Chapter 6

Fields that are not scalars

6.1 Generalities

So far we have concentrated our studies on fields that transform very simply under Lorentz transformations. Brief review: we want $\phi(x)$ to correspond to $\phi'(x')$ when $x' = \Lambda x$. For a scalar field "correspond to" means they are equal, $\phi'(x') = \phi(x)$. Tha is,

$$\phi'(x) = \phi(\Lambda^{-1}x)\,.$$

Less trivial is the case of a vector field, $A^\mu(x)$. We can obtain it from taking a derivative on the scalar field which gives, $\partial_\mu\phi'(x) = \partial_\nu\phi(\Lambda^-1x)(\Lambda^{-1})^\nu{}_\mu = \Lambda_\mu{}^\nu\partial_\nu\phi(\Lambda^-1x)$. This holds for any vector so

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \qquad \text{or} \qquad A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)\,.$$

We can generalize this to other tensors easily, by considering tensor products of vectors, *e.g.*,

$$B'^{\mu\nu\lambda}(x) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\Lambda^\lambda{}_\delta B^{\rho\sigma\delta}(\Lambda^{-1}x)$$

In general, a collection of fields ψ_α, $\alpha = 1, \dots, n$ transforms as

$$\psi'_\alpha(x) = D_{\alpha\beta}(\Lambda)\psi_\beta(\Lambda^{-1}x)\,, \tag{6.1}$$

where $D_{\alpha\beta}$ is an $n \times n$ matrix function of Λ. If $\psi'_\alpha(x') = D_{\alpha\beta}(\Lambda_1)\psi_\beta(x)$ with $x' = \Lambda_1 x$ and $\psi''_\alpha(x) = D_{\alpha\beta}(\Lambda_2)D_{\beta\gamma}(\Lambda_1)\psi_\gamma(\Lambda_1^{-1}\Lambda_2^{-1}x)$. This should equal the transformation with $x'' = \Lambda_2\Lambda_1 x$, $\psi''_\alpha(x) = D_{\alpha\beta}(\Lambda_2\Lambda_1)\psi_\beta((\Lambda_2\Lambda_1)^{-1}x)$. This imposes a requirement on the functions $D(\Lambda)$ that they furnish a *representation* of the Lorentz group:

$$D(\Lambda_2)D(\Lambda_1) = D(\Lambda_2\Lambda_1) \tag{6.2}$$

It suffices to understand the irreducible representations. Brief review/introduction. If D is a representation, so is SDS^{-1} for any invertible matrix S. If we can find an S such that SDS^{-1} is block diagonal for all Λ,

$$\begin{pmatrix} D^{(1)} & 0 & \cdots & 0 \\ 0 & D^{(2)} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & D^{(N)} \end{pmatrix}$$

then we say D is *reducible* (to be more precise, the case that $D^{(1)}$ is the only block in the block diagonal matrix should be excluded). Else, it is *irreducible*. If D is reducible it (or, rather, SDS^{-1} for some S) can be written as the *direct sum* of irreducible representations $D^{(i)}$, $i = 1, \ldots, N$, and we write this as $D = D^{(1)} \oplus D^{(2)} \oplus \cdots \oplus D^{(N)}$. The point is that one can form any reducible representation from knowledge of the possible irreducible ones. So we need only determine the fields that correspond to irreducible representations,

$$\psi'^{(1)}(x) = D^{(1)}(\Lambda)\psi^{(1)}(\Lambda^{-1}x), \ldots, \psi'^{(N)}(x) = D^{(N)}(\Lambda)\psi^{(1)}(\Lambda^{-1}x)\,,$$

and the whole collection transforms as a reducible representation

$$\begin{pmatrix} \psi'^{(1)}(x) \\ \psi'^{(2)}(x) \\ \vdots \\ \psi'^{(N)}(x) \end{pmatrix} = \begin{pmatrix} D^{(1)} & 0 & \cdots & 0 \\ 0 & D^{(2)} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & D^{(N)} \end{pmatrix} \begin{pmatrix} \psi^{(1)}(\Lambda^{-1}x) \\ \psi^{(2)}(\Lambda^{-1}x) \\ \vdots \\ \psi^{(N)}(\Lambda^{-1}x) \end{pmatrix}$$

An example: a pair of scalars, ϕ_1, ϕ_2, a vector, A^μ, and a tensor, $T^{\mu\nu}$, with the transformations given above.

If D acts on d dimensional vectors we say the dimension of D is d, $\dim(D) = d$.

(Aside: The representations may be double valued. You have seen this in QM. A spin-$\frac{1}{2}$ wave-function, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ transforms, under rotations by an angle θ about the $\hat{n}$ axis, by $\psi \to e^{i\frac{1}{2}\theta\hat{n}\cdot\vec{\sigma}}\psi$. That is $D(R) = e^{i\frac{1}{2}\theta\hat{n}\cdot\vec{\sigma}}$ is a 2-dim representation of the rotation R by an angle θ about the $\hat{n}$ axis. But R is the same for $\theta = 0$ and $\theta = 2\pi$, and $e^{i\pi\hat{n}\cdot\vec{\sigma}} = -1$.)

If $D(\Lambda)$ is a representation, so is $D^*(\Lambda)$. Proof: take the complex conjugate of $D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$. Clearly, $\dim(D^*) = \dim(D)$. D^* may or may not be equivalent to D.

If $D^{(1)}$ and $D^{(2)}$ are representations, so is the *tensor product*, $D^{(1)} \otimes D^{(2)}$. The tensor product is defined as always: if $D^{(1)}$ acts on $\psi^{(1)}$, $D^{(2)}$ on $\psi^{(2)}$, then $D^{(1)} \otimes D^{(2)}$ acts on $\psi^{(1)} \otimes \psi^{(2)}$ according to $(D^{(1)} \otimes D^{(2)})(\psi^{(1)} \otimes \psi^{(2)}) = (D^{(1)}\psi^{(1)}) \otimes$

$(D^{(2)}\psi^{(2)})$. If $d = \dim(D)$ and $d' = \dim(D')$ then $\dim(D \oplus D') = d + d'$ and $\dim(D \otimes D') = dd'$. Generally, $D \otimes D'$ is reducible, $D \otimes D' = D^{(1)} \oplus D^{(2)} \oplus \cdots \oplus D^{(N)}$, with $d_1 + d_2 + \cdots + d_N = dd'$. An example is a two index tensor. It can be obtained from the tensor product of two vectors, $C^{\mu\nu} = A^\mu B^\nu$, and is a reducible representation: under Lorentz transformations the trace $\eta_{\mu\nu}C^{\mu\nu}$, the anti-symmetric part, $C^{[\mu\nu]} = \frac{1}{2}(C^{\mu\nu} - C^{\nu\mu})$, and the symmetric traceless part, $C^{\{\mu\nu\}} - \frac{1}{4}\eta^{\mu\nu}\eta_{\lambda\sigma}C^{\lambda\sigma} = \frac{1}{2}(C^{\mu\nu} + C^{\nu\mu}) - \frac{1}{4}\eta^{\mu\nu}\eta_{\lambda\sigma}C^{\lambda\sigma}$ do not mix into each other. Moreover, one can show that the 6-dimensional antisymmetric 2-index tensor is itself the direct sum of two 3-dimensional irreducible representations, satisfying $C_\pm^{\mu\nu} = \pm\frac{1}{2}\epsilon^{\mu\nu}{}_{\lambda\sigma}C_\pm^{\lambda\sigma}$ (these are said to be self-dual and anti-self-dual, respectively). The $4 \times 4 = 16$-dimensional tensor product of two 4-vectors splits into four irreducible representations of dimension 1, 3, 3, and 9, as we have just seen. With a slight abuse of notation, this is $4 \otimes 4 = 1 \oplus 3 \oplus 3' \oplus 9$.

Here is the point. We can build up every irreducible representation (and from them every reducible representation, hence every representation) by starting from some small basic representations, and taking their tensor products repeatedly: these tensor products are direct sums of new irreducible representation, and the more basic representations we tensor-product the higher the dimension of the new irreducible representations we will find.

It turns out, as we will show below, the representations of the Lorentz groups are labeled by two half-integers, (s_+, s_-) and have dimension $(2s_+ + 1)(2s_- + 1)$. This is because the Lorentz group (or rather, its algebra) is isomorphic to two copies of spin, $SU(2) \times SU(2)$, and as you know from particle QM the representations of spin are classified by half integer $s = 0, \frac{1}{2}, 1, \ldots$ and have dimension $2s + 1 = 1, 2, 3, \ldots$ correspondingly. For example, $(0, 0)$ is a 1-dimensional representation, the scalar, $(\frac{1}{2}, \frac{1}{2})$ is a 4-dimensional representation, corresponding to vectors, A^μ. For the tensor product of two vectors, we need first, from QM, that $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. So $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (0, 0) \oplus (1, 0) \oplus (0, 1) \oplus (1, 1)$. The two 3-dimensional representations associated with the antisymmetric tensor correspond to $(1, 0)$ and $(0, 1)$ and the 9-dimensional symmetric traceless tensor is $(1, 1)$.

6.2 Spinors

Back to Lorentz group and physics. Question: other than scalars and tensors, what other representations of the Lorentz group do we have? Answer: spinors and their tensor products. Let $\sigma^\mu = (\sigma^0, \sigma^i)$, where

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let

$$P = p_\mu \sigma^\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}.$$

Then $\det(P) = p^2 = \eta_{\mu\nu} p^\mu p^\nu$. Then for any unimodular 2×2 matrix $\widetilde{M}$ we have $\det(\widetilde{M}^\dagger P \widetilde{M}) = \det(P)$. Since any hermitian matrix can be expanded in terms of σ^μ with real coefficients and $P' = \widetilde{M}^\dagger P \widetilde{M}$ is hermitian (if P is), then $P' = p'_\mu \sigma^\mu$ with $p'^2 = p^2$. That is, $\widetilde{M} = \widetilde{M}(\Lambda)$ induces a Lorentz transformation.

Now $\widetilde{M}(\Lambda)$ does not satisfy the representation defining equation (6.2). If $p'' = \Lambda_2 p'$ and $p' = \Lambda_1 p$ then $P'' = \widetilde{M}^\dagger(\Lambda_2) P' \widetilde{M}(\Lambda_2) = \widetilde{M}^\dagger(\Lambda_2) \widetilde{M}^\dagger(\Lambda_1) P \widetilde{M}(\Lambda_1) \widetilde{M}(\Lambda_2)$ and this should correspond to $p'' = \Lambda_2 \Lambda_1 p$ or $P'' = \widetilde{M}^\dagger(\Lambda_2\Lambda_1) P \widetilde{M}(\Lambda_2\Lambda_1)$. Comparing we see that,

$$\widetilde{M}(\Lambda_1)\widetilde{M}(\Lambda_2) = \widetilde{M}(\Lambda_2\Lambda_1)\,.$$

Defining $M(\Lambda) = \widetilde{M}(\Lambda^{-1})$ we obtain a representation,

$$M(\Lambda_2)M(\Lambda_1) = \widetilde{M}(\Lambda_2^{-1})\widetilde{M}(\Lambda_1^{-1}) = \widetilde{M}(\Lambda_1^{-1}\Lambda_2^{-1}) = \widetilde{M}((\Lambda_2\Lambda_1)^{-1}) = M(\Lambda_2\Lambda_1)\,.$$

So $M(\Lambda)$ gives a 2-dimensional representation of Λ. It acts on 2-dimensional *spinors* (vectors in a 2-dimensional space):

$$\psi'_\alpha = M_{\alpha\beta}(\Lambda)\psi_\beta$$

If $P' = \widetilde{M}^\dagger P \widetilde{M}$ with $P' = p'_\nu \sigma^\nu = \Lambda_\nu{}^\mu p_\mu \sigma^\nu$ then equating this to $M^\dagger P M$ for arbitrary p_μ we must have

$$\widetilde{M}(\Lambda)^\dagger \sigma^\mu \widetilde{M}(\Lambda) = \Lambda_\nu{}^\mu \sigma^\nu = (\Lambda^{-1})^\mu{}_\nu \sigma^\nu \tag{6.3}$$

or

$$M(\Lambda)^\dagger \sigma^\mu M(\Lambda) = \Lambda^\mu{}_\nu \sigma^\nu \tag{6.4}$$

As we have seen, the complex conjugate, $M^*(\Lambda)$, must also be a representation. It satisfies,

$$M^T \sigma^{\mu *} M^* = \Lambda^\mu{}_\nu \sigma^{\nu *}$$

or sandwiching with σ^2 —a similarity transformation— and defining $\bar{\sigma}^\mu = \sigma^2 \sigma^{\mu *} \sigma^2 = (\sigma^0, -\sigma^i)$, and $\overline{M} = \sigma^2 M^* \sigma^2$:

$$\overline{M}^\dagger \bar{\sigma}^\mu \overline{M} = \Lambda^\mu{}_\nu \bar{\sigma}^\nu$$

So there must be a 2-dimensional representation of the Lorentz group, of 2-component vectors, or *spinors*, that transform according to

$$\psi'_\alpha(x) = M_{\alpha\beta}(\Lambda)\psi_\beta(\Lambda^{-1}x)$$

and also a complex conjugate representation that acts on other 2-component vectors, also called a spinors, that transform according to

$$\bar{\chi}'_\alpha(x) = \overline{M}_{\alpha\beta}(\Lambda)\bar{\chi}_\beta(\Lambda^{-1}x)\,.$$

It is convenient to arrange the spinors into 2-component column vectors, and write

$$\psi' = M(\Lambda)\psi \quad \text{and} \quad \bar{\chi}' = \overline{M}(\Lambda)\bar{\chi}$$

Note that

$$\psi'^\dagger\sigma^\mu\psi' = \psi^\dagger M^\dagger\sigma^\mu M\psi = \Lambda^\mu{}_\nu\,\psi^\dagger\sigma^\nu\psi\,.$$

and

$$\bar{\chi}'^\dagger\bar{\sigma}^\mu\bar{\chi}' = \bar{\chi}^\dagger\overline{M}^\dagger\bar{\sigma}^\mu\overline{M}\bar{\chi} = \Lambda^\mu{}_\nu\,\bar{\chi}^\dagger\bar{\sigma}^\nu\bar{\chi}\,.$$

In making tensors out of these it is convenient to distinguish the indices in ψ and $\bar{\chi}$, so we write ψ_α and $\bar{\chi}_{\dot{\alpha}}$ (still with $\alpha = 1,2$ and $\dot{\alpha} = 1,2$).

We can construct a scalar out of two spinors, ψ_1 and ψ_2:

$$\psi_1'^T\sigma^2\psi_2' = \psi_1^T M^T\sigma^2 M\psi_2 = \psi_1^T\sigma^2\psi_2 \tag{6.5}$$

The last step follows from

$$\begin{aligned}(M^T\sigma^2 M)_{\alpha\beta} &= -i\epsilon_{\gamma\delta}M_{\alpha\gamma}M_{\beta\delta}\\ &= -i\epsilon_{\alpha\beta}\det(M)\\ &= \sigma^2_{\alpha\beta}\end{aligned}$$

Similarly, $\bar{\chi}_1^T\sigma^2\bar{\chi}_2$ is a scalar. Note that $\psi_1^\dagger\psi_2$ is not a scalar since M is not generally unitary. Note also that $\psi^T\sigma^2\psi = 0$, so we cannot make a scalar out of a single ψ.

Form Eq. (6.4) we can verify that

$$M = \exp\left(-\tfrac{i}{2}\vec{\alpha}\cdot\vec{\sigma}\right) = \cos(\tfrac{1}{2}\alpha) - i\hat{\alpha}\cdot\vec{\sigma}\,\sin(\tfrac{1}{2}\alpha) \tag{6.6}$$

is a representation of $\Lambda = R =$ a rotation by angle α about an axis in the $\hat{\alpha} = \vec{\alpha}/\alpha$ direction, and

$$M = \exp\left(\tfrac{1}{2}\vec{\beta}\cdot\vec{\sigma}\right) = \cosh(\tfrac{1}{2}\beta) + \hat{\beta}\cdot\vec{\sigma}\,\sinh(\tfrac{1}{2}\beta)$$

is a representation of a boost by velocity β, that is, a representation of

$$\Lambda = \begin{pmatrix}\gamma & \gamma\beta\\ \gamma\beta & \gamma\end{pmatrix}$$

where $\gamma = 1/\sqrt{1-\beta^2}$, $\vec{\beta}$ is along the x axis, and the last two rows and columns of Λ have been omitted. By complex conjugating it follows that

$$\overline{M} = \exp\left(-\tfrac{i}{2}\vec{\alpha}\cdot\vec{\sigma}\right) \tag{6.7}$$

is a representation of the same rotation R (a rotation by angle α about $\hat{\alpha}$), and

$$\overline{M} = \exp\left(-\tfrac{1}{2}\vec{\beta}\cdot\vec{\sigma}\right)$$

is a representation of a boost by velocity $\vec{\beta}$.

6.3 A Lagrangian for Spinors

We want to construct a Lagrangian density for spinors. We put the general constraints:

(i) Constructed from ψ and $\psi^\dagger$ and their first derivatives $\partial_\mu\psi$ and $\partial_\mu\psi^\dagger$.

(ii) Real, $\mathcal{L}^* = \mathcal{L}$ (or for quantum fields, hermitian, $\mathcal{L}^\dagger = \mathcal{L}$).

(iii) Lorentz invariant (at least up to total derivatives)

(iv) At most quadratic in the fields.

The last condition is not generally necessary. We impose it for simplicity. Higher orders in the fields will correspond to interactions.

From (iv) we need to construct $\mathcal{L}$ from bilinears $\psi^\dagger\otimes\psi$ and $\psi\otimes\psi$. We know we can form a vector out of these, but not a scalar. Now,

$$\partial_\mu\psi(\Lambda^{-1}x) = \partial_\mu((\Lambda^{-1})^\nu{}_\lambda x^\lambda)(\partial_\nu\psi)(\Lambda^{-1}x) = (\Lambda^{-1})^\nu{}_\mu(\partial_\nu\psi)(\Lambda^{-1}x)\,,$$

so that

$$\psi'^\dagger\sigma^\mu\partial_\mu\psi' = \psi^\dagger M^\dagger\sigma^\mu M(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi = \psi^\dagger\Lambda^\mu{}_\lambda\sigma^\lambda M(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi = \psi^\dagger\sigma^\mu\partial_\mu\psi\,.$$

Both $\psi^\dagger\sigma^\mu\partial_\mu\psi$ and $\partial_\mu\psi^\dagger\sigma^\mu\psi$ transform as scalars so they are candidates for an invariant Lagrangian. But the sum is a total derivative, $\partial_\mu(\psi^\dagger\sigma^\mu\psi)$ so it is irrelevant (it does not contribute to the equations of motion). So we take the difference,

$$\tfrac{1}{2}\psi^\dagger\sigma^\mu\overleftrightarrow{\partial_\mu}\psi = \tfrac{1}{2}\left(\psi^\dagger\sigma^\mu\partial_\mu\psi - \partial_\mu\psi^\dagger\sigma^\mu\psi\right)$$

as a possible term in the Lagrangian. It's complex conjugate is

$$\begin{aligned}
\tfrac{1}{2}(\psi^\dagger\sigma^\mu\overleftrightarrow{\partial_\mu}\psi)^* &= \tfrac{1}{2}\left(\psi^T\sigma^{\mu*}\partial_\mu\psi - \partial_\mu\psi^T\sigma^{\mu*}\psi^*\right)\\
&= \tfrac{1}{2}\left(\partial_\mu\psi^\dagger\sigma^{\mu\dagger}\psi - \psi^\dagger\sigma^{\mu\dagger}\partial_\mu\psi\right)\\
&= -\tfrac{1}{2}\psi^\dagger\sigma^\mu\overleftrightarrow{\partial_\mu}\psi
\end{aligned}$$

If we take $\mathcal{L} = A\frac{1}{2}\psi^\dagger\sigma^\mu \overleftrightarrow{\partial_\mu}\psi$ then for $\mathcal{L}^* = \mathcal{L}$ we must have $A = \pm i|A|$. Redefining $\psi \to \frac{1}{\sqrt{|A|}}\psi$ we have a candidate Lagrangian density

$$\mathcal{L} = \pm\frac{1}{2}\psi^\dagger\sigma^\mu\, i\overleftrightarrow{\partial_\mu}\psi$$

We will see below how to choose properly between the two signs. Note that if we relax assumption (iv) we could add other terms, *e.g.,* $(\psi^\dagger\sigma^\mu\psi)(\psi^\dagger\sigma_\mu\psi)$.

Equations of motion: recall, with a complex field we can take variations with respect to ψ and $\psi^\dagger$ separately, as if they were independent variables. Now in the action integral it is convenient to integrate by parts, so that

$$\int d^4x\,\mathcal{L} = \pm\int d^4x\,\frac{1}{2}\psi^\dagger\sigma^\mu\, i\overleftrightarrow{\partial_\mu}\psi = \pm\int d^4x\,\psi^\dagger\sigma^\mu\, i\partial_\mu\psi$$

Then, trivially,

$$\frac{\delta\mathcal{L}}{\delta\psi^\dagger} = 0 \quad\Rightarrow\quad \sigma^\mu\partial_\mu\psi = 0\,.$$

This is

$$(\sigma^0\partial_0 + \sigma^i\partial_i)\psi = 0 \quad\Rightarrow\quad \partial_0\psi = -\sigma^i\partial_i\psi = -\vec{\sigma}\cdot\vec{\nabla}\psi$$

so that

$$\partial_0^2\psi = -\partial_0\vec{\sigma}\cdot\vec{\nabla}\psi = -\vec{\sigma}\cdot\vec{\nabla}\partial_0\psi = (\vec{\sigma}\cdot\vec{\nabla})^2\psi = \nabla^2\psi$$

This is precisely the KG equation with $m = 0$

$$\partial^2\psi = (\partial_0^2 - \nabla^2)\psi = 0\,.$$

Each component of ψ satisfies the massless KG equation.

We can construct a Lagrangian for fields in the complex conjugate representation, $\bar{\chi}$. An analogous argument gives us

$$\mathcal{L} = \pm\bar{\chi}^\dagger\bar{\sigma}^\mu\, i\partial_\mu\bar{\chi}$$

with equation of motion

$$\bar{\sigma}^\mu\partial_\mu\bar{\chi} = 0$$

This again gives $\partial^2\bar{\chi} = 0$ but now with $\partial_0\bar{\chi} = \vec{\sigma}\cdot\vec{\nabla}\bar{\chi}$ (note the sign difference).

Plane wave expansion. Since these are complex fields we have different coefficients for the positive and negative energy components:

$$\psi_\alpha = \int (dk)\left[e^{-ik\cdot x}B_{\vec{k},\alpha} + e^{ik\cdot x}D_{\vec{k},\alpha}\right]$$

where $B_{\vec{k},\alpha}$ and $D_{\vec{k},\alpha}$ are two component operator valued objects. Since $\partial^2\psi = 0$ we must have $k^2 = E_{\vec{k}}^2 - \vec{k}^{\,2} = 0$, that is, $k^0 = E_{\vec{k}} = |\vec{k}|$ as we should for massless particles. But the equation of motion is first order in derivatives. Using $\partial_\mu e^{\mp ik\cdot x} = \mp ik_\mu e^{\mp ik\cdot x}$ we have

$$\int (dk)\left[e^{-ik\cdot x}k_\mu\sigma^\mu B_{\vec{k}} - e^{i\cdot x}k_\mu\sigma^\mu D_{\vec{k}}\right] = 0$$

That is, we need

$$\begin{pmatrix} k^0-k^3 & -k^1+ik^2 \\ -k^1-ik^2 & k^0-k^3 \end{pmatrix}\begin{pmatrix} B_{\vec{k},1} \\ B_{\vec{k},2}\end{pmatrix} = 0 \qquad \begin{array}{r} (k^0-k^3)B_{\vec{k},1} = -(-k^1+ik^2)B_{\vec{k},2} \\ (-k^1-ik^2)B_{\vec{k},1} = -(k^0+k^3)B_{\vec{k},2} \end{array}$$

and similarly for $D_{\vec{k},\alpha}$. Let

$$u_{\vec{k}} = N_{\vec{k}}\begin{pmatrix} k^1-ik^2 \\ k^0-k^3\end{pmatrix}.$$

$N_{\vec{k}}$ is a normalization factor. We could choose it to have, for example, $u_{\vec{k}}^\dagger u_{\vec{k}} = 1$, but we will wait to choose it conveniently later. Note that since $k^2 = 0$ we can also write this as

$$u_{\vec{k}} = N_{\vec{k}}\,\frac{k^0-k^3}{k^1-ik^2}\begin{pmatrix} k^0+k^3 \\ k^1+ik^2\end{pmatrix} = N'_{\vec{k}}\begin{pmatrix} k^0+k^3 \\ k^1+ik^2\end{pmatrix}.$$

For the plane wave-expansion of $\bar{\chi}$ we need to solve $k_\mu\bar{\sigma}^\mu v_{\vec{k}} = 0$. But this is just like the equation for $u_{\vec{k}}$ only replacing $-\vec{k}$ for $\vec{k}$. So $v_{\vec{k}} = u_{-\vec{k}}$ up to a phase. Since there is only one solution to the matrix equation, we rewrite our plane wave expansion as

$$\psi = \int (dk)\left[e^{-ik\cdot x}\beta_{\vec{k}}u_{\vec{k}} + e^{ik\cdot x}\delta_{\vec{k}}^\dagger u_{\vec{k}}\right]$$

where now the operators $\beta_{\vec{k}}$ and $\delta_{\vec{k}}$ are one-component objects. So other than the spinor coefficient $u_{\vec{k}}$ this expansion looks very much like the one for complex scalar fields.

6.3.1 Hamiltonian; Fermi-Dirac Statistics

Let's compute the Hamiltonian. This should allow us to fix the sign in the Lagrangian density since we want a Hamiltonian that is bounded from below. From the density

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\partial_t\psi - \mathcal{L} = \mp i\psi^\dagger\vec{\sigma}\cdot\vec{\nabla}\,\psi$$

we obtain the Hamiltonian in terms of creation/annihilation operators,

$$H = \mp \int (dk)(dk')u^\dagger_{\vec{k}'}\vec{\sigma}\cdot\vec{k}\,u_{\vec{k}}\left[(2\pi)^3\delta(\vec{k}'-\vec{k})(\delta_{\vec{k}'}\delta^\dagger_{\vec{k}} - \beta^\dagger_{\vec{k}'}\beta_{\vec{k}}) + (2\pi)^3\delta(\vec{k}'+\vec{k})(\beta^\dagger_{\vec{k}'}\delta^\dagger_{\vec{k}} - \delta_{\vec{k}'}\beta_{\vec{k}})\right]$$

Now use, $k_\mu\sigma^\mu u_{\vec{k}} = 0$ or

$$\vec{k}\cdot\vec{\sigma}\,u_{\vec{k}} = E_{\vec{k}}u_{\vec{k}} \qquad \text{with } E_{\vec{k}} = |\vec{k}|,$$

and adopt the normalization

$$u^\dagger_{\vec{k}}u_{\vec{k}} = 2E_{\vec{k}}\,.$$

Moreover, from the explicit form of $u_{\vec{k}}$ we have

$$u^\dagger_{\vec{k}}u_{-\vec{k}} = 0$$

Putting these together we have

$$H = \mp \int (dk)E_{\vec{k}}\left(-\beta^\dagger_{\vec{k}}\beta_{\vec{k}} + \delta_{\vec{k}}\delta^\dagger_{\vec{k}}\right)$$

If we take, as we have done before,

$$[\beta_{\vec{k}}, \beta^\dagger_{\vec{k}'}] = (2\pi)^3 2E_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{k}') = [\delta_{\vec{k}}, \delta^\dagger_{\vec{k}'}] \tag{6.8}$$

then

$$H = \mp \int (dk)E_{\vec{k}}\left(-\beta^\dagger_{\vec{k}}\beta_{\vec{k}} + \delta^\dagger_{\vec{k}}\delta_{\vec{k}}\right)$$

plus an infinite constant that we throw away (normal ordering). We were hoping to fix the sign, but instead we encounter a disaster! If we choose the $-$ sign in the definition of $\mathcal{L}$ the $\beta^\dagger_{\vec{k}}\beta_{\vec{k}}$ has a spectrum unbounded from below while if we choose the $+$ sign the $\delta^\dagger_{\vec{k}}\delta_{\vec{k}}$ term is unbounded form below.

But if instead of (6.8) we choose anti-commutation relations,

$$\{\beta_{\vec{k}}, \beta^\dagger_{\vec{k}'}\} = (2\pi)^3 2E_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{k}') = \{\delta_{\vec{k}}, \delta^\dagger_{\vec{k}'}\} \tag{6.9}$$

where $\{A, B\} \equiv AB + BA$, then up to an infinite constant

$$H = \pm \int (dk)E_{\vec{k}}\left(\beta^\dagger_{\vec{k}}\beta_{\vec{k}} + \delta^\dagger_{\vec{k}}\delta_{\vec{k}}\right)$$

This is bounded from below only if we take the $+$ sign in $\mathcal{L}$, unbounded form below otherwise. This fixes the sign,

$$\mathcal{L} = \psi^\dagger\, i\sigma^\mu\partial_\mu\psi\,,$$

and furthermore tells us that the fields anti-commute,

$$\{\psi^\dagger(x), \psi(y)\}|_{x^0=y^0} = \delta^{(3)}(\vec{x} - \vec{y})$$

The Fock space consists of particles $\beta^\dagger_{\vec{k}}|0\rangle$ and antiparticles, $\delta^\dagger_{\vec{k}}|0\rangle$. Two particle states satisfy $|\vec{k}_1, \vec{k}_2\rangle = \beta^\dagger_{\vec{k}_1}\beta^\dagger_{\vec{k}_2}|0\rangle = -\beta^\dagger_{\vec{k}_2}\beta^\dagger_{\vec{k}_1}|0\rangle = -|\vec{k}_2, \vec{k}_1\rangle$, so the wave function is anti-symmetric; so are $\beta^\dagger_{\vec{k}_1}\delta^\dagger_{\vec{k}_2}|0\rangle$ and $\delta^\dagger_{\vec{k}_1}\delta^\dagger_{\vec{k}_2}|0\rangle$. We have discovered Fermi-Dirac statistics! Moreover, we were forced into it by consistency of the theory. This spin-statistics connection is a theorem in QFT, rather than an *ad-hoc* rule as in particle QM.

A note on the normalization of $u_{\vec{k}}$. It seems that we chose it as $2E_{\vec{k}}$ to give the Hamiltonian as the sum of $E_{\vec{k}}$ times the number of modes. Actually, we should fix the normalization to give $\{\psi^\dagger(x), \psi(y)\}|_{x^0=y^0} = i\delta^{(3)}(\vec{x} - \vec{y})$ given that creation/annihilation operators satisfy (6.9). It is a simple exercise to check that this is the case:

$$\{\psi^\dagger_\alpha(x), \psi_\beta(y)\}|_{x^0=y^0} = \int (dk) e^{i\vec{k}\cdot\vec{k}} \left[u^*_{\vec{k}\alpha} u_{\vec{k}\beta} + v^*_{\vec{k}\alpha} v_{\vec{k}\beta}\right]$$

where we have used $v_{\vec{k}} = u_{-\vec{k}}$. We can now check by direct computation that

$$u_{\vec{k}} u^\dagger_{\vec{k}} + v_{\vec{k}} v^\dagger_{\vec{k}} = 2E_{\vec{k}} \mathbb{1} \tag{6.10}$$

giving the desired result. The relation (6.10) is in fact a completeness relation,

$$\sum \frac{|n\rangle\langle n|}{\langle n|n\rangle} = \mathbb{1} \qquad \text{is} \qquad \frac{u_{\vec{k}} u^\dagger_{\vec{k}} + v_{\vec{k}} v^\dagger_{\vec{k}}}{2E_{\vec{k}}} = \mathbb{1}$$

Helicity(η).

Helicity of a state is defined as the angular momentum along the direction of motion. Take a particle moving along the z-axis, $k^1 = k^2 = 0$, $k^3 = \pm k^0$. Then if the particle moves in the z direction, $k^3 = k^0$

$$u^{(+)} \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

while if it moves in the negative z-direction, $k^3 = -k^0$,

$$u^{(-)} \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For a rotation by α about the z-axis, Eq. (6.6) gives

$$M\begin{pmatrix}1\\0\end{pmatrix} = e^{-i\alpha/2}\begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad M\begin{pmatrix}0\\1\end{pmatrix} = e^{i\alpha/2}\begin{pmatrix}0\\1\end{pmatrix},$$

so that $u^{(\pm)}$ are eigenvectors of M with eigenvalues $e^{\mp i\alpha/2}$. But

$$M = e^{-i\alpha J^3}$$

so $u^{(+)}$ is an eigenvector of $\eta = \hat{k}\cdot\vec{J} = J^3$ with eigenvalue, or *helicity*, $\eta = \frac{1}{2}$ while $u^{(-)}$ has $\eta = \hat{k}\cdot J = -J^3$ so again $\eta = \frac{1}{2}$.

(Aside: In general

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} = \int d^3x\,[x^\mu T^{0\nu} - x^\nu T^{0\mu} - i\pi\mathcal{J}^{\mu\nu}\phi]$$

where $\phi'(x) = (1-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu})\phi(x-\omega x)$ and π is the canonical momentum conjugate to ϕ. For the complex case one must sum over π and $\pi^\dagger$. For spinors we only have $\pi = \partial\mathcal{L}/\partial(\partial_t\psi) = \psi^\dagger\, i\sigma^0 = i\psi^\dagger$, since $\pi^\dagger = \partial\mathcal{L}/\partial(\partial_t\psi^\dagger) = 0$. So for spinors, $S^{\mu\nu} = \int d^3x\,\psi^\dagger\mathcal{M}^{\mu\nu}\psi$. If $U = \exp(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu})$, then infinitesimally $U\phi(x)U^\dagger = (1-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu})\phi(x-\omega x)$. For a rotation about the z-axis by angle θ take $\omega_{12} = -\omega_{21} = \theta$, so that $U = \exp(-i\theta J^3)$. This can be verified from $UA^\lambda(0)U^\dagger = (\eta^{\lambda\sigma} + \omega^{\lambda\sigma})A_\sigma(0)$:

$$\begin{aligned} A'^1 &= A^1 + \omega^{12}A_2 = A^1 - \omega^{12}A^2 = A^1 - \theta A^2 \\ A'^2 &= A^2 + \omega^{21}A_1 = A^2 - \omega^{21}A^1 = A^2 + \theta A^1\,. \end{aligned}$$

The factor of 2 in the definition of the spin part, $S^{\mu\nu}$, is correct. This can be checked with the vector representation, for which $(\mathcal{J}^{\mu\nu})_{\lambda\sigma} = -i(\delta^\mu_\lambda\delta^\nu_\sigma - \delta^\nu_\lambda\delta^\mu_\sigma)$. End aside).

To determine the helicity of 1-particle states annihilated and created by ψ, it is convenient to project these states into or out of the vacuum. Consider then a transformation $U(R)$ by a rotation R by angle α about the z-directions and a state with momentum $\vec{k} = k\hat{z}$:

$$\langle 0|U(R)\psi(0)U^\dagger(R)|k\hat{z}\rangle = \langle 0|M(R)\psi(0)|k\hat{z}\rangle$$

The right hand side picks up only the contribution in the plane wave expansion from the annihilation operator with momentum $\vec{k} = k\hat{z}$, so the matrix M acts on this spinor giving a factor of $\exp(-i\alpha/2)$. On the left hand side we have $\langle 0|\psi(0)(U^\dagger(R)|k\hat{z}\rangle)$. Since R does not change $\vec{k} = k\hat{z}$, the state can change at most by an overall phase. Comparing we read off the phase, $U^\dagger(R)|k\hat{z}\rangle = \exp(-i\alpha/2)|k\hat{z}\rangle$ or $U(R)|k\hat{z}\rangle = \exp(i\alpha/2)|k\hat{z}\rangle$. But $U(R) = \exp(-i\alpha J^3) = \exp(-i\alpha\eta)$, so the state annihilated by ψ has $\eta = -\frac{1}{2}$. Similarly, $\langle k\hat{z}|U(R)\psi(0)U^\dagger(R)|0\rangle = \langle k\hat{z}|M\psi(0)|0\rangle = \exp(-i\alpha/2)\langle k\hat{z}|\psi(0)|0\rangle$, gives $U(R)|k\hat{z}\rangle = \exp(-i\alpha/2)|k\hat{z}\rangle$. So ψ creates states with $\eta = \frac{1}{2}$.

For $\bar{\chi}$ we use the expansion basis of spinors $v_{\vec{k}}$. Since these are obtained from $u_{\vec{k}}$ by $\vec{k} \to -\vec{k}$, we expect they have $\eta = -\frac{1}{2}$. It is trivial to verify this: from Eq. (6.7) for a rotation $\overline{M} = M$ so the computation is as above, except now for $\vec{k}$ along the$z$-axis we get $v^{(+)} \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and similarly for $\vec{k}$ pointing in the negative z-direction we have $v^{(-)} \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, both of which give positive helicity.

To summarize: *ψ annihilates states with $\eta = -\frac{1}{2}$ and creates states with $\eta = \frac{1}{2}$; $\bar{\chi}$ annihilates states with $\eta = \frac{1}{2}$ and creates states with $\eta = -\frac{1}{2}$.*

6.3.2 Weyl vs Majorana

We stated earlier (see Eq. (6.5) and comment below) that the Lagrangian for spinors does not admit a mass term. $(\sigma^2)^{\alpha\beta}\psi_\alpha\psi_\beta = 0$ because the antisymmetric matrix $(i\sigma^2)^{\alpha\beta} = \epsilon^{\alpha\beta}$ is traced with the symmetric one $\psi_\alpha\psi_\beta$. But now that we have discovered that consistent quantization requires that spinors anti-commute we must revise this assertion: for anti-commuting fields $\psi_\alpha\psi_\beta$ is anti-symmetric!

Consider then

$$\mathcal{L} = \psi^\dagger\, i\sigma^\mu\partial_\mu\psi - (m\psi^T\epsilon\psi + \text{h.c.}) \tag{6.11}$$

where $\epsilon = i\sigma^2$. Comments:

(i) Two component massless spinors are called Weyl spinors. Massive ones are called Majorana spinors.

(ii) The Lagrangian for the Majorana spinor, Eq. (6.11), has no $U(1)$ symmetry, $\psi \to e^{i\alpha}\psi$. The Weyl case does; additional interactions with other fields may or may not respect the symmetry. For example, if we have also a complex scalar ϕ an interaction term in the Lagrangian $g\phi\psi^T\epsilon\psi + g^*\phi^*\psi^\dagger\epsilon\psi^*$ respects the symmetry, but if instead we have a real scalar, then $g\phi\psi^T\epsilon\psi + g^*\phi\psi^\dagger\epsilon\psi^*$ does not. Neither Weyl nor Majorana spinors can describe the electron: one because it does not have a mass the other because it does not carry charge.

(iii) The phase of the mass term is completely arbitrary. Since the kinetic term is invariant under $\psi \to e^{i\alpha}\psi$ we can always make a redefinition of the field ψ that changes the coefficient of the mass term by a phase $m \to e^{2i\alpha}m$: we are free to choose m real and positive.

(iv) Helicity .... To be filled in later. But main points (a) same computation as before, but (b) frame dependent since massive particle can be boosted to reverse direction of motion without changing spin

6.4 The Dirac Field

We need a description of massive spinors that carry charge, like the electron. We accomplish this by a using two spinors, ψ and $\bar{\chi}$, both with the same charge, that is, both transforming the same way under a $U(1)$ transformation, $\psi \to e^{i\alpha}\psi$ and $\bar{\chi} \to e^{i\alpha}\bar{\chi}$. Then under a Lorentz transformation

$$\bar{\chi}^\dagger \psi \to \bar{\chi}^\dagger \overline{M}^\dagger M \psi = \bar{\chi}^\dagger \psi$$

The last step follows from

$$(\overline{M}^\dagger M)_{\alpha\beta} = -\epsilon_{\alpha\gamma} M_{\delta\gamma} \epsilon_{\delta\sigma} M_{\sigma\beta}$$

and since $\det(M) = 1$, $\epsilon_{\delta\sigma} M_{\delta\gamma} M_{\sigma\beta} = \epsilon_{\gamma\beta}$ so we are left with $-\epsilon_{\alpha\gamma}\epsilon_{\gamma\beta} = \delta_{\alpha\beta}$. So $\bar{c}hi^\dagger\psi$ is both Lorentz and $U(1)$ invariant.

Hence we take

$$\mathcal{L} = \psi^\dagger \sigma^\mu \, i\partial_\mu \psi + \bar{\chi}^\dagger \bar{\sigma}^\mu \, i\partial_\mu \bar{\chi} - m(\bar{\chi}^\dagger \psi + \psi^\dagger \bar{\chi})$$

Equations of Motion:

$$\sigma^\mu \, i\partial_\mu \psi = m\bar{\chi} \qquad \text{and} \qquad \bar{\sigma}^\mu \, i\partial_\mu \bar{\chi} = m\psi \, .$$

Before we solve these, note that using one in the other we have

$$\bar{\sigma}^\mu \, i\partial_\mu (\sigma^\nu \, i\partial_\nu \psi) = m^2 \psi$$

Since $\partial_\mu \partial_\nu$ is symmetric in $\mu \leftrightarrow \nu$, we can replace $\bar{\sigma}^\mu \sigma^\nu \to \frac{1}{2}\{\bar{\sigma}^\mu, \sigma^\nu\} = \eta^{\mu\nu}$, where the last step follows from the explicit form of $\bar{\sigma}^\mu$ and σ^ν. Hence the right hand side of the equation above is $i^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \psi = -\partial^2 \psi$ and we have

$$(\partial^2 + m^2)\psi = 0 \, .$$

Each component of ψ satisfies the KG equation. Similarly

$$(\partial^2 + m^2)\bar{\chi} = 0 \, .$$

6.4.1 Dirac Spinor

It is convenient to combine the two 2-component spinors into a 4-component *Dirac filed*:

$$\Psi = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$$

Let

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} .$$

This is a 4×4 matrix in 2×2 block notation. Let also

$$\gamma^0\gamma^\mu = \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \overline{\sigma}^\mu \end{pmatrix} \quad \textit{i.e.} \quad \gamma^\mu = \begin{pmatrix} 0 & \overline{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Then

$$\mathcal{L} = \Psi^\dagger\gamma^0\gamma^\mu\, i\partial_\mu\Psi - m\Psi^\dagger\gamma^0\Psi$$

or introducing the shorthand $\overline{\Psi} \equiv \Psi^\dagger\gamma^0$,

$$\mathcal{L} = \overline{\Psi}\gamma^\mu\, i\partial_\mu\Psi - m\overline{\Psi} = \overline{\Psi}(\gamma^\mu\, i\partial_\mu - m)\Psi = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi\,.$$

We have introduced the slash notation: for any vector a_μ define $a\!\!\!/ = a_\mu\gamma^\mu$.

The equation of motion is the famous Dirac equation,

$$(i\partial\!\!\!/ - m)\Psi = 0$$

Now

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

so that $(i\partial\!\!\!/)(i\partial\!\!\!/) = -\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu = -\partial^2$ and

$$(\partial^2 + m^2)\Psi = 0$$

which is the statement that all components of Ψ satisfy the KG equation, as it should since the components of ψ and $\overline{\chi}$ do.

Plane-wave expansion: we use $\alpha = 1, \ldots, 4$ for the index of Ψ,

$$\Psi_\alpha(x) = \int (dk) \sum_{s=1}^{2} \left[\beta_{\vec{k},s} u_\alpha^{(s)}(\vec{k})e^{-ik\cdot x} + \gamma^\dagger_{\vec{k},s} v_\alpha^{(s)} e^{ik\cdot x}\right]$$

where $k^2 = m^2$ and the *Dirac spinors* satisfy

$$(k\!\!\!/ - m)u^{(s)}(\vec{k}) = 0 \quad \text{and} \quad (k\!\!\!/ + m)v^{(s)}(\vec{k}) = 0\,.$$

To solve these notice that $k^2 = m^2$ gives $(k\!\!\!/ - m)(k\!\!\!/ + m) = 0$ and $(k\!\!\!/ + m)(k\!\!\!/ - m) = 0$. So take $u(\vec{k}) = (k\!\!\!/ + m)u_0$ for some u_0 such that $(k\!\!\!/ + m)u_0 \neq 0$. Noice that we have anticipated that there are two independent solutions to each of these equations. For example, if $\vec{k} = 0$, $k^0 = m$ then

$$k\!\!\!/ + m = m(\gamma^0 + 1) = m \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$$

so

$$u^{(1)} = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; u^{(2)} = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

where we have used a normalization that will give $u(\vec{k})^\dagger u(\vec{k}) = 2E_{\vec{k}}$, as will be needed for simple anti-commutation relations for $\beta_{\vec{k},s}$ and $\gamma_{\vec{k},s}$.

It will be useful to introduce for any 4×4 matrix Γ the conjugate

$$\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$$

Then $\overline{\gamma}^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ (since $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$ and the anti-commutation relations that give $(\gamma^0)^2 = \mathbb{1}$ and $\gamma^0\gamma^i = -\gamma^i\gamma^0$). Then

$$(\not{k} - m)u = 0 \quad \Rightarrow \quad \overline{u}(\not{k} - m) = 0\,,$$

and likewise $\overline{v}(\not{k} + m) = 0$.

It is easy to show that

$$\overline{u}^{(s)}(k)u^{(s')}(k) = 2m\delta^{ss'} = -\overline{v}^{(s)}(k)v^{(s')}(k)$$

and

$$\sum_s u^{(s)}(k)\overline{u}^{(s)}(k) = m + \not{k} \qquad -\sum_s v^{(s)}(k)\overline{v}^{(s)}(k) = m - \not{k} \tag{6.12}$$

Moreover,

$$\overline{u}^{(s)}(k)\gamma^\mu u^{(s)}(k) = 2k^\mu$$

In particular $\overline{u}^{(s)}(k)\gamma^0 u^{(s)}(k) = u^{(s)\dagger}(k)u^{(s)}(k) = 2E$ is not a scalar.

With these normalizations,

$$\{\Psi^\dagger(x), \Psi(y)\}|_{x^0=y^0} = \mathbb{1}\delta^{(3)}(\vec{x} - \vec{y})$$

and

$$H = \sum_s \int (dk) E_{\vec{k}} \left(\beta^\dagger_{\vec{k},s}\beta_{\vec{k},s} + \gamma^\dagger_{\vec{k},s}\gamma_{\vec{k},s}\right)$$

up to an infinite constant, removed by normal-ordering.

6.4.2 Dirac vs Weyl representations

We can always make a redefinition of the Dirac field $\Psi \to S\Psi$ by a unitary matrix S. Then we change $\gamma^\mu \to S^\dagger\gamma^\mu S$. This allows us to choose a different, convenient basis of Dirac gamma matrices. For example we take

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}$$

In this *DIrac representation* we have

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}.$$

The basis we had before is called the *Weyl representation*:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}.$$

In any basis,

$$\gamma^{0\dagger} = \gamma^0, \gamma^{i\dagger} = -\gamma^i, \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

and

$$\mathrm{Tr}\, \not{a} = 0 \tag{6.13}$$

$$\mathrm{Tr}\, \not{a}\not{b} = 4a \cdot b \tag{6.14}$$

$$\mathrm{Tr}\, \not{a}_1 \cdots \not{a}_{2n+1} = 0 \tag{6.15}$$

$$\mathrm{Tr}\, \not{a}\not{b}\not{c}\not{d} = 4(a \cdot bc \cdot d + a \cdot db \cdot c - a \cdot cb \cdot d) \tag{6.16}$$

6.4.3 Wick's Theorem, T-product, Perturbation theory

Take $x_1^0 > x_2^0$, $\psi = \psi^{(+)} + \psi^{(-)}$, with $\psi^{(+)}$ and $\psi^{(-)\dagger}$ annihilation operators. Then

$$\begin{aligned} \psi(x_1)\psi^\dagger(x_2) &= (\psi_1^{(+)} + \psi_1^{(-)})(\psi_2^{(+)\dagger} + \psi_2^{(-)\dagger}) \\ &= \psi_1^{(+)}\psi_2^{(-)\dagger} + \{\psi_1^{(+)}, \psi_2^{(+)\dagger}\} - \psi_2^{(+)\dagger}\psi_1^{(+)} + \psi_1^{(-)}\psi_2^{(+)\dagger} + \psi_1^{(-)}\psi_2^{(-)\dagger} \\ &= {:}\psi(x_1)\psi^\dagger(x_2){:} + c\text{-number} \end{aligned} \tag{6.17}$$

with the understanding that in the normal ordering we pick up a minus sign any time we move an operator through another. So we define the T-ordered product for two anti-commuting fields $A(x)$ and $B(y)$ as

$$T(A(x)B(y)) = \theta(x^0 - y^0)A(x)B(y) - \theta(y^0 - x^0)B(y)A(x)\,.$$

Then the c-number is $\langle 0|T\psi(x_1)\psi^\dagger(x_2)|0\rangle = \overbracket{\psi(x_1)\psi^\dagger}(x_2)$ and Wick's theorem is just as before with the caveat that we must include minus signs for anti-commutations. For example,

$$\begin{aligned} &\langle 0|T\,\psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)|0\rangle \\ &\qquad = \overbracket{\psi(x_1)\psi^\dagger}(x_4)\overbracket{\psi(x_2)\psi^\dagger}(x_3) - \overbracket{\psi(x_1)\psi^\dagger}(x_3)\overbracket{\psi(x_2)\psi^\dagger}(x_4) \end{aligned} \tag{6.18}$$

The basic quantity we will need to compute amplitudes and for our Feynman rules is the two point function:

$$\langle 0|T\,\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} i \frac{(\not{k} + m)_{\alpha\beta}}{k^2 - m^2 + i\epsilon}$$

Note this is not symmetric under $k \to -k$. You can verify this by writing explicitly the plane-wave expansion of the Dirac field (if you try this you will need to use (6.12)).

To understand Feynman rules we work in a specific context. Let ψ be a Dirac spinor of mass m and ϕ a real scalar of mass M, and take

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}M^2\phi^2 - g\phi\overline{\psi}\psi$$

The last term is called a *Yukawa interaction* and the coefficient g a *Yukawa coupling constant*. Then, as before, Green functions are

$$G^{(n,m,l)}(x_1,\ldots,x_n,y_1,\ldots,y_m,z_1,\ldots,z_l) = \langle 0|T\left(\psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_m)\phi(z_1)\cdots\phi(z_l)\right)|0\rangle$$

and in perturbation theory this equals

$$\frac{\langle 0|T\left(\psi_{\rm in}(x_1)\cdots\phi_{\rm in}(z_l)e^{-i\int d^4x g\phi_{\rm in}(x)\overline{\psi}_{\rm in}(x)\psi_{\rm in}(x)}\right)|0\rangle}{\langle 0|T\left(e^{-i\int d^4x g\phi_{\rm in}(x)\overline{\psi}_{\rm in}(x)\psi_{\rm in}(x)}\right)|0\rangle}$$

This can be expanded using Wick's theorem as above. But note that $[\phi,\psi] = 0 = [\phi,\psi^\dagger]$ so there is no sign change when moving ψ or $\psi^\dagger$ through ϕ.

For example, the simplest non-trivial Green function is $G^{(1,1,1)}$, which to lowest order in an expansion in g is

$$\begin{aligned}G^{(1,1,1)}_{\alpha\beta}(x,y,z) &= -ig\int d^4w\langle 0|T\psi_{{\rm in}\alpha}(x)\psi^\dagger_{{\rm in}\beta}(y)\phi_{\rm in}(z)\phi_{\rm in}(w)\overline{\psi}_{{\rm in}\gamma}(w)\psi_{{\rm in}\gamma}(w)|0\rangle\\ &= -ig\int d^4w(-1)^2\,\overbrace{\psi_{{\rm in}\alpha}(x)\overline{\psi}_{{\rm in}\gamma}(w)}\,\overbrace{\psi_{{\rm in}\gamma}(w)\overline{\psi}_{{\rm in}\beta}(y)}\,\overbrace{\phi_{\rm in}(z)\phi_{\rm in}(w)}\end{aligned}$$

and the rest as before. In particular $G(\{x\}) = \int\prod d^4k e^{i\sum k\cdot x}(2\pi)^4\delta^{(4)}(\sum k)\widetilde{G}(\{k\})$:

$$\begin{aligned}G^{(1,1,1)} &= -ig\int d^4w\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-w)}\frac{i(k\!\!\!/ + m)_{\alpha\gamma}}{k^2-m^2+i\epsilon}\\ &\qquad\times\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(w-y)}\frac{i(p\!\!\!/ + m)_{\gamma\beta}}{p^2-m^2+i\epsilon}\int\frac{d^4q}{(2\pi)^4}e^{iq\cdot(z-w)}\frac{i}{q^2-M^2+i\epsilon}\\ &= -ig\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}e^{-ik\cdot x+ip\cdot y-iq\cdot z}(2\pi)^4\delta^{(4)}(k-p+q)\\ &\qquad\times i^2\frac{[(k\!\!\!/ + m)(k\!\!\!/ + m)]_{\alpha\beta}}{(k^2-m^2+i\epsilon)(p^2-m^2+i\epsilon)}\frac{i}{q^2-M^2+i\epsilon}\end{aligned}$$

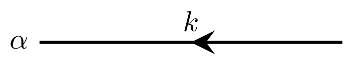
so that

$$\widetilde{G}^{(1,1,1)}(-k,p,-q) = -ig\frac{i}{k\!\!\!/ - m}\frac{i}{p\!\!\!/ - m}\frac{i}{q^2-M^2}$$

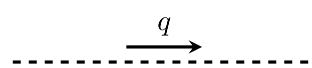
where we have omitted the $i\epsilon$ and used

$$\frac{1}{\not{k} - m} = \frac{\not{k} + m}{k^2 - m^2}\,.$$

We can represent this graphically as follows:

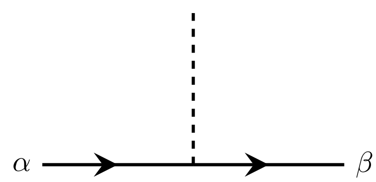


$$= \left(\frac{i}{\not{k} - m + i\epsilon}\right)_{\alpha\beta}$$



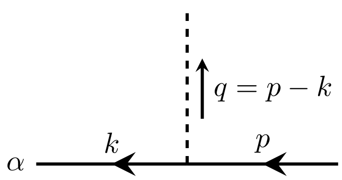
$$= \frac{i}{q^2 - M^2 + i\epsilon}$$

and for the vertex



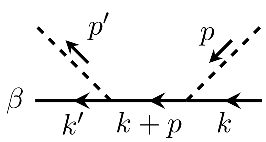
$$= -ig\delta_{\alpha\beta}\,.$$

So the computation above is



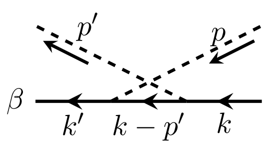
$$= \left(\frac{i}{\not{k} - m + i\epsilon}\right)_{\alpha\gamma}(-ig\delta_{\gamma\delta})\left(\frac{i}{\not{p} - m + i\epsilon}\right)_{\delta\beta}\frac{i}{q^2 - M^2 + i\epsilon}$$

Here is an example relevant to $\phi\psi \to \phi\psi$ scattering (or rather scattering of the quanta of these fields). To lowest order in an expansion in the coupling constant, the contributions to $\widetilde{G}^{(1,1,2)}$ are



$$= (-ig)^2\left[\frac{i}{\not{k}' - m}\,\frac{i}{(\not{k} + \not{p}) - m}\,\frac{i}{\not{k} - m}\right]_{\beta\alpha}\frac{i}{p^2 - M^2}\,\frac{i}{p'^2 - M^2}$$

and



$$= (-ig)^2\left[\frac{i}{\not{k}' - m}\,\frac{i}{(\not{k} - \not{p}') - m}\,\frac{i}{\not{k} - m}\right]_{\beta\alpha}\frac{i}{p^2 - M^2}\,\frac{i}{p'^2 - M^2}$$

6.4.4 LSZ-reduction for spinros, stated

The LSZ reduction formula for spinors is exactly as for scalars, except

(i) Amputate with $i\partial\!\!\!/ - m$ rather than $\partial^2 + m^2$

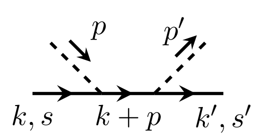
(ii) An "in" 1-particle state $|\vec{k}, s\rangle_{\text{in}}$ gives $u_\alpha^{(s)}(\vec{k})$, where α is contracted with the corresponding index in the Green's function. This comes from

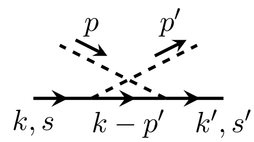
$$\psi_\alpha^{(+)}|\vec{k}, s\rangle_{\text{in}} = \int (dk') \sum_{s'} e^{-ik'\cdot x} u_\alpha^{(s')}(\vec{k}')\beta_{\vec{k}',s'}|\vec{k}, s\rangle_{\text{in}} \,.$$

And, similarly, $\bar{u}^{(s)}(\vec{k})$ for ${}_{\text{out}}\langle\vec{k}, s|$, $\bar{v}^{(s)}(\vec{k})$ for antiparticle $|\vec{k}, s\rangle_{\text{in}}$, and $v^{(s)}(\vec{k})$ for antipartilce ${}_{\text{out}}\langle\vec{k}, s|$.

(iii) Possibly $(-1)^p$ for some p, from anti-commuting states.

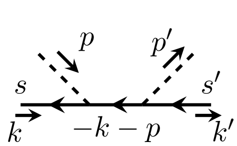
Example: in the sample Yukawa theory above, from the computation of $\widetilde{G}^{(1,1,2)}$, we can obtain the amplitude for the scattering of a spin-0 particle with a spin-$\frac{1}{2}$ particle:

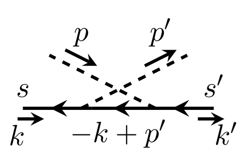
${}_{\text{out}}\langle\vec{k}', s'; \vec{p}'|\vec{k}, s; \vec{p}\rangle_{\text{in}} =$


$+$


$$= -ig^2\bar{u}^{(s')}(\vec{k}')\left[\frac{1}{(k\!\!\!/ + p\!\!\!/) - m} + \frac{1}{(k\!\!\!/ - p\!\!\!/') - m}\right]u^{(s)}(\vec{k})$$

Note the convention here: while time ir ordered later to earlier as we read left to right in ${}_{\text{out}}\langle\vec{k}', s'; \vec{p}'|\vec{k}, s; \vec{p}\rangle_{\text{in}}$, the Feynman diagram for the amplitude is ordered earlier to later (in to out) as we read left to right. But the expression for the amplitude is ordered, in this case, in the opposite sense: the Dirac spinor for the out state is on the left while the one for the in state is on the right. Generally a line representing a spinor that eneters the diagram from the left and has an arrow pointing right represents an in-particle, while if the arrow is pointing left it represent an in-antiparticle. A line exiting on the right represents an out-particle if the arrow is pointing right, and an out-antiparticle if pointing left. Here is an example of antiparticle scattering off the scalar:

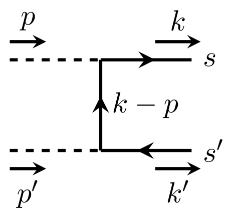


$+$


$$= -ig^2\bar{v}^{(s)}(\vec{k})\left[\frac{1}{(-k\!\!\!/ - p\!\!\!/) - m} + \frac{1}{(-k\!\!\!/ + p\!\!\!/') - m}\right]v^{(s')}(\vec{k}')$$

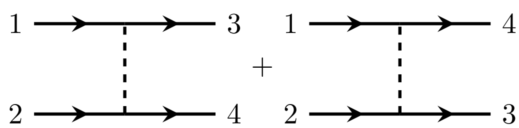
Note that while the particle and antiparticle scattering amplitudes of the scalar appear superficially different, they are the same (up to a sign) once $\not{k}u(\vec{k}) = mu(\vec{k})$ and $\not{k}v(\vec{k}) = -mv(\vec{k})$ are used.

Here is an example which involves both u and v spinors: scalar-scalar scattering into particle -antiparticle pair (of fermions):



$$= -ig^2 \bar{u}^{(s)}(\vec{k}) \left[\frac{1}{(\not{k} - \not{p}) - m} \right] v^{(s')}(\vec{k}')$$

And, finally, here is an example with a sign from anti-commuting external states, two particles in the initial state scattering into two particles:



$$= -ig^2 \left[\frac{\bar{u}_3 u_1 \bar{u}_4 u_2}{(k_1 - k_3)^2 - M^2} - \frac{\bar{u}_4 u_1 \bar{u}_3 u_2}{(k_1 - k_4)^2 - M^2} \right]$$

Here we used the compact notation $u_1 = u^{(s_1)}(\vec{k}_1)$, etc. The relative sign between the two terms is a reflection of Dirac statistics of the external states.

6.5 Generators

Let's study the generators of the Lorentz group in the representation of Weyl spinors:

$$M = 1 - \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}$$

corresponding to $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ with $\omega^{\mu\nu} = -\omega^{\nu\mu}$ infinitesimal. So $\mathcal{M}^{\mu\nu}$ are six 2×2 matrices that we want to characterize. First we note that $\det(M) = 1$ implies $\mathrm{Tr}(\mathcal{M}^{\mu\nu}) = 0$. Next, determine $\mathcal{M}^{\mu\nu}$, using the known transformation properties of vectors. On one hand

$$\begin{aligned} M^\dagger P M &= ((\Lambda^{-1})_\mu{}^\nu p_\nu)\sigma^\mu \\ &= p^\nu \sigma^\mu(\eta_{\mu\nu} - \omega_{\mu\nu}) \end{aligned}$$

and on the other

$$\begin{aligned} M^\dagger P M &= (1 + \tfrac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\dagger\mu\nu})p_\lambda\sigma^\lambda(1 - \tfrac{i}{2}\omega_{\sigma\rho}\mathcal{M}^{\sigma\rho}) \\ &= p_\mu\sigma^\mu + \tfrac{i}{2}\omega_{\mu\nu}p_\lambda(\mathcal{M}^{\dagger\mu\nu}\sigma^\lambda - \sigma^\lambda\mathcal{M}^{\mu\nu}) \,. \end{aligned}$$

Equating,

$$\tfrac{i}{2}\omega_{\mu\nu}p_\lambda(\mathcal{M}^{\dagger\mu\nu}\sigma^\lambda - \sigma^\lambda\mathcal{M}^{\mu\nu}) = -p^\nu\sigma^\mu\omega_{\mu\nu} = -\omega_{\mu\nu}p_\lambda\eta^{\lambda\nu}\sigma^\mu$$

and since this must hold for arbitrary $\omega_{\mu\nu}$ and p_λ we have

$$\mathcal{M}^{\dagger\mu\nu}\sigma^\lambda - \sigma^\lambda\mathcal{M}^{\mu\nu} = -i(\eta^{\lambda\mu}\sigma^\nu - \eta^{\lambda\nu}\sigma^\mu),.$$

The solution to this is straightforward:

$$\mathcal{M}^{0i} = -\mathcal{M}^{i0} = \tfrac{i}{2}\sigma^i, \quad \mathcal{M}^{ij} = -\mathcal{M}^{ji} = \tfrac{1}{2}\epsilon^{ijk}\sigma^k,$$

(Aside: to determine this, expand $\mathcal{M}^{\mu\nu}$ in the basis of σ^μ. Since $\operatorname{Tr}\mathcal{M}^{\mu\nu} = 0$ we have $\mathcal{M}^{\mu\nu} = (a_j^{\mu\nu} + ib_j^{\mu\nu})\sigma^j$ and $\mathcal{M}^{\dagger\mu\nu} = (a_j^{\mu\nu} - ib_j^{\mu\nu})\sigma^j$. This gives

$$a_j^{\mu\nu}[\sigma^j,\sigma^\lambda] - ib_j^{\mu\nu}\{\sigma^j,\sigma^\lambda\} = -i(\eta^{\lambda\mu}\sigma^\nu - \eta^{\lambda\nu}\sigma^\mu)\,.$$

Setting $\lambda = 0$ gives an equation for the b_j's:

$$2b_j^{\mu\nu}\sigma^j = -(\delta^{\nu 0}\sigma^\mu - \delta^{\mu 0}\sigma^\nu)$$

and taking $\mu = 0$ and $\nu = k$ we have $2b_j^{0k}\sigma^j = \sigma^k$ from which $b_j^{0k} = -b_j^{k0} = \frac{1}{2}\delta_j^k$ follows. Similarly we obtain $b_j^{ik} = 0$. For the $a's$ set $\lambda = i$. Then

$$a_j^{\mu\nu}2i\epsilon^{jik}\sigma^k - ib_j^{\mu\nu}2\delta^{ij} = -i(\eta^{i\mu}\sigma^\nu - \eta^{i\nu}\sigma^\mu)$$

Then setting $\mu = 0$ gives $a_j^{0\nu} = 0$ and setting $\mu = l$ and $\nu = m$ gives $a_j^{lm} = \frac{1}{2}\epsilon^{lmj}$.)

Let

$$J^k = \tfrac{1}{2}\epsilon^{kij}\mathcal{M}^{ij} = \tfrac{1}{2}\sigma^k, \quad \text{and} \quad K^i = \mathcal{M}^{0i} = \tfrac{i}{2}\sigma^i\,.$$

Note that $J^{i\dagger} = J^i$ while $K^{i\dagger} = -K^i$. These satisfy,

$$[J^i,J^j] = i\epsilon^{ijk}J^k, \quad [J^i,K^j] = i\epsilon^{ijk}K^k, \quad [K^i,K^j] = -i\epsilon^{ijk}J^k\,.$$

Defining

$$J_\pm^i \equiv \tfrac{1}{2}(J^i \pm iK^i)$$

we have

$$[J_+^i,J_+^j] = i\epsilon^{ijk}J_+^k, \quad [J_-^i,J_-^j] = i\epsilon^{ijk}J_-^k, \quad [J_+^i,J_-^j] = 0\,.$$

You recognize these are two mutually commuting copies of (the algebra of) $SO(3) \sim SU(2)$. You know this from the rotation group in QM: for spinors, $\vec{S} = \frac{1}{2}\vec{\sigma}$, and $[S^i,S^j] = i\epsilon^{ijk}S^k$, while for vectors $(L^i)_{jk} = -i\epsilon^{ijk}$, with $[L^i,L^j] = i\epsilon^{ijk}L^k$. The irreducible representations of $SU(2)$ are $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ of dimension $2\ell + 1 = 1, 2, 3, 4, \ldots$. You recognize $\ell = 0$ as a scalar, $\ell = \frac{1}{2}$ a spinor, $\ell = 1$ a vector, etc.

Comments:

(i) For any representation $D(\Lambda)$ we have infinitesimal generators. If $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ then $D(\Lambda)_{\alpha\beta} = \delta_{\alpha\beta} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}$. The six matrices $\mathcal{J}^{\mu\nu}$ satisfy the same commutation relations as $\mathcal{M}^{\mu\nu}$, they are fixed by the multiplication "table" of the Lorentz group, which itself follows form requiring $D(\Lambda)$ be a representation, Eq. (6.2).

(ii) The *defining* representation is the 4-dim representation acting on vectors, p^μ, like Λ itself. That is, $D(\Lambda)^\mu{}_\nu p^\nu = \Lambda^\mu{}_\nu p^\nu \Rightarrow -\frac{i}{2}\omega_{\lambda\sigma}(\mathcal{J}^{\lambda\sigma})_{\mu\nu}p^\nu = \omega_{\mu\nu}p^\nu$, from which we read off, for the 4-dimensional (defining) representation of the Lorentz group: $(\mathcal{J}^{\lambda\sigma})_{\mu\nu} = -i(\delta^\lambda_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\lambda_\nu)$. We can compute easily the same commutation relations satisfied by the $\mathcal{J}^{\lambda\sigma}$ and of course they are the same as those satisfied by $\mathcal{M}^{\lambda\sigma}$.

(iii) The commutation relations for J^i and K^i derived above may seem ambiguous since they were found from comparing with Pauli matrices but both J^i and K^i are given in terms of Pauli matrices. For example, $[K^i, K^j] = -i\epsilon^{ijk}\frac{1}{2}\sigma^k$ was written as $-i\epsilon^{ijk}J^k$ rather than $-\epsilon^{ijk}K^k$. There is a simple argument why $[K, K] = -K$ is excluded. Under parity $\mathcal{M}^{ij} \to (-1)^2\mathcal{M}^{ij}$, $\mathcal{M}^{0i} \to (-1)\mathcal{M}^{0i}$. Hence $J^i \to J^i$ and $K^i \to -K^i$. This gives, $[J, J] \sim J$ but not K, $[J, K] \sim K$ but not J and $[K, K] \sim J$ but not K.

(iv) We would not have faced this ambiguity (nor would we have had to use the parity argument to sort it out) had we studied the commutation relations for arbitrary representations.

6.5.1 All the representations of the Lorentz Group

Since the same commutation relations must hold for any representation we use that to construct them. We build on our knowledge of the representations of $SU(2)$.

For J^i_+ the irreducible representations are classified by $s_+ = 0, \frac{1}{2}, 1, \ldots$. The $(2s_++1)\times(2s_++1)$ matrices are labeled $J^i_{s_+}$. For J^i_- the irreducible representations are classified by $s_- = 0, \frac{1}{2}, 1, \ldots$. The $(2s_-+1)\times(2s_-+1)$ matrices are labeled $J^i_{s_-}$. Moreover, since the representation matrices of the generators of the Lorentz group, J^i_+ and J^i_-, satisfy $[J^i_+, J^i_-] = 0$, they are tensor products, $J^i_+ = J^i_{s_+} \otimes \mathbb{1}_{s_-}$, where $\mathbb{1}_{s_-}$ is the $(2s_- + 1) \times (2s_- + 1)$ identity, matrix, and, similarly, $J^i_- = \mathbb{1}_{s_+} \otimes J^i_{s_-}$. The irreducible representation are labeled by (s_+, s_-) and have generators $J^i = J^i_+ + J^i_- = J^i_{s_+} \otimes \mathbb{1}_{s_-} + \mathbb{1}_{s_+} \otimes J^i_{s_-}$ and similarly for K^i. They have dimension $(2s_+ + 1)(2s_- + 1)$.

For example, $(0, 0)$ is a representation of dimension 1, a scalar.

$(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ have dimension 2. They are spinors. They are two different spinor representations. Consider $(\frac{1}{2}, 0)$:

$$J^i_{s_+} = \tfrac{1}{2}\sigma^i,\; J^i_{s_-} = 0, \;\; \Rightarrow \vec{J} = \vec{J}_+ + \vec{J}_- = \tfrac{1}{2}\vec{\sigma} \otimes \mathbb{1},\; \vec{K} = \tfrac{1}{i}(\vec{J}_+ - \vec{J}_-) = -\tfrac{i}{2}\vec{\sigma} \otimes \mathbb{1}$$

Similarly, for $(0,\frac{1}{2})$

$$J^i_{s_+}=0,\ J^i_{s_-}=\tfrac{1}{2}\sigma^i,\ \ \Rightarrow \vec{J}=\vec{J}_++\vec{J}_-=\mathbb{1}\otimes\tfrac{1}{2}\vec{\sigma},\ \vec{K}=\tfrac{1}{i}(\vec{J}_+-\vec{J}_-)=\mathbb{1}\otimes\tfrac{i}{2}\vec{\sigma}$$

But this is where we started from. From $\psi'=M\psi$ we obtained $\vec{J}=\frac{1}{2}\vec{\sigma}$ and $\vec{K}=\frac{i}{2}\vec{\sigma}$ so this is the $(0,\frac{1}{2})$ representation. So what is $(\frac{1}{2},0)$? We already know that $\overline{M}$ is a representation. Look more closely:

$$M=e^{-i\vec{\theta}\cdot(\frac{1}{2}\vec{\sigma})-i\vec{\gamma}\cdot(\frac{i}{2}\vec{\sigma})}=e^{-i\vec{\alpha}\cdot(\frac{1}{2}\vec{\sigma})}\quad\text{with } \vec{\alpha}=\vec{\theta}+i\vec{\gamma},\ (\alpha^i\in\mathbb{C}).$$

The representation we are looking for should have group elements

$$e^{-i\vec{\theta}\cdot(\frac{1}{2}\vec{\sigma})-i\vec{\gamma}\cdot(-\frac{i}{2}\vec{\sigma})}=e^{-i\vec{\alpha}^*\cdot(\frac{1}{2}\vec{\sigma})}$$

while

$$M^*=e^{-i\vec{\theta}\cdot(-\frac{1}{2}\vec{\sigma}^*)+i\vec{\gamma}\cdot(-\frac{i}{2}\vec{\sigma}^*)}=e^{-i\vec{\alpha}^*\cdot(-\frac{1}{2}\vec{\sigma}^*)}\,.$$

This is close. In fact, it is what we want up to a similarity transformation: using $\sigma^2\vec{\sigma}^*\sigma^2=-\vec{\sigma}$ we have

$$\overline{M}=\sigma^2M^*\sigma^2=e^{-i\vec{\alpha}^*\cdot(\frac{1}{2}\vec{\sigma})}\,.$$

That is, M^* is in the equivalence class of $(\frac{1}{2},0)$.

Generalize: note that since $M=\mathbb{1}-i\omega_{\mu\nu}\mathcal{M}^{\mu\nu}$ we have $M^*=\mathbb{1}-i\omega_{\mu\nu}(-\mathcal{M}^{\mu\nu*})$. The matrices $-\mathcal{M}^{\mu\nu*}$ satisfy the same commutation relations as the $\mathcal{M}^{\mu\nu}$. More genrally, if $[T^a,T^b]=if^{abc}T^c$ then $[-T^{a*},-T^{b*}]=if^{abc*}(-T^{c*})$, so $-T^{a^*}$ satisfy the same commutation relations as T^a if f^{abc} are real. In our case $(J^i_\pm)$ the f^{abc} are ϵ^{ijk}. Since for $SU(2)$ the only irreducible representation of dimension $2s+1$ is generated by $\vec{J}_s$ it must be that $S(-\vec{J}^*_s)S^{-1}=\vec{J}_s$ for some invertible matrix S. Now

$$D(\Lambda)=e^{-i\vec{\alpha}\cdot\vec{J}_+-i\vec{\alpha}^*\cdot\vec{J}_-}$$

so that

$$SD^*(\Lambda)S^{-1}=e^{-i\vec{\alpha}^*\cdot\vec{J}_+-i\vec{\alpha}\cdot\vec{J}_-}$$

The role of J_+ and J_- has been exchanged. To be more precise, the matrix S that acts on the tensor product exchanges the $+$ and $-$ sectors. We therefore have,

$$(s_+,s_-)^*\sim(s_-,s_+)$$

Note that for $s_+=s_-$ the complex conjugate representation is similar to itself. This is a *real representation* and the vectors on which it acts can be taken to have real components. For example, let's investigate the $(\frac{1}{2},\frac{1}{2})$ representation. It is 2×2 dimensional. It smells like a 4-vector. Let's show it is. It is an object with indices $\dot{\alpha}\alpha$ as in $V_{\dot{\alpha}\alpha}$, with

$$V'_{\dot{\alpha}\alpha}=\overline{M}_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta}V_{\dot{\beta}\beta}$$

This transforms like $\bar{\chi}_{\dot{\alpha}}\psi_{\alpha}$. We have already seen that this 4-component object can be arranged into a 4-vector; more specifically $\psi^{\dagger}\sigma^{\mu}\psi$ is a 4-vector and the relation between $\psi^{\dagger} = \bar{\chi}^{\dagger}\sigma^2$, so consider $V^{\mu} = V_{\dot{\alpha}\alpha}(\sigma^2\sigma^{\mu})_{\dot{\alpha}\alpha}$. Then

$$\begin{aligned} V'^{\mu} &= \overline{M}_{\dot{\alpha}\dot{\beta}} M_{\alpha\beta} V_{\dot{\beta}\beta}(\sigma^2\sigma^{\mu})_{\dot{\alpha}\alpha} \\ &= (\overline{M}^T \sigma^2 \sigma^{\mu} M)_{\dot{\beta}\beta} V_{\dot{\beta}\beta} \\ &= (\sigma^2 M^{\dagger}\sigma^{\mu} M)_{\dot{\beta}\beta} V_{\dot{\beta}\beta} \\ &= \Lambda^{\mu}{}_{\nu}(\sigma^2\sigma^{\nu})_{\dot{\beta}\beta} V_{\dot{\beta}\beta} \\ &= \Lambda^{\mu}{}_{\nu} V^{\nu} \end{aligned}$$

More generally, $X_{\alpha_1\cdots\alpha_{s_+}\dot{\alpha}_1\cdots\alpha_{s_-}}$ with all α indices symmetrized and all $\dot{\alpha}$ indices symmetrized is in the (s_+, s_-) representation.