1. (i) Scale transformation $\phi \to \phi'(x) = \lambda^D \phi(\lambda x)$, $\mathcal{L} \to \mathcal{L}'(x) = \lambda^{\tilde{D}} \mathcal{L}(\lambda x)$.

Now $S' = \int d^4x\, \mathcal{L}'(x) = \int d^4x\, \lambda^{\tilde{D}} \mathcal{L}(\lambda x)$. Change variables: let $x'^\mu = \lambda x^\mu \Rightarrow$

$S' = \int d^4x'\, \lambda^{-4} \lambda^{\tilde{D}} \mathcal{L}(x') = \int d^4x'\, \mathcal{L}(x') = S$ if $\underline{\tilde{D} = 4}$. That is, for $\tilde{D} = 4$ $S' = S$

Next, $\mathcal{L} = \frac{1}{2} \partial_\mu \phi(x) \partial_\nu \phi(x) g^{\mu\nu} \to \mathcal{L}' = \frac{1}{2} \lambda^{2D} \dfrac{\partial \phi(\lambda x)}{\partial x^\mu} \dfrac{\partial \phi(\lambda x)}{\partial x^\nu} g^{\mu\nu}$

Now, if $x'^\mu = \lambda x^\mu$, then $\dfrac{\partial \phi(\lambda x)}{\partial x^\mu} = \dfrac{\partial x'^\sigma}{\partial x^\mu} \dfrac{\partial \phi(x')}{\partial x'^\sigma} = \dfrac{\partial(\lambda x^\sigma)}{\partial x^\mu} \partial_\sigma \phi(x') = \lambda\, \partial_\mu \phi(x')$

where $\partial_\mu \phi(x')$ means ∂_μ is with respect to the argument of ϕ, namely, x'^μ. Then

$\mathcal{L}' = \frac{1}{2} \lambda^{2D+2} \partial_\mu \phi(x') \partial_\nu \phi(x') g^{\mu\nu}$. This equals $\lambda^{\tilde{D}} \mathcal{L}(\lambda x) = \lambda^{\tilde{D}} \mathcal{L}(x')$ iff $2D + 2 = \tilde{D}$

$\Rightarrow \underline{D = \frac{\tilde{D} - 2}{2}}$. Or, with $\tilde{D} = 4$, $\underline{D = 1}$

(ii) Now take $\lambda = 1 + \epsilon$, with ϵ infinitesimal (i.e., work to linear order in ϵ).

$$\phi'(x) = (1+\epsilon)^1 \phi(1+\epsilon) = \phi(x) + \epsilon \phi(x) + \epsilon x^\mu \partial_\mu \phi(x) \Rightarrow \delta\phi = \epsilon(1 + x\cdot\partial)\phi$$

Similarly $\delta\mathcal{L} = \epsilon(4 + x\cdot\partial)\mathcal{L} = \epsilon\, \partial_\mu (x^\mu \mathcal{L})$ (where I've used $\partial_\mu x^\mu = \delta^\mu_\mu = 4$).

Then $\delta\mathcal{L} = \epsilon\, \partial_\mu \mathcal{J}^\mu$ with $\mathcal{J}^\mu = x^\mu \mathcal{L}$.

Noether: $\boxed{S^\mu = \Pi^\mu (1 + x\cdot\partial)\phi - x^\mu \mathcal{L}}$ where $\Pi^\mu \equiv \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$

Recall $T^{\mu\nu} = \Pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \to x_\nu T^{\mu\nu} = \Pi^\mu x\cdot\partial\phi - x^\mu \mathcal{L}$

$\Rightarrow S^\mu = x_\nu T^{\mu\nu} + \Pi^\mu \phi = x^\nu T^\mu{}_\nu + \frac{1}{2} \partial^\mu \phi^2$

(iii) Now $\mathcal{L}(x) = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$. We have already checked that $\frac{1}{2}(\partial_\mu \phi)^2$

transforms correctly, that is, $\frac{1}{2}(\partial_\mu \phi(x))^2 \to \lambda^4 \frac{1}{2}(\partial_\mu \phi(x'))^2$

Now, we want $V(\phi) \to V'(\phi) = \lambda^4 V(\phi(x'))$. But this must result from $V'(\phi) = V(\lambda\phi(x'))$

That is $V(\lambda\phi) = \lambda^4 V(\phi)$. $V(\phi)$ is a homogeneous function of degree 4:

$\Rightarrow V(\phi) = g\phi^4$ where g is a constant.

Explicit calculation:

$$\partial_\mu S^\mu = \partial_\mu \left[\partial^\mu \phi (1 + x\cdot\partial)\phi - x^\mu \left(\tfrac{1}{2}(\partial_\nu \phi)^2 - V(\phi)\right)\right]$$

$$= \Box\phi (1+x\cdot\partial)\phi + \partial^\mu\phi\, \partial_\mu \phi + \partial^\mu\phi\, (1+x\cdot\partial)\partial_\mu\phi - 4\left(\tfrac{1}{2}(\partial_\nu\phi)^2 - V(\phi)\right)$$

$$- x\cdot\partial\left(\tfrac{1}{2}(\partial_\nu\phi)^2 - V(\phi)\right)$$

$$= \Box\phi (1+x\cdot\partial)\phi \quad + 4V(\phi) + x\cdot\partial V(\phi)$$

The EOM is $\Box\phi + V'(\phi) = 0$ where $V'(\phi) = \frac{dV}{d\phi}$. So

$$\partial_\mu S^\mu = -V'(\phi)(1+x\cdot\partial)\phi + 4V + x\cdot\partial V$$

This does not vanish in general. But if $V = g\phi^4$, then $V' = 4g\phi^3$ and $x\cdot\partial V = 4g\phi^3 x\cdot\partial\phi$ and $\partial_\mu S^\mu = -4g\phi^3(1+x\cdot\partial)\phi + 4g\phi^4 + 4g\phi^3 x\cdot\partial\phi = 0$.

(iv) Now $\tilde{D} = d$ and $2D + 2 = d \Rightarrow D = \frac{d-2}{2}$.

Also $V(\lambda^{\frac{d-2}{2}}\phi) = \lambda^d V(\phi)$; with $\xi = \lambda^{\frac{d-2}{2}}$ so $\lambda = \xi^{\frac{2}{d-2}}$ we have $V(\xi\phi) = \xi^{\frac{2d}{d-2}} V(\phi)$ or $V(\phi) = g\,\phi^{\frac{2d}{d-2}}$.

(v) For $d=3$, $V(\phi) = g\phi^6$. For $d=6$ $V(\phi) = g\phi^3$ (and $V = g\phi^4$ for $d=4$).

From Assignment 1, problem 2, these are all cases with $[g] = 0$, that is g is dimensionless (a pure number).

This makes sense: a scale invariant theory should not have parameters that introduce a preferred scale.

2. (i) Calculate $\partial_\mu \Theta^\mu{}_\nu$ and show it vanishes. Since $\partial_\mu T^\mu{}_\nu = 0$, we only need

$$\partial_\mu \kappa (\partial^\mu \partial^\nu - g^{\mu\nu}\Box)\phi^2 = \kappa(\Box\partial^\nu - \partial^\nu\Box)\phi^2 = 0 \quad (\text{using } \Box\partial^\nu = \partial^\nu\Box).$$

Next we check that the added piece does not contribute to the charge:

$$\int d^3x \; \kappa(\partial^0\partial^\nu - g^{0\nu}\Box)\phi^2$$

for $\nu=0$ this $= \int d^3x \; \kappa(\partial^0\partial^0 - \partial^0\partial^0 + \nabla^2)\phi^2 = \kappa\int d^3x \, \vec\nabla\cdot(\vec\nabla\phi^2) = \kappa\int d^2S\, \hat n\cdot\vec\nabla\phi^2$

a pure surface term, with the surface at infinity. But ϕ vanishes at ∞.

for $\nu=i$ this $= \int d^3x \; \kappa\, \partial^0\partial^i\phi^2 = \kappa\partial^0\int d^3x\, \partial^i\phi^2$ = a surface term again.

(ii) $\Theta^\mu{}_\mu = T^\mu{}_\mu + \kappa(\Box - 4\Box)\phi^2 = T^\mu{}_\mu - 3\kappa\Box\phi^2$

Now $T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac12(\partial_\lambda\phi)^2 - V(\phi)\right)$ so

$$T^\mu{}_\mu = -(\partial_\lambda\phi)^2 + 4V(\phi)$$

Also, the EOM is $\Box\phi + V'(\phi) = 0$. So

$$\Theta^\mu{}_\mu = -(\partial_\lambda\phi)^2 + 4V - 6\kappa\phi\Box\phi - 6\kappa(\partial_\lambda\phi)^2 = -(6\kappa+1)(\partial_\lambda\phi)^2 + 4V + 6\kappa\phi V'(\phi)$$

This vanishes if $6\kappa+1=0 \Rightarrow \kappa = -\frac16$ and $4V - V' = 0 \Rightarrow V = g\phi^4$, g = const.

(iii) $S^\mu = \pi^\mu(1+x\cdot\partial)\phi - x^\mu\mathcal{L} = \partial^\mu\phi(1+x\cdot\partial)\phi - x^\mu\mathcal{L}$

$$= \tfrac12\partial^\mu\phi^2 + x^\nu(\partial_\nu\phi\,\partial^\mu\phi - \delta^\mu_\nu\mathcal{L}) = \tfrac12\partial^\mu\phi^2 + x^\nu T^\mu{}_\nu$$

Note that $(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)(x_\nu\phi^2) - x_\nu(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)\phi^2 = \partial^\mu x_\nu\,\partial^\nu\phi^2 + \partial^\nu x_\nu\,\partial^\mu\phi^2 - g^{\mu\nu}2\,\partial_\lambda x_\nu\,\partial^\lambda\phi^2$

$= \partial^\mu\phi^2 + 4\partial^\mu\phi^2 - 2\partial^\mu\phi^2 = 3\partial^\mu\phi^2$. So we have

$$S^\mu = x^\nu T^\mu{}_\nu + \tfrac16\left[(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)(x_\nu\phi^2) - x_\nu(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)\phi^2\right] = x^\nu\Theta^\mu{}_\nu + \tfrac16(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)(x_\nu\phi^2)$$

Defining $\tilde S^\mu = S^\mu - \frac16(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)(x_\nu\phi^2)$ we have $\tilde S^\mu = x^\nu\Theta^\mu{}_\nu$.

Then $\partial_\mu\tilde S^\mu = \partial_\mu x^\nu\,\Theta^\mu{}_\nu + x^\nu\partial_\mu\Theta^\mu{}_\nu = \delta^\nu_\mu\Theta^\mu{}_\nu + 0 = \Theta^\mu{}_\mu$. Hence $\partial_\mu\tilde S^\mu = 0 \Leftrightarrow \Theta^\mu{}_\mu = 0$.

(iv) Now let $V(\phi) = \frac{1}{2}m^2\phi^2 + g\phi^3 + \lambda\phi^4$. We have

$$\Theta^\mu{}_\mu = T^\mu{}_\mu - \frac{1}{6}(\partial^\mu\partial_\mu - 4\Box)\phi^2 = \left((\partial^\mu\phi)^2 - 4\mathcal{L}\right) + \frac{1}{2}\Box\phi^2$$

$$= (\partial_\mu\phi)^2 - 2(\partial_\mu\phi)^2 + 4V + \phi\Box\phi + (\partial\phi)^2$$

$$= 4V + \phi\Box\phi$$

Now the EOM are: $\Box\phi + V'(\phi) = 0$ so we have

$\Theta^\mu{}_\mu = 4V - \phi V'(\phi) = \Delta \neq 0$ in general.

Explicitly: $\Theta^\mu{}_\mu = 4\left(\frac{1}{2}m^2\phi^2 + g\phi^3 + \lambda\phi^4\right) - \phi\left(m^2\phi + 3g\phi^2 + 4\lambda\phi^3\right) = m^2\phi^2 + g\phi^3$

This is consistent with what was found in Problem 1, (v), above. It shows that dimensional parameters in $\mathcal{L}$ break scale invariance.

Conformal transformations:

(i) $x^\mu \to -\dfrac{x^\mu}{x^2} \to -\dfrac{x^\mu + a^\mu}{(x+a)^2} \to -\dfrac{-x^\mu/x^2 + a^\mu}{\left(-\frac{x}{x^2}+a\right)^2} = x^2\,\dfrac{x^\mu - a^\mu x^2}{(x-a x^2)^2} = \dfrac{x^\mu - a^\mu x^2}{1-2a\cdot x + a^2x^2}$

$$x'^\mu = \frac{x^\mu - a^\mu x^2}{1-2a\cdot x + a^2x^2}\,, \quad \delta x'^\mu = 2a\cdot x\, x^\mu - x^2 a^\mu + \mathcal{O}(a^2)$$

(ii) Assume under $x \to x' = x + \delta x$ that $\phi(x) \to \phi'(x) = (1 + C a\cdot x)\,\phi(x+\delta x)$ for some constant C (to be determined).

So we have

$$\phi'(x) = (1 + C a\cdot x)\,\phi(x+\delta x) = \left(1 + C x\cdot a + (2a\cdot x\, x^\lambda - a^\lambda x^2)\partial_\lambda\right)\phi$$

$$\Rightarrow \partial_\mu \phi' = \left[1 + C x\cdot a + (2a\cdot x\, x^\lambda - x^2 a^\lambda)\partial_\lambda\right]\partial_\mu\phi + \left[C a_\mu + 2a_\mu x^\lambda + 2a\cdot x\, g^{\mu\lambda} - 2a^\lambda x_\mu)\partial_\lambda\right]\phi$$

Now, the "kinetic energy" term in $\mathcal{L}$ transforms as follows:

$$\tfrac{1}{2}\partial_\mu\phi'\partial_\mu\phi' = \tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi + C x\cdot a(\partial_\mu\phi)^2 + \partial_\mu\phi\,(2a\cdot x\, x^\lambda - x^2 a^\lambda)\partial_\lambda\partial_\mu\phi + \tfrac{1}{2}C a\cdot\partial\phi^2 + 2a\cdot x(\partial_\mu\phi)^2$$

or

$$\delta\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi')^2 - \tfrac{1}{2}(\partial_\mu\phi)^2 = (C+2)a\cdot x(\partial\phi)^2 + (2a\cdot x\, x^\lambda - a^\lambda x^2)\tfrac{1}{2}\partial_\lambda\left((\partial_\mu\phi)^2\right) + \tfrac{1}{2}C a\cdot\partial\phi^2$$

$$= \left[2(C+2)a\cdot x + 2a\cdot x\, x\cdot\partial - x^2 a\cdot\partial\right]\tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{2}C a\cdot\partial\phi^2$$

The last term is already of the form of a total derivative. To see that the 1st term is too, for a special choice of C, consider

$$\partial_\mu\left[(2a\cdot x\, x^\mu - a^\mu x^2)\mathcal{L}\right] - (2a\cdot x\, x\cdot\partial - x^2 a\cdot\partial)\mathcal{L} = \mathcal{L}\,\partial_\mu(2a\cdot x\, x^\mu - x^2 a^\mu) = 8a\cdot x\,\mathcal{L}$$

So we need $2(C+2) = 8 \Rightarrow C = 2$.

Finally we have

$$\boxed{\delta\mathcal{L} = \delta\left(\tfrac{1}{2}(\partial_\mu\phi)^2\right) = \partial_\mu\left[(2a\cdot x\, x^\mu - x^2 a^\mu)\mathcal{L} + a^\mu\phi^2\right]}$$

((Little cheat/trick: the transformation $\phi \to \phi'$ can be obtained by treating $x^\mu \to -\frac{x^\mu}{x^2}$ as if it were a scale transformation,

$$\phi(x) \to \frac{1}{x^2}\phi(-\tfrac{x}{x^2}) \to \frac{1}{x^2}\phi(-\tfrac{x}{x^2}+a^\mu) \to \frac{1}{x^2}\frac{1}{(-\frac{x}{x^2}+a)^2}\phi\left(-\frac{-\frac{x}{x^2}+a}{(-\frac{x}{x^2}+a)^2}\right) = \frac{1}{1-2a\cdot x+a^2x^2}\phi(x') \text{))}.$$

(iii) The Noether currents can now be constructed

$$J^\mu(a) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left[2x\cdot a + (2a\cdot x\, x^\lambda - x^2 a^\lambda)\partial_\lambda\right]\phi - \left(2a\cdot x\, x^\mu - a^\mu x^2\right)\mathcal{L} - a^\mu\phi^2$$

$$= a_\nu\left[2\pi^\mu x^\nu\phi + (2x^\nu x^\lambda - g^{\nu\lambda}x^2)(\pi^\mu\partial_\lambda\phi - \delta^\mu_\lambda\mathcal{L}) - g^{\mu\nu}\phi^2\right]$$

here $\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$

or

$$K^{\mu\nu} = (2x^\nu x^\lambda - g^{\nu\lambda}x^2)T^\mu{}_\lambda + 2x^\nu\pi^\mu\phi - g^{\mu\nu}\phi^2$$

(iv) Can we "improve" this?

Note that $(2x^\nu x^\lambda - g^{\nu\lambda}x^2)\,6\,(T^\mu{}_\lambda - \Theta^\mu{}_\lambda) = (2x^\nu x^\lambda - g^{\nu\lambda}x^2)(\partial^\mu\partial_\lambda - \delta^\mu_\lambda\partial^2)\phi^2$

So we can try adding to $K^{\mu\nu}$ something like (a multiple of)

$$(\partial^\mu\partial_\lambda - \delta^\mu_\lambda\partial^2)\left[(2x^\nu x^\lambda - g^{\nu\lambda}x^2)\phi^2\right] = (2x^\nu x^\lambda - g^{\nu\lambda}x^2)(\partial^\mu\partial_\lambda - \delta^\mu_\lambda\partial^2)\phi^2 + \Delta^{\mu\nu}$$

$$\Delta^{\mu\nu} = \phi^2(\partial^\mu\partial_\lambda - \delta^\mu_\lambda\partial^2)(2x^\nu x^\lambda - g^{\nu\lambda}x^2) + (\partial^\mu\phi^2)\partial_\lambda[2x^\nu x^\lambda - g^{\nu\lambda}x^2] + (\partial_\lambda\phi^2)\partial^\mu[2x^\nu x^\lambda - g^{\nu\lambda}x^2]$$
$$- 2\delta^\mu_\lambda(\partial_\sigma\phi^2)\partial^\sigma(2x^\nu x^\lambda - g^{\nu\lambda}x^2)$$

Need $\partial^\sigma(2x^\nu x^\lambda - g^{\nu\lambda}x^2) = 2(g^{\nu\sigma}x^\lambda + g^{\lambda\sigma}x^\nu - g^{\nu\lambda}x^\sigma)$, $\partial_\lambda(2x^\nu x^\lambda - g^{\nu\lambda}x^2) = 8x^\nu$

$$\partial^2(2x^\nu x^\lambda - g^{\nu\lambda}x^2) = 2g^{\nu\lambda}(1+1-4) = -4g^{\nu\lambda}$$

$$\Delta^{\mu\nu} = 12\phi^2 g^{\mu\nu} + 8x^\nu\partial^\mu\phi^2 + 2(g^{\nu\mu}x^\lambda + g^{\lambda\mu}x^\nu - g^{\nu\lambda}x^\mu)\partial_\lambda\phi^2 - 4(g^{\nu\sigma}x^\mu + g^{\mu\sigma}x^\nu - g^{\mu\nu}x^\sigma)\partial_\sigma\phi^2$$
$$= 12\phi^2 g^{\mu\nu} + 6x^\nu\partial^\mu\phi^2 - 6x^\mu\partial^\nu\phi^2 + 6g^{\mu\nu}x\cdot\partial\phi^2$$
$$= 6\left[2x^\nu\pi^\mu\phi + 2\phi^2 g^{\mu\nu} - \partial^\nu(x^\mu\phi^2) + g^{\mu\nu}\phi^2 + g^{\mu\nu}\partial_\lambda(x^\lambda\phi^2) - 4g^{\mu\nu}\phi^2\right]$$
$$= 6\left[2x^\nu\pi^\mu\phi - g^{\mu\nu}\phi^2 - (\delta^\mu_\lambda\partial^\nu - g^{\mu\nu}\partial_\lambda)(x^\lambda\phi^2)\right]$$

So

$$\tilde{K}^{\mu\nu} \equiv K^{\mu\nu} - \frac{1}{6}\left(\partial^\mu\partial_\lambda - \delta^\mu_\lambda \Box\right)\left[\left(2x^\nu x^\lambda - g^{\nu\lambda}x^2\right)\phi^2\right] - \left(\delta^\mu_\lambda \partial^\nu - g^{\mu\nu}\partial_\lambda\right)\left(x^\lambda \phi^2\right)$$

$$= \left(2x^\nu x^\lambda - g^{\nu\lambda}x^2\right)\left[T^\mu{}_\lambda - \frac{1}{6}\left(\partial^\mu\partial_\lambda - \delta^\mu_\lambda\Box\right)\phi^2\right]$$

or

$$\tilde{K}^{\mu\nu} = \left(2x^\nu x^\lambda - g^{\nu\lambda}x^2\right)\Theta^\mu{}_\lambda$$

Note that $\tilde{K}^{\mu\nu} - K^{\mu\nu}$, that is, the term we add to $K^{\mu\nu}$ to define $\tilde{K}^{\mu\nu}$, is automatically conserved, e.g.,

$$\partial_\mu\left(\delta^\mu_\lambda\partial^\nu - g^{\mu\nu}\partial_\lambda\right)(\text{anything}) = 0$$

One should also check that $K^\mu = \int d^3x\, K^{0\mu}$ is unaltered. The first term added to $K^{\mu\nu}$ gives $\int d^3x \left[-\frac{1}{6}\left(\partial^0\partial_\lambda - \delta^0_\lambda\Box\right)\left(\left(2x^\mu x^\lambda - g^{\mu\lambda}x^2\right)\phi^2\right)\right.$

$$= -\frac{1}{6}\int d^3x \left[\vec{\nabla}^2\left(\left(2x^\mu x^0 - g^{\mu 0}x^2\right)\phi^2\right) - \frac{1}{6}\partial^0\int d^3x\left(\partial_i\left(2x^\mu x^i - g^{\mu i}x^2\right)\phi^2\right)\right.$$

which are pure surface terms (they vanish if $\phi \to 0$ faster than $|x^\mu|$).

The second term added gives $-\int d^3x \left(\delta^0_\lambda\partial^\mu - g^{0\mu}\partial_\lambda\right)\left(x^\lambda\phi^2\right)$. The $\mu=0$ component is $-\int d^3x \left(\delta^0_\lambda\partial^0 - \partial_\lambda\right)\left(x^\lambda\phi^2\right) = +\int d^3x\, \partial_i\left(x^i\phi^2\right)$ = a surface term.

The $\mu=i$ components are $-\int d^3x \left(\delta^0_\lambda\partial^i\right)\left(x^\lambda\phi^2\right) = -x^0\int d^3x\, \partial^i\phi^2$ = a surface term.

Finally $\partial_\mu\tilde{K}^{\mu\nu} = 0 \Leftrightarrow \partial_\mu\left(2x^\nu x^\lambda - g^{\nu\lambda}x^2\right)\Theta^\mu{}_\lambda = \left(2\delta^\nu_\mu x^\lambda + 2x^\nu\delta^\lambda_\mu - 2g^{\nu\lambda}x_\mu\right)\Theta^\mu{}_\lambda$

$$= 2\left(x_\lambda\Theta^{\nu\lambda} + x^\nu\Theta^\lambda{}_\lambda - x_\lambda\Theta^{\lambda\nu}\right)$$

$$= 2x^\nu\Theta^\lambda{}_\lambda \quad \left(\text{using } \Theta^{\lambda\nu} = \Theta^{\nu\lambda}\right).$$

Also, recall, $\partial_\mu \tilde{S}^\mu = \Theta^\lambda{}_\lambda \Rightarrow$ both vanish if and only if $\Theta^\lambda{}_\lambda = 0$.

4. We look for symmetries of the form

$$A \to e^{i\alpha} A, \quad B \to e^{i\beta} B \quad \text{and} \quad C \to e^{i\gamma} C. \qquad (*)$$

(There may be more complicated symmetries involving mixing the fields among themselves; back to this later).

Now, clearly the kinetic energy part of $\mathcal{L}$ is symmetric under $(*)$

So consider V:

(i) $V = \lambda_1 A B^2 + \lambda_2 B C^2 + h.c \longrightarrow \lambda_1 e^{i(\alpha+2\beta)} A B^2 + \lambda_2 e^{i(\beta+2\gamma)} B C^2 + h.c.$

This is invariant if

$$\alpha + 2\beta = 0 \quad \text{and} \quad \beta + 2\gamma = 0$$

That is, $\mathcal{L}$ is invariant under $A \to e^{4i\gamma} A$, $B \to e^{-2i\gamma} B$ and $C \to e^{i\gamma} C$.

Infinitesimally $\delta A = 4i\epsilon A$, $\delta B = -2i\epsilon B$, $\delta C = i\epsilon C$ and h.c.'s of these

Hence, Noether current is

$$J^\mu = \partial^\mu A^* (4iA) + \partial^\mu A(-4iA^*) + \partial^\mu B^* (-2iB) + \partial^\mu B(2iB^*) + \partial^\mu C^* (iC) + \partial^\mu C(-iC^*)$$

$$= -i \left[4 (A^* \overleftrightarrow{\partial^\mu} A) - 2 (B^* \overleftrightarrow{\partial^\mu} B) + (C^* \overleftrightarrow{\partial^\mu} C) \right]$$

If ψ is a complex field, $\psi = \int (dk) \left(e^{ikx} b_{\vec{k}} + e^{-ikx} c^\dagger_{\vec{k}} \right)$

$\Rightarrow \partial^0 \psi = i \int (dk) E_k \left(e^{ikx} b_{\vec{k}} - e^{-ikx} c^\dagger_{\vec{k}} \right)$

$\Rightarrow -i \int d^3x \left(\psi^* \partial^0 \psi - \psi \leftrightarrow \psi^* \right) = \int d^3x \int (dk')(dk) E_k \left(e^{-ik'x} b^\dagger_{k'} + e^{ik'x} c_{k'} \right) \left(e^{ikx} b_k - e^{-ikx} c^\dagger_k \right) \quad - \psi \leftrightarrow \psi^*$

$\int (dk')(dk)\, E_k (2\pi)^3 \left[\delta^3(k-k') \left(b^\dagger_{k'} b_k + b_{k'} b^\dagger_k - c_{k'} c^\dagger_k - c^\dagger_k c_{k'} \right) \right]$

$= \int (dk) \left(b^\dagger_k b_k - c^\dagger_k c_k \right) = N_+ - N_-$

So denoting by a superscript "A", "B" or "C" the creation/annihilation/number operators, as in $A = \int (dk) \left(e^{ikx} b^A_{\vec{k}} + e^{-ikx} c^{A\dagger}_{\vec{k}} \right)$, $N^A_+ = \int (dk)\, b^{A\dagger}_{\vec{k}} b^A_{\vec{k}}$

we have $Q = \int d^3x\, J^0 = 4 (N^A_+ - N^A_-) - 2 (N^B_+ - N^B_-) + N^C_+ - N^C_-$

Q is the conserved "charge". N_+^A counts the number of A particles and N_-^A the number of A antiparticles.

In a reaction pairs of A, B or C can be created/annihilated (since $N_+^x - N_-^x = (N_+^x+n) - (N_-^x+n)$ for $x = A, B, C$ separately and n an integer.

Moreover one can have $B \to 2\bar{C}$ and $A \to 2\bar{B}$ (plus obvious variations, eg, $B + C \to \bar{C}$, $\bar{A} \to 2B$), since

$$\underbrace{2N_+^B + N_-^C}_{\text{initial}} = \underbrace{2(N_+^B - 1) + (N_-^C + 2)}_{\text{final}}$$

(ii) $V = \lambda_1 AB^3 + \lambda_2 (B^*)^2 C^2 + \text{c.c.}$

Now $\alpha + 3\beta = 0$ $\quad -2\beta + 2\gamma = 0$ $\Rightarrow$ $\beta = \gamma$, $\alpha = -3\beta$

$\Rightarrow$ $j^\mu = -i\,[-3A^* \overleftrightarrow{\partial^\mu} A + B^* \overleftrightarrow{\partial^\mu} B + C^* \overleftrightarrow{\partial^\mu} C]$

$Q = -3(N_+^A - N_-^A) + (N_+^B - N_-^B) + (N_+^C - N_-^C) = \text{constant}$

So now $A \leftrightarrow 3B$ and $B \leftrightarrow C$ (+ pair creation)

(iii) $V = \lambda ABC + \text{c.c.}$

$\alpha + \beta + \gamma = 0$

This does not fix two in terms of the third $\Rightarrow$ more symmetry!

We can take, for example $\gamma = 0$ and $\beta = -\alpha$ and $\beta = 0$ and $\gamma = -\alpha$.

This gives two independent symmetry generators (and any other choice, eg, $\alpha = 0$, $\beta = -\gamma$, gives a generator that is a combination of the ones we have already selected).

So we have

$$J_1^\mu = -i\left[A^* \overleftrightarrow{\partial}^\mu A - B^* \overleftrightarrow{\partial}^\mu B\right] \qquad\qquad Q_1 = (N_+^A - N_-^A) - (N_+^B - N_-^B)$$

$$J_2^\mu = -i\left[A^* \overleftrightarrow{\partial}^\mu A - C^* \overleftrightarrow{\partial}^\mu C\right] \quad \text{with} \quad Q_2 = (N_+^A - N_-^A) - (N_+^C - N_-^C)$$

Note that $Q_1 - Q_2 = (N_+^C - N_-^C) - (N_+^B - N_-^B)$ (which corresponds to $\alpha=0$, $\beta=-\gamma$).

Now $A \to \bar{B}$ or $A \to \bar{C}$ plus pair creation and obvious variations (e.g., $\bar{A} \to B$, $AB \to C\bar{C}, \ldots$)

• Adding a function of $|\phi|^2$ where $\phi = A, B, C$ changes nothing since it is automatically invariant under $\phi \to e^{i\varphi}\phi$.

• Adding $\tilde{g}A^4 + \text{c.c.}$ breaks any symmetry that involves $A \to e^{i\alpha}A$.
So J^μ is no longer conserved, and Q is no longer a constant, in cases (i) & (ii).
In case (iii) we do have a symmetry: the case $\alpha=0$, $\beta=-\gamma$ still gives a conserved current, namely $J_1^\mu - J_2^\mu$, and a charge $Q_1 - Q_2$.