B** are that normal component of **B** vanishes at metal surfaces, i.e.

$B_x = 0$ at x=0 and x=a; B_y=0 at y=0 and y=b; and B_z=0 at z=0 and z=C

and given above expressions and conditions on k_x, k_y, k_z in part (a), we see that these are identically satisfied.

(d) Since c $\geq$ b > a, from Eq. (1) lowest frequency in cavity is for l=0,m=1,n=1 (since 2 of these integers cannot simultaneously be 0) or $\omega = c[(\pi/b)^2 + (\pi/C)^2]^{1/2}$ where we distinguish between velocity of light c and dimension of box C.

Problem (3): (20 points)

Consider a photon of energy E_γ incident on a stationary proton (rest mass $=m_p$). For sufficiently large E_γ a π meson (rest mass $=m_\pi$) can be produced according to the reaction

$$\gamma + p \rightarrow p + \pi$$

Calculate E_γ, the threshold photon energy for this reaction to occur in terms of m_p and m_π.
(Hint: At the threshold, the π meson will have vanishingly small momentum)

Solution:

Let photon initially have been traveling along x-direction. Initial momentum of photon along x = E_γ/c and along y =0. At threshold, after reaction, let momentum of proton be $\mathbf{p_p}$

momentum of pion is $\mathbf{p_\pi} = 0$, so $\mathbf{p_p}$ must be along x and = E_γ/c by conservation of momentum.

Conservation of energy and using above result gives

$$E_\gamma + m_p c^2 = m_\pi c^2 + \sqrt{m_p^2 c^4 + p_p^2 c^2} = m_\pi c^2 + \sqrt{m_p^2 c^4 + E_\gamma^2}$$

so

$$E_\gamma + m_p c^2 - m_\pi c^2 = \sqrt{m_p^2 c^4 + E_\gamma^2}$$

Squaring both sides yields the solution

$$E_\gamma = \frac{m_\pi(2m_p - m_\pi)}{2(m_p - m_\pi)}c^2$$

Problem (4): (20 points)

"Derive" the Lorentz Force Law as follows: Let charge q be at rest in frame $\tilde{S}$, so $\tilde{\mathbf{F}} = q\tilde{\mathbf{E}}$ and let frame $\tilde{S}$ move with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ with respect to S. Use the transformation rules (Eqs. (12.67) and (12.109)) to rewrite $\tilde{\mathbf{F}}$ in terms of $\mathbf{F}$, and $\tilde{\mathbf{E}}$ in terms of **E** and **B**. From these, deduce the formula for F in terms of **E** and **B**.

Solution:

From Eq. (12.67) we have

$$\tilde{F}_x = F_x$$

$$\tilde{F}_y = \frac{1}{\gamma} F_y$$

$$\tilde{F}_z = \frac{1}{\gamma} F_z$$

So

$$F_x = q\tilde{E}_x = qE_x$$

$$F_y = \frac{q}{\gamma}\tilde{E}_y = \frac{q}{\gamma}[\gamma(E_y - vB_z)]$$

$$F_z = \frac{q}{\gamma}\tilde{E}_z = \frac{q}{\gamma}[\gamma(E_z + vB_y)]$$

using Eqs. (12.109) for forces and fields in frame S, where charge is moving with velocity v along x-axis

We can rewrite above 3 equations as

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

Problem (5): (20 points)

A model for electric quadrupole radiation is two oppositely directed dipoles separated by a distance d (see Fig.) oscillating as shown. Use the results of Secn. 11.1.2 for the potentials of each dipole but note that they *are not located* at the origin. Assume that $r >> \omega/c >> d$, then keeping only terms of first order in d,

(a) Find the scalar and vector potentials. (5 points)

(b) Find the electric and magnetic fields. (10 points)

(c) Find the time-averaged Poynting vector as a function of r and θ, and the total power radiated. (5 points)

Problem 5

In the approximation $r \gg c/\omega \gg d$, the potentials for an electric dipole are,

$$V_2(r,\,\theta,\,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\frac{\cos\theta}{r}\sin\left(\omega t - \frac{\omega r}{c}\right),$$
$$\vec{A}_2(r,\,\theta,\,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left(\omega t - \frac{\omega r}{c}\right)\hat{z}.$$

The scalar potential for the problem at hand is given by,

$$V_4(r,\,\theta,\,t) = V_2(r_+,\,\hat{r}_+\cdot\hat{z},\,t) - V_2(r_-,\,\hat{r}_-\cdot\hat{z},\,t),$$

where $\vec{r}_\pm = \vec{r} \pm d\hat{z}/2$. Similarly, the vector potential is given by,

$$\vec{A}_4(r,\,\theta,\,t) = \vec{A}_2(r_+,\,\hat{r}_+\cdot\hat{z},\,t) - \vec{A}_2(r_-,\,\hat{r}_-\cdot\hat{z},\,t).$$

Consider the follow series of approximations. Approximation 1: $r \gg d$,

$$V_2(r_\pm,\,\hat{r}_\pm\cdot\hat{z},\,t) \approx -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r} \mp \frac{d}{2r^2}\cos(2\theta)\right)\sin\left(\omega t - \frac{\omega r_\pm}{c}\right)$$

(Note that $\cos 2\theta = \cos^2\theta - \sin^2\theta$). Approximation 2: $c/\omega \gg d$,

$$V_2(r_\pm,\,\hat{r}_\pm\cdot\hat{z},\,t) \approx -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r} \mp \frac{d}{2r^2}\cos(2\theta)\right)\left[\sin\left(\omega t - \frac{\omega r}{c}\right) \mp \frac{\omega d}{2c}\cos\theta\cos\left(\omega t - \frac{\omega r}{c}\right)\right]$$

Using these first two approximations V_4 becomes,

$$V_4(r,\,\theta,\,t) \approx \frac{p_0\omega}{4\pi\epsilon_0 c}\frac{d}{r}\left[\frac{1}{r}\cos(2\theta)\sin\left(\omega t - \frac{\omega r}{c}\right) + \frac{\omega}{c}\cos^2(\theta)\cos\left(\omega t - \frac{\omega r}{c}\right)\right].$$

Here we see that the dipole term has vanished. Approximation 3: $r \gg c/\omega$,

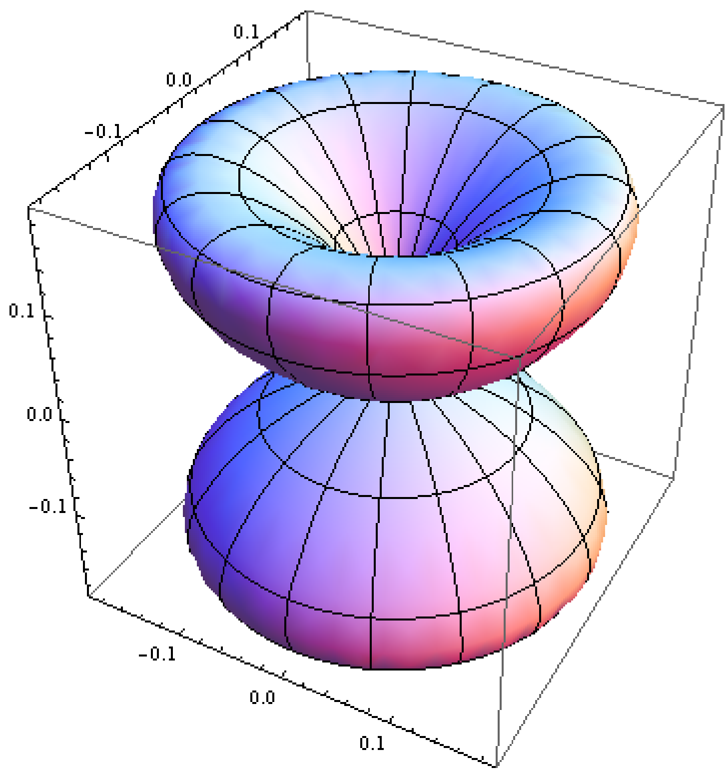
$$V_4(r,\,\theta,\,t) \approx \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\frac{d}{r}\cos^2\theta\cos\left(\omega t - \frac{\omega r}{c}\right).$$

The same series of approximations leads to the following expression for the vector potential,

$$\vec{A}_4(r,\,\theta,\,t) \approx \frac{\mu_0 p_0\omega^2}{4\pi c}\frac{d}{r}\cos\theta\cos\left(\omega t - \frac{\omega r}{c}\right)\hat{z}.$$

The electric and magnetic fields are

$$\vec{E} = -\vec{\nabla}V_4 - \frac{\partial\vec{A}_4}{\partial t}$$
$$\vec{E} = -\frac{\mu_0 p_0\omega^3}{4\pi c}\frac{d}{r}\sin\theta\cos\theta\sin\left(\omega t - \frac{\omega r}{c}\right)\hat{\theta}$$
$$\vec{B} = \vec{\nabla}\times\vec{A}_4$$
$$\vec{B} = -\frac{\mu_0 p_0\omega^3}{4\pi c^2}\frac{d}{r}\sin\theta\cos\theta\sin\left(\omega t - \frac{\omega r}{c}\right)\hat{\phi}$$



The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

$$\vec{S} = \frac{\mu_0 p_0^2 \omega^6}{16\pi^2 c^3}\frac{d^2}{r^2}\sin^2\theta\cos^2\theta\sin^2\left(\omega t - \frac{\omega r}{c}\right)\hat{r}$$

$$\langle\vec{S}\rangle = \frac{\mu_0 p_0^2 \omega^6}{32\pi^2 c^3}\frac{d^2}{r^2}\sin^2\theta\cos^2\theta\,\hat{r}$$

The total power radiated is

$$\langle P\rangle = \int \langle\vec{S}\rangle \cdot d\vec{a}$$

$$\langle P\rangle = \frac{\mu_0 p_0^2 \omega^6}{60\pi c^3}d^2$$

The intensity profile is shown above.