PHYSICS 215A: PARTICLES AND FIELDS, FALL 2014
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Assignment 3

November 3 (Due November 17)

1. Some systems are "scale-invariant." This is a symmetry under blowing-up spacetime by a factor, $x^\mu \to \lambda^{-1} x^\mu$, and rescaling fields by some corresponding factor. Derive the Noether current and associated constant of the motion, or charge, for scale-invariant theories. More specifically:

(i) Consider the transformation of some field

$$\phi(x) \to \phi'(x) = \lambda^D \phi(\lambda x)$$

where D is a constant and λ is the parameter of the transformation. Similarly, for the Lagrangian density

$$\mathscr{L}(x) \to \mathscr{L}'(x) = \lambda^{\tilde{D}} \mathscr{L}(\lambda x)$$

If this is to be a symmetry of the action integral, $S = \int dt d^3x \, \mathscr{L}$, what value must you choose for $\tilde{D}$? What value of D must you choose for the kinetic part of the Lagrangian, $\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$, to give a scaling factor of $\tilde{D}$, and what is this when $\tilde{D}$ is chosen to make the action integral invariant?

(ii) Derive the form of the conserved Noether current for the class of theories for which S is invariant under the transformation $\phi(x) \to \phi'(x) = \lambda^D \phi(\lambda x)$ (use the values for D and $\tilde{D}$ derived in part (i)). This current is alternatively called the "scale," "dilation" or even "dilatation" current in the literature and we will denote it as S^μ. Eliminate the term in S^μ that depends explicitly on the Lagrangian density by re-writing it in terms of the stress-energy tensor.

(iii) Consider the theory specified by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

What must be the form of the "potential" function $V(\phi)$ so that the action integral remains invariant? Verify by explicit calculation (using the equations of motion) that the current is conserved.

(iv) Repeat the above steps for the case of an d-dimensional spacetime (that is $d-1$ space dimensions plus one time). How does D depend on d and what is the form of $V(\phi)$ consistent with invariance of $S = \int d^d x \, \mathscr{L}$?

(v) Compare the results here (for the form of $V(\phi)$) with the results of Assignment 1, question 2, on the mass dimension of various parameters in the Lagrangian. What conclusions do you draw?

2. *Ambiguities in Conserved Currents.* If J^μ is the conserved current in Noether's theorem then $J^\mu + \Delta J^\mu$ is also a conserved current with the same symmetry generator $T = \int d^3x\, J^0$ if $\partial_\mu \Delta J^\mu = 0$ and $\int d^3x\, \Delta J^0 = 0$. (If $\partial_\mu \Delta J^\mu = 0$ and $\Delta T = \int d^3x\, \Delta J^0 \neq 0$ then ΔT itself may generate a distinct new symmetry). In this problem you will construct a new "improved" stress-energy tensor $\Theta^{\mu\nu}$ for the theory specified by

$$\mathscr{L} = \tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

such that

$$\partial_\mu \Theta^{\mu\nu} = 0 \quad \text{and} \quad P^\mu = \int d^3x\, \Theta^{0\mu} = \int d^3x\, T^{0\mu}. \tag{1}$$

(i) Show that

$$\Theta^{\mu\nu} = T^{\mu\nu} + \kappa(\partial^\mu \partial^\nu - g^{\mu\nu}\Box)\phi^2,$$

where κ is a constant and $\Box = \partial^2 = g^{\lambda\sigma}\partial_\lambda \partial_\sigma$, satisfies the conditions in (1).

(ii) Show that for a particular value of κ and a particular form of the potential $V(\phi)$ the trace of this tensor vanishes, $\Theta^\mu{}_\mu = g_{\mu\nu}\Theta^{\mu\nu} = 0$, provided ϕ satisfies the equations of motion. What is the value of κ and the form of $V(\phi)$ for which this happens? (By a "form" of $V(\phi)$ I mean a specific functional dependence). For this particular value of κ the currents $\Theta^{\mu\nu}$ are called "improved," and we refer to $\Theta^{\mu\nu}$ as the "improved energy-momentum tensor" or the "improved stress-energy tensor."

(iii) Show that the scale current S^μ can be written as

$$S^\mu = x^\nu \Theta^\mu{}_\nu + \tfrac{1}{6}(\partial^\mu \partial^\nu - g^{\mu\nu}\Box)(x_\nu \phi^2),$$

and that there is an improved dilatation current $\tilde{S}^\mu$ such that $\partial_\mu \tilde{S}^\mu = \Theta^\mu{}_\mu$ so that $\tilde{S}^\mu$ is conserved if and only if $\Theta^{\mu\nu}$ is traceless.

(iv) If $V(\phi) = \frac{1}{2}m^2\phi^2 + g\phi^3 + \lambda\phi^4$ where m, g and λ are constants, show that $\Theta^\mu{}_\mu = \Delta \neq 0$ and use the equations of motion to give Δ as a polynomial in ϕ.

3. *Conformal Transformations.* Another interesting set of transformations that may be a symmetry of a theory is the set of conformal transformations. These can be seen as the composition of an inversion, $x^\mu \to -x^\mu/x^2$, followed by a translation $x^\mu \to x^\mu + a^\mu$, followed by another inversion. Although this involves discrete transformations, it still defines a set of transformations continuously connected to the identity transformation (at $a^\mu = 0$).

(i) Show that the infinitesimal form of the transformation is

$$\delta x^\mu = x'^\mu - x^\mu = 2a\cdot x x^\mu - x^2 a^\mu.$$

(ii) Assume that under a transformation $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$, the scalar field is transformed as follows:

$$\phi(x) \to \phi'(x) = (1 + Ca\cdot x)\phi(x + \delta x).$$

Here C is a constant to be determined and the transformation is given to lowest order in a^μ. How does $\partial_\mu\phi$ transform? In preparation for applying Noether's theorem consider, at first, the simplest Lagrangian density: $\mathscr{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. For what value of the constant C is $\delta\mathscr{L}$ a total derivative, that is, $\delta\mathscr{L} = \partial_\mu\mathscr{F}^\mu$? What is $\mathscr{F}^\mu$? Generalize this result to the case where $\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$, with $V(\phi) = \lambda\phi^4$. (Why not $g\phi^3$?).

(iii) Use Noether's theorem to determine the set of four conserved currents $K^{\mu\nu}$ (and the corresponding "charges," K^μ). Show that the the explicit dependence on $\mathscr{L}$ can be eliminated in favor of energy-momentum tensor, $T^{\mu\nu}$:

$$K^{\mu\nu} = (2x^\nu x^\lambda - g^{\nu\lambda}x^2)T^\mu{}_\lambda + 2x^\nu(\partial^\mu\phi)\phi - g^{\mu\nu}\phi^2.$$

(iv) Show that one can improve the currents $K^{\mu\nu}$ (much as we improved the stress-energy tensor in problem 2 above, by adding automatically conserved currents that do not modify the charges $K^\mu = \int d^3x\, K^{0\mu}$) in such a way that the improved currents $\tilde{K}^{\mu\nu}$ satisfy

$$\tilde{K}^{\mu\nu} = (2x^\nu x^\lambda - g^{\nu\lambda}x^2)\Theta^\mu{}_\lambda.$$

What is the relation between conservation of $\tilde{K}^{\mu\nu}$ and conservation of $\tilde{S}^\mu$ (the improved dilation current)?

4. Consider the following Lagrangian density, involving complex scalar fields $A(x)$, $B(x)$ and $C(x)$,

$$\mathscr{L} = \partial_\mu A^*\partial^\mu A + \partial_\mu B^*\partial^\mu B + \partial_\mu C^*\partial^\mu C - V(A, B, C).$$

For each form of the potential energy function V listed below, find the internal symmetries of the field theory, and compute the corresponding conserved currents. For each compute the charge (that is, the symmetry generator $\int d^3x\, J^0$) in terms of creation and annihilation operators, and discuss possible conservation of particle number/type that must be present for any interaction as a result of the symmetry. Note, in all the expressions below "+c.c." means "add the complex conjugate of the preceding expression."

(i) $V = \lambda_1 AB^2 + \lambda_2 BC^2 + \text{c.c.}$

(ii) $V = \lambda_1 AB^3 + \lambda_2 (B^*)^2C^2 + \text{c.c.}$

(iii) $V = \lambda ABC + \text{c.c.}$

Discuss further:

- How are your conclusions changed if you add to each of the above potentials a term $\Delta V = m_A^2|A|^2 + m_B^2|B|^2 + m_C^2|C|^2 + g_A|A|^4$?
- How are your conclusions changed if you add to each of the above potentials a term $\tilde{g}A^4 + \text{c.c.}$ in addition to ΔV?