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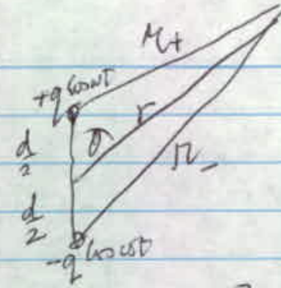
Electric Dipole Radiation



$$q(t) = q_0 \cos \omega t$$

$$\vec{P}(t) = p_0 \cos \omega t\, \hat{z} \qquad p_0 = q_0 d$$

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q_0 \cos(\omega t - \mathcal{R}_+/c)}{\mathcal{R}_+} - \frac{q_0 \cos(\omega t - \mathcal{R}_-/c)}{\mathcal{R}_-} \right\}$$

$$\mathcal{R}_\pm = \sqrt{r^2 \mp r d \cos\theta + \left(\tfrac{d}{2}\right)^2} \quad \text{from triangle above}$$

approx's: (1) $d \ll r$ (2) $d \ll \frac{c}{\omega}$ $(d \ll \lambda)$ (3) $r \gg \frac{c}{\omega}$ $(r \gg \lambda)$

$$\mathcal{R}_\pm \simeq r\left(1 \mp \frac{d}{2r}\cos\theta\right) \quad \text{by (1)} \quad ①$$

$$\frac{1}{\mathcal{R}_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \quad \text{by (1)} \quad ②$$

$$\cos\left(\omega t - \frac{\mathcal{R}_\pm}{c}\right) = \cos\left[\left(\omega t - \frac{r}{c}\right) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos\left(\omega t - \frac{r}{c}\right)\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin\left(\omega t - \frac{r}{c}\right)\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

$$(\text{by approx}^t\ (2)) \simeq \cos\left(\omega t - \frac{r}{c}\right) \mp \frac{\omega d}{2c}\cos\theta \sin\left(\omega t - \frac{r}{c}\right) \quad ③$$

$$V(\vec{r}, t) = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r}\left\{ -\frac{\omega}{c}\sin\left[\omega\left(t - \frac{r}{c}\right)\right] + \frac{1}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}$$

$$\omega \to 0, \quad V(r, t) \to \frac{p_0 \cos\theta}{4\pi\epsilon_0 r^2} \quad [\text{static dipole result}]$$

Now use approx^t (3),

$$\boxed{V(r,t) = -\frac{p_0 \cos\theta}{4\pi\epsilon_0 r}\frac{\omega}{c}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]} = -\frac{p_0 \omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - \frac{r}{c}\right)\right]$$

$$I(t) = \frac{dq}{dt}\hat{z} = -q_0 \omega \sin(\omega t)\,\hat{z}$$

$$\therefore A(\vec{r}, t) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}\left(t - \frac{\mathcal{R}}{c}\right)}{\mathcal{R}}\, dx\, dy\, dz = \iint \frac{\vec{J}\left(t - \frac{\mathcal{R}}{c}\right)}{\mathcal{R}} dx\, dy$$

$$\simeq I\left(t - \frac{\mathcal{R}}{c}\right)\hat{z}$$

$$\therefore \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{-q_0 \omega \sin\left[\omega\left(t - \frac{\mathcal{R}}{c}\right)\right]}{\mathcal{R}}\, dz\, \hat{z}$$

$$\rightarrow \boxed{A(\vec{r}, t) \simeq -\frac{\mu_0}{4\pi}\frac{p_0 \omega}{r}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{z}} = -\frac{\mu_0}{4\pi r}p_0 \omega \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

(2b)

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \cancel{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}}^{\,0}\hat{\phi}$$

$$= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\left[\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-\tfrac{r}{c})] - \frac{\omega}{rc}\cos[\omega(t-\tfrac{r}{c})]\right)\hat{r} - \frac{\sin\theta}{r^2}\sin[\omega(t-\tfrac{r}{c})]\,\hat{\theta}\right\}$$

negligible by Approx (3).

$$\simeq -\frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-\tfrac{r}{c})]\,\hat{r}$$

$$\frac{\partial A}{\partial t} \simeq -\frac{\mu_0 p_0^2\omega^2}{4\pi r}\cos[\omega(t-\tfrac{r}{c})]\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$[\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}]$$

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\sin\theta\cdot\cos[\omega(t-\tfrac{r}{c})]$$

$$= -\frac{\mu_0 p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\tfrac{r}{c})]\,\hat{\theta}$$

$$\nabla\times\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi} + \frac{1}{r\sin\theta}\left[\cancel{\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}}\right]\hat{r} + \frac{1}{r}\left[\cancel{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)}\right]\hat{\phi}$$

$A_\phi = 0$ No dependence on ϕ

$$= -\frac{\mu_0 p_0\omega}{4\pi r^2}\left[\frac{\partial}{\partial r} r\left[-\sin\theta\sin[\omega(t-\tfrac{r}{c})]\right] - \frac{\partial}{\partial\theta}\{\cos\theta\sin[\omega(t-\tfrac{r}{c})]\}\right]\hat{\phi}$$

$$= -\frac{\mu_0 p_0\omega}{4\pi r^2}\left[-\sin\theta\sin[\omega(t-\tfrac{r}{c})] + r\sin\theta\frac{\omega}{c}\cos[\omega(t-\tfrac{r}{c})] + \sin\theta\sin[\omega(t-\tfrac{r}{c})]\right]$$

$$A_r = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\tfrac{r}{c})]\cos\theta$$

$$A_\theta = +\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\tfrac{r}{c})]\sin\theta \qquad rA_\theta = \frac{\mu_0 p_0\omega}{4\pi}\sin[\omega(t-\tfrac{r}{c})]\sin\theta$$

$$A_\phi = 0 \qquad \frac{\partial}{\partial r}(rA_\theta) = -\frac{\mu_0 p_0\omega}{4\pi}\left(\frac{\omega}{c}\right)\sin\theta\cos[\omega(t-\tfrac{r}{c})]$$

$$\frac{\partial A_r}{\partial\theta} = +\frac{\mu_0 p_0\omega}{4\pi r}\sin\theta\sin[\omega(t-\tfrac{r}{c})]$$

$$\nabla\times A = \left\{\frac{\mu_0 p_0\omega}{4\pi r}\left(\frac{\omega}{c}\right)\sin\theta\cos[\omega(t-\tfrac{r}{c})]\hat{\phi} - \frac{\mu_0 p_0\omega}{4\pi r}\frac{\sin\theta}{r}\sin[\omega(t-\tfrac{r}{c})]\right\}\hat{\phi}$$

$$= -\frac{\mu_0 p_0\omega}{4\pi r}\left[\frac{\omega}{c}\sin\theta\cos[\omega(t-\tfrac{r}{c})] + \underbrace{\cancel{\frac{\sin\theta}{r}\sin[\omega(t-\tfrac{r}{c})]}}_{\text{neglect by (3)}}\right]\hat{\phi}$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\phi}$$

$$S(\vec{r},t) = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \left(\frac{\mu_0 p_0 \omega^2}{4\pi}\right)^2 \frac{1}{\mu_0}\left(\frac{\sin\theta}{r}\right)^2 \cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r} \times \frac{\omega^2}{c}\left(\frac{\sin\theta}{r}\right)^2$$

$$= \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c}\cdot\frac{\sin^2\theta}{r^2}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}$$

$$\boxed{\langle S\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r}}$$

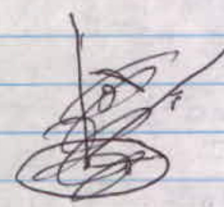
$$\langle P\rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta\, d\theta\, d\phi = \boxed{\frac{\mu_0 p_0^2\omega^4}{12\pi c}}$$

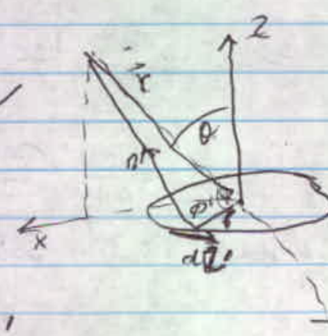
$$\left[\int_0^\pi \sin^2\theta \sin\theta\, d\theta = \int_{-1}^{1}[1-z^2]\,dz = \left[z-\frac{1}{3}z^3\right]_{-1}^{1} = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}.\right]$$

$$\int d\phi = 2\pi$$

No radiation along dipole axis ($\hat{z}$) → Doughnut shaped

Magnetic Dipole Radiation





$$I(t) = I_0 \cos\omega t$$

$$\vec{m}(t) = \pi b^2 I(t)\hat{z} = m_0\cos\omega t\,\hat{z}$$

$$m_0 = \pi b^2 I_0$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{I_0\cos\left[\omega\left(t-\frac{\mathscr{r}}{c}\right)\right]}{\mathscr{r}}\,d\vec{l}'$$

By symmetry $A(\vec{r},t)$. ($\vec{r}$ directly above x-axis) must point along $\hat{y}$.

$$A(\vec{r},t) = \frac{b\mu_0 I_0\hat{y}}{4\pi}\int_0^{2\pi}\frac{\cos\left[\omega\left(t-\frac{\mathscr{r}}{c}\right)\right]}{\mathscr{r}}\cos\phi'\,d\phi'$$

$$\mathscr{r} = \sqrt{r^2+b^2-2rb\cos\psi} \qquad \psi \text{ angle between } \vec{r}, + \vec{b}$$

$$\vec{r} = r\sin\theta\,\hat{x} + r\cos\theta\,\hat{z}$$

$$\vec{b} = b\cos\phi'\,\hat{x} + b\sin\phi'\,\hat{y}$$

$$rb\cos\psi = \vec{r}\cdot\vec{b} = rb\sin\theta\cos\phi'$$

$$\mathscr{r} = \sqrt{r^2+b^2-2rb\sin\theta\cos\phi'}$$

approx^ns ① $b \ll r$ By ① $\mathscr{r} = r\left(1-\frac{b}{r}\sin\theta\cos\phi'\right)$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r}\left(1+\frac{b}{r}\sin\theta\cos\phi'\right)$$

$$\cos\left[\omega\left(t-\frac{R}{c}\right)\right] \simeq \cos\left[\omega\left(t-\frac{r}{c}\right) + \frac{\omega b}{c}\sin\theta\cos\phi'\right]$$
$$= \cos\omega\left(t-\frac{r}{c}\right)\cos\left[\frac{\omega b}{c}\sin\theta\cos\phi'\right]$$
$$- \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\frac{\omega b}{c}\sin\theta\cos\phi'\right]$$

Approx^n ②, $b \ll \frac{c}{\omega}$ $\quad (b \ll \lambda)$

$$\cos\left[\omega\left(t-\frac{R}{c}\right)\right] \simeq \cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega b}{c}\sin\theta\cos\phi'\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

$$\cancel{A(\vec{r},t) = \frac{\mu_0 I_0 b}{4\pi}\int_0^{2\pi}\left\{\cos\omega\left(t-\frac{r}{c}\right) + b\sin\theta\cos\phi\left(\frac{1}{r}\right.\right.}$$

$$A(\vec{r},t) = \hat{y}\frac{\mu_0 I_0 b}{4\pi}\int_0^{2\pi}\frac{1}{r}\left(1+\frac{b}{r}\sin\theta\cos\phi'\right)\left[\cos\omega\left(t-\frac{r}{c}\right) - \omega\frac{b}{c}\sin\theta\cos\phi'\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right]\cos\phi'\,d\phi'$$

$$= \hat{y}\frac{\mu_0 I_0 b}{4\pi}\int_0^{2\pi}\left\{\frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right) + \frac{b}{r^2}\sin\theta\cos\phi'\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega b}{rc}\sin\theta\cos\phi'\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right.$$
$$\left.+\cancel{\frac{\omega b^2}{rc}\sin\theta\cos\phi'\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}\right\}\cos\phi'\,d\phi$$

$$= \frac{\mu_0 I_0 b\hat{y}}{4\pi r}\int_0^{2\pi}\left[\cos\left[\omega\left(t-\frac{r}{c}\right)\right] + b\sin\theta\cos\phi'\left(\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\right]\cos\phi'\,d\phi'$$

$$\int_0^{2\pi}\cos\phi'\,d\phi' = 0 \qquad \int_0^{2\pi}\cos^2\phi'\,d\phi' = \pi$$

$$\vec{A} = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\phi}$$

Static Limit $\omega = 0$, $\vec{A}(r,\theta) = \frac{\mu_0}{4\pi} m_0 \frac{\sin\theta}{r^2}\hat{\phi}$

Approx^n ③ $\quad r \gg c/\omega$

$$\vec{A} = -\frac{\mu_0 m_0\omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\phi}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\phi}$$

$$\vec{B} = \nabla\times\vec{A} = \frac{\mu_0 m_0\omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$$

$$\boxed{\rightarrow \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta}}$$

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left[\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right]^2\hat{r}$$

$$\langle S\rangle = \frac{\mu_0 m_0^2\omega^4}{32\pi^2c^3}\frac{\sin^2\theta}{r^2}\hat{r}$$

$$\langle P\rangle = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3}$$

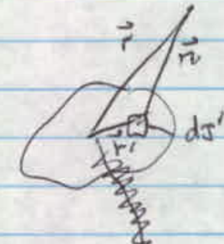
$$\frac{P_{mag}}{P_{electric}} = \left(\frac{m_0}{p_0 c}\right)^2 \qquad m_0 = \pi b^2 I_0 \qquad I_0 = q_0\omega$$

$$p_0 = q_0 d \to \left(\frac{I_0}{\omega}\right)(\pi b) \qquad \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\left(2\pi\frac{b}{\lambda}\right)^2 \longleftarrow \left(\frac{\omega b}{c}\right)^2 \longleftarrow \left(\frac{\omega\pi b^2 I_0}{I_0\pi b c}\right)^2 \sim \left(\frac{\omega b}{c}\right)^2$$

so $P_{mag} \ll P_{electric}$

<u>arbitrary source</u>



$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t-\mathscr{r}/c)}{\mathscr{r}}\,d\tau'$$

$$\mathscr{r} = \sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'} \qquad \text{approx}^n\ (1)\ \ r' \ll r$$

$$\mathscr{r} \simeq r\left(1-\frac{\vec{r}\cdot\vec{r}'}{r^2}\right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1+\frac{\vec{r}\cdot\vec{r}'}{r^2}\right)$$

$$\rho\left(r', t-\frac{\mathscr{r}}{c}\right) = \rho\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right) \qquad t_0 = t-\frac{r}{c}$$

$$\rho\left(r', t-\frac{\mathscr{r}}{c}\right) \simeq \rho(\vec{r}', t_0) + \dot{\rho}(\vec{r}', t_0)\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right) + \dots \left[\tfrac{1}{2}\ddot{\rho}\ \hat{r}\right.$$

$$\left(\text{assume } r' \ll \frac{c}{|\dot{\rho}/\rho|}, \frac{c}{(\ddot{\rho}/\rho)^{1/2}}, \dots\right)$$

Keeping only 1st order terms in r'

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0 r}\left[\int\rho(\vec{r}',t_0)\,d\tau' + \frac{\hat{r}}{r}\cdot\int\vec{r}'\rho(\vec{r}',t_0)\,d\tau' + \frac{\hat{r}}{c}\cdot\frac{d}{dt}\int\vec{r}'\rho(\vec{r}',t_0)\,d\tau'\right]$$

$$\simeq \frac{1}{4\pi\epsilon_0}\left[\underbrace{\frac{Q}{r}}_{\text{Monopole}} + \underbrace{\frac{\hat{r}\cdot\vec{p}(t_0)}{r^2}}_{\text{Dipole}} + \frac{\hat{r}\cdot\dot{p}(t_0)}{rc}\right]$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t-\mathscr{r}/c)}{\mathscr{r}}\, d\tau'$$

replace $\mathscr{r}$ by r

$$A(\vec{r},t) \simeq \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{r}', t_0)\, d\tau'$$

$$\int \vec{J}(\vec{r}', t_0)\, d\tau' = \dot{\vec{p}}(t_0)$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{p}(t_0)}{r}}$$

$\nabla\cdot J = -\frac{\partial \rho}{\partial t}$

$$0 = \int \nabla\cdot(x\vec{J})\, d\tau$$

$$\int x\vec{J}\cdot d\vec{s} = \int \left[x(\nabla\cdot\vec{J}) + \vec{J}\cdot\nabla x\right] d\tau$$

$$= \int \left[x\left(-\frac{\partial\rho}{\partial t}\right) + \vec{J}\cdot\hat{x}\right] d\tau$$

$$= -\frac{\partial}{\partial t}\int x\rho\, d\tau + \int J_x\, d\tau$$

$$\int J_x\, d\tau = \frac{\partial}{\partial t}\int x\rho\, d\tau = \frac{\partial}{\partial t}P_x$$

Calculate fields

Keep only terms $\sim \frac{1}{r}$ in $\vec{E}, \vec{B}$

$r = (x^2+y^2+z^2)^{\frac{1}{2}}$

$\frac{\partial r}{\partial x} = \frac{2x}{2r} = \frac{x}{r}$, etc

$$\nabla t_0 = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{r}$$

$$\nabla V = \nabla\left[\frac{1}{4\pi\epsilon_0}\hat{r}\cdot\frac{\dot{\vec{p}}(t_0)}{rc}\right] \simeq \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{rc}\nabla t_0$$

(Monopole term does not contribute to radiation)

$$= -\frac{1}{4\pi\epsilon_0 c^2}\left[\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{r}\right]\hat{r} = -\frac{\mu_0}{4\pi}\left[\frac{\hat{r}\cdot\ddot{\vec{p}}}{r}\right]\hat{r}$$

$$\frac{\partial A}{\partial t} = \frac{\mu_0}{4\pi}\frac{\ddot{p}(t_0)}{r}$$

$$\vec{E} = \frac{\mu_0}{4\pi r}\left[(\hat{r}\cdot\ddot{\vec{p}})\hat{r} - \ddot{\vec{p}}\right] = \frac{\mu_0}{4\pi r}\left[\hat{r}\times(\hat{r}\times\ddot{\vec{p}})\right]$$

prove

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \dot{p}_x & \dot{p}_y & \dot{p}_z \end{vmatrix}$$

$\frac{\partial \dot{p}_z}{\partial y} - \frac{\partial \dot{p}_y}{\partial z} = \ddot{p}_z\frac{\partial t_0}{\partial y} - \ddot{p}_y\frac{\partial t_0}{\partial z}$

$$\begin{vmatrix} i & j & k \\ \frac{\partial t_0}{\partial x} & \frac{\partial t_0}{\partial y} & \frac{\partial t_0}{\partial z} \\ \ddot{p}_x & \ddot{p}_y & \ddot{p}_z \end{vmatrix}$$

$$\vec{B} = \nabla\times\vec{A} \simeq \frac{\mu_0}{4\pi r}\left[\nabla\times\dot{\vec{p}}(t_0)\right] = \frac{\mu_0}{4\pi r}\left[\nabla t_0 \times \ddot{\vec{p}}(t_0)\right]$$

$$= -\frac{\mu_0}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}(t_0)\right]$$

Spherical Polar Coords

z axis along $\ddot{\vec{p}}(t_0)$

$$\vec{E}(r,\theta,t) = \frac{\mu_0\ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta}$$

$$\vec{B}(r,\theta,t) = \frac{\mu_0\ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\phi}$$

$$S(\vec{r},t) \simeq \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$

$$P(r,t) = \oint_S S(\vec{r},t)\, dA = \frac{\mu_0}{6\pi c}\left[\ddot{p}\left(t-\frac{r}{c}\right)\right]^2$$

$$P_{rad}(t_0) = \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2$$

Examples

Oscillating Electric Dipole $p(t) = p_0\cos\omega t$ $\quad\ddot{p}(t) = -\omega_0^2 p_0\cos(\omega t)$

Single charge $\vec{p} = q\vec{d}(t)$

$\ddot{\vec{p}}(t) = q\vec{a}(t)$

$P = \frac{\mu_0 q^2 a^2}{6\pi c}$ — Larmor formula

Point charges

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\imath}{(\vec{\imath}\cdot\vec{u})^3}\left[\vec{\imath}\times(\vec{u}\times\vec{a})\right] \qquad \vec{u} = c\hat{\imath} - \vec{v}$$

$$\vec{B} = \frac{1}{c}\hat{\imath}\times\vec{E}(\vec{r},t)$$

$\vec{E} \perp$ to $\hat{\imath}$

$$\vec{S} = \frac{1}{\mu_0 c}\left[\vec{E}\times(\hat{\imath}\times\vec{E})\right] = \frac{1}{\mu_0 c}\left[E^2\hat{\imath} - \cancel{(\hat{\imath}\cdot\vec{E})\vec{E}}\right]$$

$$= \frac{1}{\mu_0} E_{rad}^2\,\hat{\imath}$$

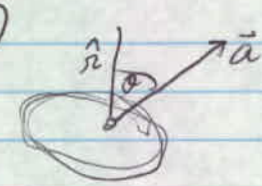
If charge is instantaneously at rest (at time t_r) $\vec{u} = c\hat{\imath}$

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2\imath}\left[\hat{\imath}\times(\hat{\imath}\times\vec{a})\right] = \frac{\mu_0 q}{4\pi\imath}\left[(\hat{\imath}\cdot\vec{a})\hat{\imath} - \vec{a}\right]$$

$$\left((\hat{\imath}\cdot\vec{a})\hat{\imath} - \vec{a}\right)\cdot\left((\hat{\imath}\cdot\vec{a})\hat{\imath} - \vec{a}\right) = -(\hat{\imath}\cdot\vec{a})^2 + a^2$$

$$S_{rad} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\imath}\right)^2\left[a^2 - (\hat{\imath}\cdot a)^2\right]\hat{\imath}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\imath} \qquad (A)$$



Total Power radiated

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\theta}{\imath^2}\imath^2\sin\theta\, d\theta\, d\phi \qquad (B)$$

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}} \quad \text{— Larmor formula}$$

HW Problems 11.3, 11.4, 11.8, 11.9, 11.10

Eqs. (A) + (B) hold accurately ~~except when~~ as long as $v \ll c$

If $\frac{dW}{dt}$ is rate at which energy passes through the sphere at radius $\imath$

then rate at which energy left the charge was

$$t_r = t - \frac{\imath}{c}$$

$$\frac{dW}{dt_r} = \frac{dW/dt}{\partial t_r/\partial t} = \frac{dW}{dt}\frac{\vec{\imath}\cdot\vec{u}}{\imath c} \qquad \frac{\partial t_r}{\partial t} = \frac{\imath c}{\vec{\imath}\cdot\vec{u}}$$

$$\frac{\vec{\imath}\cdot\vec{u}}{\imath c} = 1 - \frac{\hat{\imath}\cdot\vec{v}}{c}$$

Power radiated into a patch of area $\imath^2\sin\theta\, d\theta\, d\phi = \imath^2 d\Omega$

$$\frac{dP}{d\Omega} = \frac{\vec{\imath}\cdot\vec{u}}{\imath c}\frac{1}{\mu_0 c}E_{rad}^2\imath^2 = \frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{\imath}\times(\vec{u}\times\vec{a})|^2}{(\hat{\imath}\cdot\vec{u})^5} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$P = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$$

Example $\vec{v}$ + $\vec{a}$ collinear

$$\vec{u} \times \vec{a} = c(\hat{r} \times \vec{a})$$

$$\frac{dP}{d\Omega} = \frac{q^2 c^2}{16\pi^2 \epsilon_0} \frac{|\hat{r} \times (\hat{r} \times a)|^2}{(c - \hat{r}\cdot\vec{v})^5}$$

$$\vec{r}\cdot\vec{u} = cr - \vec{r}\cdot\vec{v}$$
$$= cr - rv\cos\theta = cr(1-\beta\cos\theta)$$
$$\hat{r}\cdot\vec{v} = rv\cos\theta$$
$$c - \hat{r}\cdot\vec{v} = c - v\cos\theta = c(1-\beta\cos\theta)$$

$$\hat{r} \times (\hat{r} \times \vec{u}) = (\vec{r}\cdot\vec{a})\hat{r} - \hat{r} \quad \rightarrow \quad |\hat{r} \times (\hat{r} \times u)|^2 = a^2 - (\vec{r}\cdot\vec{a})^2$$
$$= a^2 \sin^2\theta$$

Take $\vec{v}$ along z-axis

$$\frac{dP}{dt} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad \beta = \frac{v}{c}$$