PHYSICS 215A: PARTICLES AND FIELDS, FALL 2014
DEPARTMENT OF PHYSICS, UCSD
BENJAMIN GRINSTEIN

Assignment 1

October 13, 2014 (Due October 22, 2014)

1. Consider a system consisting of pendulums hanging from a horizontal wire. Each pendulum consists of a massless rigid rod of length ℓ with a pointlike body of mass m at its free end. The fixed end of each pendulum is welded to the wire. The pendulums are free to oscillate on a vertical plane, perpendicular to the wire. These planes are separated by a fixed distance a (that is, the pendulums are separated by this distance). Of course, this is in the presence of a uniform gravitational field (with acceleration g pointing down). As the pendulums swing the horizontal wire twists, producing a restoring torque of magnitude $\kappa\times$ angular twisting between one pendulum and the next (two contributions to the torque, one from each nearest neighbor pendulum) — κ is just the analogue of the spring constant k, for the case of twist rather than stretch. You may take the system finite (N pendulums hanging from a wire of length Na) or infinite, your choice.

(i) What are the dynamical variables of this system? What is the range of values they may each take?

(ii) What is the Lagrangian for the system? Calculate the equations of motion.

(iii) Formulate the continuum limit of the system. Specify the limiting procedure (for variables and for the parameters m, ℓ, κ, g and a). Calculate the Lagrangian density in this limit. Compute the equations of motion. Compute the Hamiltonian density. *Note: The formulation of this question is somewhat ambiguous: you can define your limiting procedure to get rid of some terms. The aim is to retain as much of the interesting dynamics in the continuum limit as possible.*

(iv) Under what conditions is the system invariant under boosts along the direction along the wire?

(v) Read about the sine-Gordon equation in, e.g., Wikipedia:
http://en.wikipedia.org/wiki/Sine-Gordon_equation

2. The action integral has units of $\hbar = 1$, that is, it is dimensionless (in the units adopted for this course). Consider the following Lagrangian density:

$$\mathscr{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi + \tfrac{1}{2}\kappa(\partial^\mu\phi\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - g\phi^3 - \lambda\phi^4 - \sigma\phi^6$$

where $\phi = \phi(x)$ is the dynamical variable and κ, g, m, λ and σ are constants.

(i) Suppose $\mathscr{L}$ describes a system in 3 space dimensions. What are the mass dimensions of ϕ, κ, g, m, λ and σ? By mass dimensions we mean, e.g., the mass dimension of x^μ is -1, or $[x^\mu] = -1$ for short.

(ii) Repeat the above for the case of 2 spatial dimensions.

(iii) And again, now with 5 dimensions of space.

This is not just busy work. Identifying terms in the Lagrangian with couplings of negative mass dimension is central to the question of renormalizability and the construction of (quantum) effective field theories.

3. Relativistic gymnastics:

(i) Given an infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}\,,$$

show that the infinitesimal parameters $\omega_{\mu\nu}$ are antisymmetric.

(ii) The Levi-Civita tensor, $\epsilon^{\mu\nu\lambda\sigma}$, is a totaly antisymmetric tensor, with $\epsilon^{0123} = +1$. Prove that for any two index tensor $X^{\mu}{}_{\nu}$,

$$\epsilon_{\alpha\beta\gamma\delta} X^{\alpha}{}_{\mu} X^{\beta}{}_{\nu} X^{\gamma}{}_{\lambda} X^{\delta}{}_{\sigma} = \epsilon_{\mu\nu\lambda\sigma} \det(X)$$

where $X^{\mu}{}_{\nu}$ are the elements of the matrix X.

(iii) Show that the Kronecker δ symbol and Levi-Civita ϵ symbol are form invariant under infinitesimal Lorentz transformations. What about finite transformations?