1. (i) Without loss of generality consider $x^0 > y^0$ in

$$\langle 0| T\phi_1(x)\phi_2(y)|0\rangle = \langle 0|\phi_1(x)\phi_2(y)|0\rangle$$

Since these are free fields ("in") we can expand in modes

$$\phi_n(x) = \int (dk)\left(e^{-ik\cdot x}\alpha_{n,\vec{k}} + e^{ik\cdot x}\alpha^\dagger_{n,\vec{k}}\right)$$

with $[\alpha_{n\vec{k}}, \alpha^\dagger_{n'\vec{k}'}] = \delta_{nn'}\, 2E(2\pi)^3\delta^3(\vec{k}-\vec{k}')$, $[\alpha_{n\vec{k}}, \alpha_{n'\vec{k}'}] = 0$

In $\langle 0|\phi_1(x)\phi_2(y)|0\rangle$ we have 4 terms in the mode expansion:

$\langle 0|\alpha_{1\vec{k}}\alpha_{2\vec{k}'}|0\rangle = 0 = \langle 0|\alpha^\dagger_{1\vec{k}}\alpha_{2\vec{k}'}|0\rangle$ vanish because $\alpha_{2\vec{k}}|0\rangle = 0$

$\langle 0|\alpha^\dagger_{1\vec{k}}\alpha^\dagger_{2\vec{k}'}|0\rangle = 0$ vanishes because $\alpha_{1\vec{k}}|0\rangle = 0$

and $\langle 0|\alpha_{1\vec{k}}\alpha^\dagger_{2\vec{k}'}|0\rangle = 0$ because $\alpha_{1\vec{k}}\alpha^\dagger_{2\vec{k}'} = \alpha^\dagger_{2\vec{k}'}\alpha_{1\vec{k}}$ and then $\alpha_{1\vec{k}}|0\rangle = 0$.

$\Rightarrow \langle 0|T(\phi_1(x)\phi_2(y))|0\rangle = 0$

(ii) $\psi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \Rightarrow \langle 0|T(\psi(x)\psi(y))|0\rangle = \frac{1}{2}\langle 0|T\phi_1(x)\phi_1(y)|0\rangle - \frac{1}{2}\langle 0|T\phi_2(x)\phi_2(y)|0\rangle$

$$- \frac{i}{2}\langle 0|T\phi_1(x)\phi_2(y)|0\rangle + \frac{i}{2}\langle 0|T\phi_2(x)\phi_1(y)|0\rangle$$

The first two terms each give $\pm\frac{1}{2}G^{(2)}(x,y) = \pm\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\frac{i}{p^2-m^2+i\epsilon}$ so they cancel.

The last two terms vanish by part (i).

Similarly $\langle 0|T(\psi(x)\psi^\dagger(y))|0\rangle = \frac{1}{2}\langle 0|T\phi_1(x)\phi_1(y))|0\rangle + \frac{1}{2}\langle 0|T\phi_2(x)\phi_2(y)|0\rangle = G^{(2)}(x,y)$

(iii) We need to understand the difference between

$T(\psi_{in}(x_1)\cdots\psi_{in}(x_k)\,\psi^\dagger_{in}(y_1)\cdots\psi^\dagger_{in}(y_\ell))$ and $:\psi_{in}(x_1)\cdots\psi_{in}(x_k)\psi^\dagger_{in}(y_1)\cdots\psi^\dagger_{in}(y_\ell):$

Better start with two fields; recall

$$\psi_{in}(x) = \int (dk)\left(\beta_k e^{-ik\cdot x} + \gamma^\dagger_k e^{ik\cdot x}\right) \equiv \psi^{(+)}_{in}(x) + \psi^{(-)}_{in}(x)$$

Suppress the "in" label. Use shorthand ψ_i for $\psi(x_i)$.

Now take $x_1^0 > x_2^0$:

$$T\psi_1\psi_2 - :\psi_1\psi_2: = \psi_1\psi_2 - :\psi_1\psi_2:$$

$$= \psi_1^+\psi_2^+ + \psi_1^+\psi_2^- + \psi_1^-\psi_2^+ + \psi_1^-\psi_2^- - (\psi_1^+\psi_2^+ + \psi_2^-\psi_1^+ + \psi_1^-\psi_2^+ + \psi_1^-\psi_2^-)$$

$$= [\psi_1^+, \psi_2^-]$$

$$= 0 \quad (\text{since } [\beta_k, \delta_{k'}^\dagger] = 0).$$

This is different than in the real scalar case, for which $\phi^- = (\phi^+)^\dagger$.

Similarly $T\psi_1^\dagger\psi_2^\dagger = :\psi_1^\dagger\psi_2^\dagger:$

The non-trivial case is

$$T\psi_1\psi_2^\dagger - :\psi_1\psi_2^\dagger: = [\psi_1^{(+)}, \psi_2^{(+)\dagger}] = \text{a c-number} = \langle 0|T\psi_1\psi_2^\dagger|0\rangle \equiv \overbrace{\psi_1\psi_2^\dagger}$$

Wick's theorem goes as before

$$T(\psi_1\cdots\psi_n\psi_{n+1}^\dagger\cdots\psi_m^\dagger) = :\psi_1\cdots\psi_n\psi_{n+1}^\dagger\cdots\psi_m^\dagger: + \text{all possible contractions } \overbrace{\psi\psi^\dagger}$$

In particular $\langle 0|T\psi_1\cdots\psi_n\psi_{n+1}^\dagger\cdots\psi_m^\dagger|0\rangle = 0$ if $m \neq n$, and

$$\langle 0|T\psi(x_1)\cdots\psi(x_n)\psi^\dagger(y_1)\cdots\psi^\dagger(y_n)|0\rangle = \sum_{\substack{\text{permutations}\\ \pi}} \overbrace{\psi(x_1)\psi^\dagger(y_{\pi(1)})}\cdots\overbrace{\psi(x_n)\psi^\dagger(y_{\pi(n)})}$$

2. $\mathcal{L} = (\partial_\mu \psi)^* \partial^\mu \psi - m^2 \psi^* \psi + \rho \psi + \rho^* \psi^\dagger$

I will analyze this two different ways. First I will use what we know: real scalars with source. So write $\psi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$ and $\rho = \frac{J_1 - i J_2}{\sqrt{2}}$

Then

$$\mathcal{L} = \sum_{i=1}^{2} \left[\tfrac{1}{2} (\partial_\mu \phi_i)^2 - \tfrac{1}{2} m^2 \phi_i^2 + J_i \phi_i \right]$$

This is the sum of 2 copies of the case analyzed in class. And the copies are independent (the operators commute, $[\phi_1, \phi_2] = [\phi_1, \Pi_2] = [\phi_2, \Pi_1] = [\Pi_2, \Pi_1] = 0$)

So the S-matrix is

$$S = T e^{i\int d^4x \, \mathcal{L}'_{in}} = T e^{i\int d^4x \,(J_1 \phi_{1in} + J_2 \phi_{2in})} = T e^{i\int d^4x\, J_1 \phi_{1in}} \; T e^{i\int d^4x\, J_2 \phi_{2in}}$$

Now, this form is appropriate for computing matrix elements between particles labeled by "1" or "2", created by $\phi_1^{(+)}$ or $\phi_2^{(+)}$, respectively. (And you see immediately that the probability of creating n_1 & n_2 particles out of the vacuum is $P_{n_1 n_2} = \left(\frac{1}{n_1!} \xi_1^{n} e^{-\xi_1}\right)\left(\frac{1}{n_2!} \xi_2^{n} e^{-\xi_2}\right)$ with $\xi_i = \int (dk) \, |\tilde{J}_i(k)|^2$.

The problem with this is that the particles we created are neither "+" nor "−" but rather superpositions of them. After all $\beta_{\vec{k}} = \frac{\alpha_{1\vec{k}} + i\alpha_{2\vec{k}}}{\sqrt{2}}$ $\quad \gamma_{\vec{k}} = \frac{\alpha_{1\vec{k}} - i\alpha_{2\vec{k}}}{\sqrt{2}}$

so the particles created by $\alpha_k^\dagger$ are

$$\alpha^\dagger_{1\vec{k}} |0\rangle = \frac{\beta^\dagger_{\vec{k}} + \gamma^\dagger_{\vec{k}}}{\sqrt{2}} |0\rangle = \frac{1}{\sqrt{2}} \left(|\vec{k}+\rangle + |\vec{k}-\rangle \right)$$

This first way of computing was included because it is instructional and because it gives, with little computation, $p_0 = \text{prob}(0 \to 0) = e^{-\xi_1 - \xi_2} = \exp\left(-\int (dk) \left(|J_1(k)|^2 + |J_2(k)|^2\right)\right)$

We'll compare with this result, below.

We really want the probability of making charged states, created by $\phi_1^{(+)} + i\phi_2^{(+)}$. So we write

$$S = T\, e^{i\int d^4x\,(J_1\phi_{1_{in}} + J_2\phi_{2_{in}})} = T\, e^{i\int d^4x\,(\rho\psi_{in} + \rho^*\psi_{in}^\dagger)}$$

We need to compute the matrix element of S between $|0\rangle_{in}$ and a final state with n particles of type "+" (created by $\psi_{in}^{(+)}$). Now $\psi_{in} = \psi_{in}^{(+)} + \psi_{in}^{(-)}$ where $\psi_{in}^{(+)}$ annihilates "+" states and $\psi_{in}^{(-)}$ creates "−" states, while $\psi_{in}^{(+)\dagger}$ creates "+" states and $\psi^{(-)\dagger}$ annihilates "−" states. We need a version of Wick's theorem appropriate to this case.

$$\text{Now}\quad S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n\, T\big((\rho_1\psi_1 + \rho_1^*\psi_1^\dagger)\cdots(\rho_n\psi_n + \rho_n^*\psi_n^\dagger)\big)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{i^n}{n!}\binom{n}{k}\int d^4x_1\cdots d^4x_n\, \rho_1\cdots\rho_k\rho_{k+1}^*\cdots\rho_n^*\, T\big(\psi_1\cdots\psi_k\psi_{k+1}^\dagger\cdots\psi_n^\dagger\big)$$

Wick's theorem now gives $T(\cdots) = :(\cdots): +$ contractions. The 1st contraction involves one pair $\psi, \psi^\dagger$. Combinatorics: $k(n-k)$. The prefactor is $\frac{1}{n!}\binom{n}{k}k(n-k) = \frac{1}{(n-k-1)!\,(k-1)!} = \frac{1}{(n-2)!}\binom{n-2}{k-1}$

so $S(\text{one contraction terms}) = \xi \sum_{n=0}^{\infty} \frac{i^{n-2}}{(n-2)!}\binom{n-2}{k-2}\int d^4x_1\cdots d^4x_{n-2}\, \rho_1\cdots\rho_{k-1}\rho_k^*\cdots\rho_{n-2}^* \; :\psi_1\cdots\psi_{k-1}\psi_k^\dagger\cdots\psi_{n-2}^\dagger:$

where $\xi = i^2\int d^4x\, d^4y\; \rho(x)\rho^*(y)\,\langle 0|T\psi(x)\,\psi(y)^\dagger|0\rangle$

Aside: Let's compute ξ:

$$\xi = -\int d^4x\, d^4y\; \rho(x)\rho^*(y)\int\frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\,\frac{i}{p^2-m^2+i\epsilon}$$

$$= -i\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2-m^2+i\epsilon}\int d^4x\, e^{-ip\cdot x}\rho(x)\int d^4y\, e^{ip\cdot y}\rho^*(y)$$

$$= -i\int\frac{d^4p}{(2\pi)^4}\,\frac{|\tilde\rho(p)|^2}{p^2-m^2+i\epsilon}$$

We will later need

$\mathrm{Re}\,\xi = \mathrm{Im}\int\frac{d^4p}{(2\pi)^4}\frac{|\tilde\rho(p)|^2}{p^2-m^2+i\epsilon}$; use $\mathrm{Im}\frac{1}{\omega \pm i\epsilon} = \mp\pi\delta(\omega)$ to get

$$\mathrm{Re}\,\xi = -\frac{1}{2}\int(dp)\,\big(|\tilde\rho(p)|^2 + |\tilde\rho(-p)|^2\big).$$

To complete the calculation we need similar combinatorics for the term with m Wick contractions.

How many equivalent m-contractions in $\psi_1 \cdots \psi_k \psi^\dagger_{k+1} \cdots \psi^\dagger_n$?

We can choose $\binom{k}{m}$ fields in $\psi_1 \cdots \psi_k$, and $\binom{n-k}{m}$ in $\psi^\dagger_{k+1} \cdots \psi^\dagger_n$.

We are left with two sets of m objects, and ask how many ways of pairing them:

$(a_1 \cdots a_m)(b_1 \cdots b_m)$: there are $m!$ ways of pairing all a's with b's.

So the term in S' has factor

$$\frac{1}{n!}\binom{n}{k}\binom{k}{m}\binom{n-k}{m} m! = \frac{1}{n!}\frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!}\frac{(n-k)!}{m!(n-k-m)!} m! = \frac{1}{m!(k-m)!(n-k-m)!}$$

$$\text{so} \quad S(m\text{ contractions}) = \frac{1}{m!}\zeta^m \sum_{n=0}^{\infty}\sum_k \frac{i^{n-2m}}{(n-2m)!}\binom{n-2m}{k-m}\int d^4x_1 \cdots d^4x_{n-2m}\, \rho_1 \cdots \rho_{k-m} \rho^*_{k-m+1} \cdots \rho^*_{n-2m} :\psi_1 \cdots \psi^\dagger_{n-2m}:$$

Summing over all contractions, $S = :e^{i\int d^4x(\rho\psi + \rho^*\psi^\dagger)}: e^{\zeta}$

Compute:

vac $\to$ vac: $\langle 0|S|0\rangle = \langle 0| :e^{i\int d^4x(\rho\psi + \rho^*\psi^\dagger)}: e^{\zeta}|0\rangle = e^{\zeta}$

vac $\to$ "+" : $\langle \vec{k}+|S|0\rangle = \langle \vec{k}+| :i\int d^4x\,(\rho\psi + \rho^*\psi^\dagger)|0\rangle e^{\zeta}$

$$= i\int d^4x\,\rho^*(x)\,\langle \vec{k}+|\,\psi^\dagger(x)\,|0\rangle\, e^{\zeta}$$

$$= i\int d^4x\,\rho^*(x)\,\langle 0|\,\beta_{\vec{k}}\,\psi^\dagger(x)|0\rangle\, e^{\zeta}$$

$$= i\int d^4x\,\rho^*(x)\,e^{ik\cdot x}\, e^{\zeta}$$

$$= i\,\tilde{\rho}^*(k)\,e^{\zeta}$$

and $\int (dk)\,|\langle \vec{k}+|S|0\rangle|^2 = e^{\zeta+\zeta^*}\int (dk)\,|\tilde{\rho}(k)|^2$

More usefully, with $\psi(x) = \int (dp)\left(e^{-ip\cdot x}\beta_p + e^{ip\cdot x}\gamma_p^{+}\right)$

and $|\vec{k}+\rangle = \beta^{+}_{\vec{k}}|0\rangle$, $|\vec{k}-\rangle = \gamma^{+}_{\vec{k}}|0\rangle$

$$\beta_{\vec{k}}\, S = \beta_k :\exp\left(i\int d^4x\, \rho\psi + \rho^{*}\psi^{+}\right): e^{\xi}$$

Sketch:

$$\sum_n \frac{1}{n!}\beta_k (A\beta + C\beta^{+})^n = \sum_n \frac{1}{n!}\left[n[\beta_k, A\beta + C\beta^{+}](A\beta + C\beta^{+})^{n-1} + (A\beta + C\beta^{+})^n \beta_k\right]$$

so $\beta_k S = S\left(\beta_k + [\beta_k, i\int d^4x\, \rho^{*}(x)\psi^{+}(x)]\right)$

$$= S\left(\beta_k + i\int d^4x\, \rho^{*}(x)\, e^{ik\cdot x}\right) = S\left(\beta_k + i\tilde{\rho}^{*}(k)\right)$$

And $\beta_{k_1}\cdots\beta_{k_n} S = S(\beta_{k_1} + i\tilde{\rho}^{*}(k_1))\cdots(\beta_{k_n} + i\tilde{\rho}^{*}(k_n))$

The rest of the computation is identical to that in lecture...

prob of vac $\to$ n particles "+" $= \frac{1}{n!}\left(\int (dk)\, |\tilde{\rho}(k)|^2\right)^n e^{2\xi}$

If we creat "−" particles, the calculation is as above, but $\rho\psi$ rather than $\rho^{*}\psi^{*}$:

$$\gamma_{\vec{k}}\, S = S\left(\gamma_k + i\int d^4x\, \rho(x)\, e^{ik\cdot x}\right) = S\left(\gamma_k + i\tilde{\rho}(-k)\right)$$

Since $[\gamma_{\vec{k}}, \beta_{\vec{k}'}] = 0$ we have

prob vac $\to$ n_+ "+" particles and n_- "−" particles

$$= \left[\frac{1}{n_+!}\left(\int (dk)\, |\tilde{\rho}(k)|^2\right)^{n_+}\right]\left[\frac{1}{n_-!}\left(\int (dk)\, |\tilde{\rho}(-k)|^2\right)^{n_-}\right] e^{2\,\mathrm{Re}\,\xi}$$

The integrals are not the same. While we can change variables $\vec{k} \to -\vec{k}$, the 0-th component $k^0 = E_{\vec{k}}\ (= \sqrt{\vec{k}^2 + m^2})$ is positive in $\tilde{\rho}(k)$ and negative in $\tilde{\rho}(-k)$.

Finally let's compare $0 \to 0$ here with the calculation using $\phi_{1,2}$ (real fields).

With $\phi_1 \phi_2$ we obtained

$$\log p_0 = -\int (dk)\, \left(|\tilde{J}_1(k)|^2 + |\tilde{J}_2(k)|^2 \right)$$

With ψ we obtained

$$\log p_0 = 2 \operatorname{Re} S = -\int (dk) \left(|\tilde{\rho}(k)|^2 + |\tilde{\rho}(-k)|^2 \right)$$

Now, recall $\rho(x) = \frac{J_1 + i J_2}{\sqrt{2}}$

So $\tilde{\rho}(k) = \frac{1}{\sqrt{2}} \left(\tilde{J}_1(k) + i\, \tilde{J}_2(k) \right)$

and
$$|\tilde{\rho}(k)|^2 + |\tilde{\rho}(-k)|^2 = \frac{1}{2} \left| \tilde{J}_1(k) + i\, \tilde{J}_2(k) \right|^2 + \frac{1}{2} \left| \tilde{J}_1(-k) + i\, \tilde{J}_2(-k) \right|^2$$
$$= \frac{1}{2} \left| \tilde{J}_1(k) + i\, \tilde{J}_2(k) \right|^2 + \frac{1}{2} \left| \tilde{J}_1(k) - i\, \tilde{J}_2(k) \right|^2$$

where we have used $\tilde{J}_{1,2}^*(k) = \tilde{J}_{1,2}(-k)$.

Hence

$$|\tilde{\rho}(k)|^2 + |\tilde{\rho}(-k)|^2 = |\tilde{J}_1(k)|^2 + |\tilde{J}_2(k)|^2$$

and the two ways of computing p_0 coincide.

3. We have $|g\rangle = \int (dk)\, g(\vec{k})\, \alpha^\dagger_{\vec{k}} |0\rangle$

We will need $1 = \langle g|g\rangle = \int (dk')(dk)\, g^*(\vec{k}')\, g(\vec{k}) \langle 0|\alpha_{\vec{k}'} \alpha^\dagger_{\vec{k}}|0\rangle = \int (dk) |g(\vec{k})|^2$

(i) For final state $|f\rangle_{out} = S^\dagger |f\rangle_{in}$ we compute

$${}_{out}\langle f|g\rangle_{in} = {}_{in}\langle f|S|g\rangle_{in} = \langle f|S|g\rangle \quad \text{for short.}$$

Now

$$S = e^{i\int d^4x\, \phi^-_{in}(x) J(x)}\, e^{i\int d^4x\, \phi^+_{in}(x) J(x)}\, e^{-\frac{1}{2}\zeta} \quad \text{, with } \zeta = \int (dk) |\tilde{J}(k)|^2$$

We'll need $S\alpha^\dagger_k = (S\alpha^\dagger_k S^{-1})S$ so consider

$$e^{i\int d^4x\, \phi^+_{in} J}\, \alpha^\dagger_k\, e^{-i\int d^4x\, \phi^+_{in} J} = \alpha^\dagger_k + i\int d^4x\, J(x)\, [\phi^+_{in}(x), \alpha^\dagger_k]$$

$$= \alpha^\dagger_k + i\int d^4x\, J(x)\, e^{-ik\cdot x}$$

$$= \alpha^\dagger_k + i\,\tilde{J}(k)$$

So we have

$$\langle 0|S|g\rangle = \int (dk)\, g(\vec{k}) \langle 0|S\alpha^\dagger_k|0\rangle$$

$$= \int (dk)\, g(\vec{k}) \langle 0|(\alpha^\dagger_k + i\tilde{J}(k))S|0\rangle$$

$$= i\int (dk)\, g(\vec{k})\, \tilde{J}(k)\, e^{-\frac{1}{2}\zeta}$$

So $p_{-1} = \left|\int (dk)\, g(\vec{k})\, \tilde{J}(k)\right|^2 e^{-\zeta}$

Now for $p_{0,1}$ we also need $S^{-1}\alpha_k S = \alpha_k + i\,\tilde{J}(-k)$ (from lecture)

So $\langle \vec{k}|S|g\rangle = \langle 0|\alpha_k S|g\rangle = \langle 0|S\alpha_k|g\rangle + i\,\tilde{J}(-k)\langle 0|S|g\rangle$

We already computed the 2^{nd} term. The first is

$\int (dp)\, g(\vec{p}) \langle 0|S\alpha_k\alpha^\dagger_p|0\rangle = g(\vec{k}) \langle 0|S|0\rangle = g(\vec{k})\, e^{-\frac{1}{2}\zeta}$

Combining,

$$\langle \vec{k}|S|g\rangle = e^{-\frac{1}{2}\zeta}\left(g(\vec{k}) - \tilde{J}(-k)\int (dp)\, g(\vec{p})\, \tilde{J}(p)\right)$$

and $p_0 = \int (dk)\, |\langle \vec{k}|S|g\rangle|^2 = e^{-\zeta}\int (dk) \left| g(\vec{k}) - \tilde{J}(-k)\int (dp)\, g(\vec{p})\tilde{J}(p)\right|^2$

This can be simplified. let $\chi = \int (dp)\, g(\vec{p})\, \tilde{J}(p)$

Then $|g(\vec{k}) - \tilde{J}(-k)\chi|^2 = |g(\vec{k})|^2 + |\chi|^2 |\tilde{J}(k)|^2 - g^*(\vec{k})\, \tilde{J}(-k)\chi - g(\vec{k})\, \tilde{J}^*(-k)\, \chi^*$

Integrate, using $\langle g|g\rangle = 1$ and reality $\tilde{J}^*(-k) = J(k)$,

$$\int (dk)\, |g - \chi \tilde{J}|^2 = 1 + |\chi|^2 \int (dp)\, |\tilde{J}(p)|^2 - \chi \int (dk)\, g^*(\vec{k})\, \tilde{J}^*(k) - \chi^* \int (dk)\, g(\vec{k})\, \tilde{J}(k)$$

or

$$p_0 = e^{-S}\left(1 + S|\chi|^2 - 2|\chi|^2\right) =$$

Finally, we need

$$\langle \vec{k}_1 \vec{k}_2 | S | g\rangle = \langle 0 | \alpha_{\vec{k}_1} \alpha_{\vec{k}_2} S | g \rangle = \langle 0 | S (\alpha_{\vec{k}_1} + i\tilde{J}(-k_1))(\alpha_{\vec{k}_2} + i\tilde{J}(-k_2)) | g\rangle$$

$$= \langle 0 | S \alpha_{k_1}\alpha_{k_2} | g\rangle + i\tilde{J}(-k_1)\langle 0|S\alpha_{k_2}|g\rangle + i\tilde{J}(-k_2)\langle 0|S\alpha_{k_1}|g\rangle$$

$$+ i^2 \tilde{J}(-k_1)\tilde{J}(-k_2)\langle 0|S|g\rangle$$

Only the 1st term is new:

$$\langle 0|S\alpha_{k_1}\alpha_{k_2}|g\rangle = \int (dp)\, g(\vec{p})\, \langle 0|S\alpha_{k_1}\alpha_{k_2}\alpha_p^\dagger|0\rangle = g(\vec{k}_2)\langle 0|S\alpha_{k_1}|0\rangle = 0$$

So

$$\langle \vec{k}_1\vec{k}_2|S|g\rangle = \left[i\tilde{J}(-k_1)\, g(\vec{k}_2) + i\tilde{J}(-k_2)\, g(\vec{k}_1) - i\tilde{J}(-k_1)\tilde{J}(-k_2)\chi\right] e^{-\frac{1}{2}S}$$

Finally

$$p_{+1} = \frac{1}{2}\int (dk_1)(dk_2)\, |\langle k_1 k_2|S|g\rangle|^2 = \frac{1}{2}e^{-S}\int (dk_1)(dk_2)\, |\tilde{J}(-k_1)\, g(k_2) + \tilde{J}(-k_2)\, g(\vec{k}_1) - \chi\tilde{J}(-k_1)\tilde{J}(-k_2)|^2$$

$$= \frac{1}{2}e^{-S}\left[2S + |\chi|^2 S^2 + 2|\chi|^2 - 4S|\chi|^2\right]$$

Note that

$$p_{-1} + p_0 + p_1 = e^{-S}\left[|\chi|^2 + 1 + S|\chi|^2 - 2|\chi|^2 + S + \tfrac{1}{2}|\chi|^2 S^2 + |\chi|^2 - 2S|\chi|^2\right]$$

$$= e^{-S}\left[1 + S - S|\chi|^2 + \tfrac{1}{2}S^2|\chi|^2\right]$$

For a weak source $S \sim |\chi|^2 \ll 1$, to lowest order

$$= (1 - S + \cdots)(1 + S + \cdots) = 1.$$

4. (i) In the time ordered product use $:\phi_1\phi_2: = T(\phi_1\phi_2) - \overbrace{\phi_1\phi_1}$

$$T(:\phi_1\phi_2::\phi_3\phi_4:) = T\left[\left(T(\phi_1\phi_2) - \overbrace{\phi_1\phi_2}\right)\left(T(\phi_3\phi_4) - \overbrace{\phi_3\phi_4}\right)\right]$$

$$= T\left[T(\phi_1\phi_2)T(\phi_3\phi_4)\right] - T\left[T\phi_1\phi_2\right]\overbrace{\phi_3\phi_4} - T\left[T(\phi_3\phi_4)\right]\overbrace{\phi_1\phi_1} + \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}$$

$$= T(\phi_1\phi_2\phi_3\phi_4) - T(\phi_1\phi_1)\overbrace{\phi_3\phi_4} - T(\phi_3\phi_4)\overbrace{\phi_1\phi_2} + \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}$$

$$= T(\phi_1\phi_2\phi_3\phi_4) - \left(:\phi_1\phi_2: + \overbrace{\phi_1\phi_2}\right)\overbrace{\phi_3\phi_4} - \left(:\phi_3\phi_4: + \overbrace{\phi_3\phi_4}\right)\overbrace{\phi_1\phi_2} + \overbrace{\phi_1\phi_1}\,\overbrace{\phi_3\phi_4}$$

$$= T(\phi_1\phi_2\phi_3\phi_4) - :\phi_1\phi_2\,\overbrace{\phi_3\phi_4}: - :\overbrace{\phi_1\phi_2}\,\phi_3\phi_4: - :\overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}:$$

That is, from $T(\phi_1\phi_2\phi_3\phi_4)$ which is $:\phi_1\phi_2\phi_3\phi_4:$ plus all possible contractions take away terms with contractions of $\overbrace{\phi_1\phi_2}$ or $\overbrace{\phi_3\phi_4}$ or both. Explicitly:

$$= :\phi_1\phi_2\phi_3\phi_4: + :\overbrace{\phi_1\phi_2}\phi_3\phi_4: + :\overbrace{\phi_1\phi_2\phi_3\phi_4}: + :\phi_1\overbrace{\phi_2\phi_3}\phi_4: + :\phi_1\overbrace{\phi_2\phi_3\phi_4}:$$

(ii) Already stated above: missing are $\overbrace{\phi_1\phi_2}$ and $\overbrace{\phi_3\phi_4}$

(iii) In $T(:\phi_1\cdots\phi_k: \;\cdots\; :\phi_\ell\cdots\phi_n:)$

we omit any contractions among fields that appear within any one normal ordered product.