Chapter 4

Interaction with External Sources

4.1 Classical Fields and Green's Functions

We start our discussion of sources with classical (non-quantized) fields. Consider

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi(x))^2 - \tfrac{1}{2}m^2\phi(x)^2 + J(x)\phi(x)$$

Here $J(x)$ is a non-dynamical field, the *source*. The equation of motion for the dynamical field ($\phi(x)$) is

$$(\partial^2 + m^2)\phi(x) = J(x)\,. \tag{4.1}$$

The terminology comes from the more familiar case in electromagnetism. The non-homogeneous Maxwell equations,

$$\vec{\nabla}\cdot\vec{E} = \rho, \qquad \vec{\nabla}\times\vec{B} - \partial_0\vec{E} = \vec{J}$$

have as sources of the electric and magnetic fields the charge and current densities, ρ and $\vec{J}$, respectively. We can go a little further in pushing the analogy. In terms of the electric and vector potentials, A_0 and $\vec{A}$, respectively, the fields are

$$\vec{E} = -\partial_0\vec{A} - \vec{\nabla}A_0, \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

so that Gauss's law becomes

$$-\partial_0\vec{\nabla}\cdot\vec{A} - \nabla^2 A_0 = \rho$$

In *Lorentz gauge*, $\vec{\nabla}\cdot\vec{A} + \partial_0 A_0 = 0$ this takes the form

$$\partial^2 A_0 = \rho\,.$$

This is exactly for the form of (4.1), for a massless field with the identification A_0 for ϕ and ρ for J. In Lorentz gauge the equations satisfied by the vector potential are again of this form,

$$\partial^2 \vec{A} = \vec{J}\,.$$

The four vector $J^\mu = (\rho, J^i)$ serves as source of the four-vector potential A^μ. Writing

$$\partial^2 A^\mu = J^\mu\,.$$

is reassuringly covariant under Lorentz transformations, as it should be since this is where relativity was discovered! Each of the four components of A^μ satisfies the massless KG equation with source.

Consider turning on and off a localized source. The source is "on," that is non-vanishing, only for $-T < t < T$. Before J is turned on ϕ is evolving as a *free* KG field (meaning free of sources or interactions), so we can think of the initial conditions as giving $\phi(x) = \phi_{\text{in}}(x)$ for $t < -T$, with $\phi_{\text{in}}(x)$ a solution of the free KG equation. We similarly have that for $t > T$ $\phi(x) = \phi_{\text{out}}(x)$ where $\phi_{\text{out}}(x)$ is a solution of the free KG equation. Both $\phi_{\text{in}}(x)$ and $\phi_{\text{out}}(x)$ are solutions of the KG equations for all t but they agree with ϕ only for $t < -T$ and $t > T$, respectively. Our task is to find $\phi_{\text{out}}(x)$ given $\phi_{\text{in}}(x)$ (and of course the source $J(x)$).

Solve (4.1) using Green functions:

$$\phi(x) = \phi_{\text{hom}}(x) + \int d^4y\, G(x-y)J(y) \tag{4.2}$$

where the Green function satisfies

$$(\partial^2 + m^2)G(x) = \delta^{(4)}(x) \tag{4.3}$$

and $\phi_{\text{hom}}(x)$ is a solution of the associated homogeneous equation, which is the KG equation,

$$(\partial^2 + m^2)\phi_{\text{hom}}(x) = 0\,.$$

We can use the freedom in $\phi_{\text{hom}}(x)$ to satisfy boundary conditions. We can determine the Green function by Fourier transform,

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\widetilde{G}(k) \qquad \delta^{(4)}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}$$

Then

$$(\partial^2 + m^2)\int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\widetilde{G}(k) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}(-k^2 + m^2)\widetilde{G}(k) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}$$

so that

$$\widetilde{G}(k) = -\frac{1}{k^2 - m^2}\,.$$

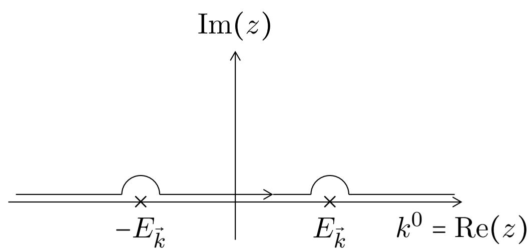
So preliminarily take

$$G(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 - m^2}$$

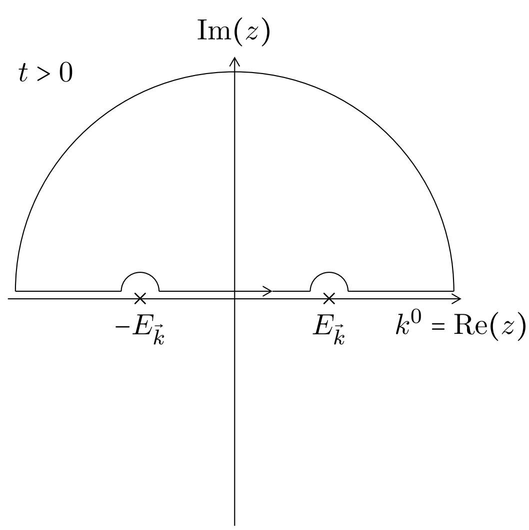
However, note this is ill-defined since the integrand diverges at $k^2 = m^2$. The integral over k^0 diverges at $k^0 = \pm\sqrt{\vec{k}^2 + m^2}$, or $k^0 = \pm E_{\vec{k}}$ for short:

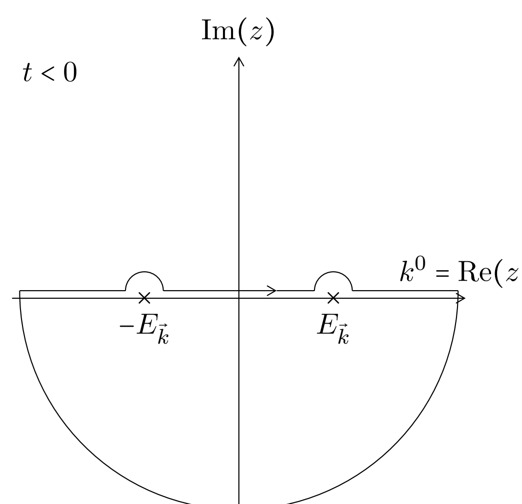
$$\int dk^0 \frac{e^{ik^0 t}}{(k^0 - E_{\vec{k}})(k^0 + E_{\vec{k}})}\,.$$

This integral can be thought of as an integral over a complex variable z along a contour on the real axis, $\mathrm{Re}(z) = k^0$, from $-\infty$ to ∞. Then the points $z = \pm E_{\vec{k}}$ are locations of simple poles of the integrand, and we can define the integral by deforming the contour to go either above or below these poles. For example we can take the following contour:

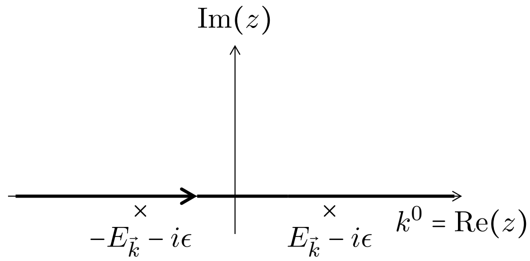


Regardless of which deformation of the contour we choose, the contour can be closed with a semicircle of infinite radius centered at the origin on the upper half-plane if $t > 0$ and in the lower half-plane if $t < 0$:





because he integral along the big semicircle vanishes as the radius of the circle is taken infinitely large. For the choice of contour in this figure no poles are enclosed for $t > 0$ so the integral vanishes. This defined the *advanced* Green's function, $G_{\text{adv}}(x) = 0$ for $t > 0$. Alternatively we can "displace" the poles by an infinitesimal amount $-i\epsilon$, with $\epsilon > 0$, so they lie just below the real axis,



Then we have

$$G_{\text{adv}}(x) = -\int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x} \frac{1}{(k^0 + i\epsilon)^2 - \vec{k}^{\,2} - m^2}$$

Similarly, the *retarded* Green's function is

$$G_{\text{ret}}(x) = -\int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x} \frac{1}{(k^0 - i\epsilon)^2 - \vec{k}^{\,2} - m^2}$$

with the contour bellow the two poles. It has $G_{\text{ret}}(x) = 0$ for $t < 0$. For later use it is convenient to write

$$\begin{aligned} G_{\text{ret}}(x) &= -\theta(x^0) \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot\vec{x}} \left[2\pi i \frac{1}{2E_{\vec{k}}} \left(e^{iE_{\vec{k}}t} - e^{-iE_{\vec{k}}t} \right) \right] \\ &= -i\theta(x^0) \int (dk)\,(e^{ik\cdot x} - e^{-ik\cdot x}) \end{aligned} \tag{4.4}$$

$$\begin{aligned} G_{\text{adv}}(x) &= \theta(-x^0) \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot\vec{x}} \left[2\pi i \frac{1}{2E_{\vec{k}}} \left(e^{iE_{\vec{k}}t} - e^{-iE_{\vec{k}}t} \right) \right] \\ &= i\theta(-x^0) \int (dk)\,(e^{ik\cdot x} - e^{-ik\cdot x}) \end{aligned} \tag{4.5}$$

We also note that one may choose a contour that goes above one pole and below the other. For example, going below $-E_{\vec{k}}$ and above $+E_{\vec{k}}$ we have

$$\begin{aligned} G_F(x) &= -\int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x} \frac{1}{(k^0 - E_{\vec{k}} + i\epsilon)(k^0 + E_{\vec{k}} - i\epsilon)} \\ &= -\int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x} \frac{1}{(k^2 - m^2 + i\epsilon)} \\ &= i\left[\theta(x^0) \int (dk)\, e^{-ik\cdot x} + \theta(-x^0) \int (dk)\, e^{ik\cdot x} \right] \end{aligned} \tag{4.6}$$

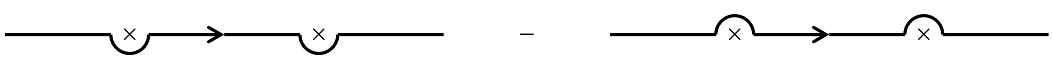
In the solution (4.2), for $x^0 < -T$ the integral over y^0 only has contributions from $x^0 - y^0 < 0$, and for $x^0 > T$ it has contributions only from $x^0 - y^0 > 0$. So we have

$$\begin{aligned}\phi(x) &= \phi_{\text{in}}(x) + \int d^4y\, G_{\text{ret}}(x-y)J(y) \\ &= \phi_{\text{out}}(x) + \int d^4y\, G_{\text{adv}}(x-y)J(y)\end{aligned}$$

Hence we can write

$$\begin{aligned}\phi_{\text{out}}(x) &= \phi_{\text{in}}(x) + \int d^4y \left[G_{\text{ret}}(x-y) - G_{\text{adv}}(x-y)\right]J(y) \\ &= \phi_{\text{in}}(x) + \int d^4y\, G^{(-)}(x-y)J(y) \qquad (4.7)\end{aligned}$$

$G^{(-)}$ can be obtained from the difference of (4.4) and (4.5). But more directly we note that



is the same as

The straight segments cancel and we are left with

This gives

$$\begin{aligned}G^{(-)}(x) &= -\int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot\vec{x}} \left[2\pi i \frac{1}{2E_{\vec{k}}}\left(e^{iE_{\vec{k}}t} - e^{-iE_{\vec{k}}t}\right)\right] \\ &= -i\int (dk)\left(e^{iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}} - e^{-iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}}\right) \\ &= -i\int \frac{d^4k}{(2\pi)^3}\theta(k^0)\delta(k^2-m^2)\left(e^{ik\cdot x} - e^{-ik\cdot x}\right) \\ &= -i\int \frac{d^4k}{(2\pi)^3}\varepsilon(k^0)\delta(k^2-m^2)e^{ik\cdot x}\end{aligned}$$

where

$$\varepsilon(k^0) = \begin{cases} +1 & k^0 > 0 \\ -1 & k^0 < 0 \end{cases}$$

You may have noticed that this looks a lot like the function $\Delta_+(x) = [\phi^{(+)}(x), \phi^{(-)}(0)]$. In fact, computing the commutator of free fields at arbitrary times,

$$\begin{aligned}[\phi(x), \phi(y)] &= \int (dk)(dk')[\alpha_{\vec{k}} e^{-ik\cdot x} + \alpha^\dagger_{\vec{k}} e^{ik\cdot x}, \alpha_{\vec{k}'} e^{-ik'\cdot y} + \alpha^\dagger_{\vec{k}'} e^{ik'\cdot y}] \\ &= \int (dk)\big[e^{ik\cdot(y-x)} - e^{-ik\cdot(y-x)}\big] \\ &= iG^{(-)}(y-x) = -iG^{(-)}(x-y)\end{aligned}$$

4.2 Quantum Fields

We now consider equation (4.1) for the KG field with a source in the case that the field is an operator on the Hilbert space $\mathcal{F}$. The source $J(x)$ is a a c-number, a *classical source*, and still take it to be localized in space-time. Suppose the system is in some state initially, well before the source is turned on. Let's take the vacuum state for definiteness, although any other state would be just as good. As the system evolves, the source is turned on and then off, and we end up with the system in some final state. Now, in the classical case if we start from nothing and turn the source on and then off radiation is produced, emitted out from the region of the localized source. In the quantum system we therefore expect we will get a final state with any number of particles emitted from the localized source region.

Then $\phi(x) = \phi_{\text{in}}(x)$ is the statement that for $t < -T$ ϕ is the same operator as a solution to the free KG equation. Likewise for $\phi = \phi_{\text{out}}$ for $t > T$. Does this mean $\phi_{\text{out}} = \phi_{\text{in}}$? Surely not, it is not true in the classical case. Since ϕ satisfies the equal time commutation relations, $i[\partial_t\phi(x), \phi(y)] = \delta^{(3)}(\vec{x} - \vec{y})$, etc, so do ϕ_{out} and ϕ_{in}. Since the latter solve the free KG equation, they have expansions in creation and annihilation operators, and the same Hilbert space. More precisely, we can construct a Fock space $\mathcal{F}_{\text{in}}$ out of $\alpha^\dagger_{\text{in}}$, and a space $\mathcal{F}_{\text{out}}$ out of $\alpha^\dagger_{\text{out}}$. These spaces are just Hilbert spaces of the free KG equation, and therefore they are isomorphic, $\mathcal{F}_{\text{in}} \approx \mathcal{F}_{\text{out}} \approx \mathcal{F}_{\text{KG}}$. So there is some linear, invertible operator $S : \mathcal{F}_{\text{KG}} \to \mathcal{F}_{\text{KG}}$, so that $|\psi\rangle_{\text{out}} = S^{-1}|\psi\rangle_{\text{in}}$. Since the states are normalized, S must preserve normalization, so S is unitary, $S^\dagger S = SS^\dagger = 1$. The operator S is called the *S-matrix*.

Let's understand the meaning of $|\psi\rangle_{\text{out}} = S^\dagger|\psi\rangle_{\text{in}}$. The state of the system initially (far past) is $|\psi\rangle_{\text{in}}$. It evolves into a state $|\psi\rangle_{\text{out}} = S^\dagger|\psi\rangle_{\text{in}}$ at late times, well after the source is turned off. We can expand it in a basis of the Fock space, the states $|\vec{k}_1, \ldots, \vec{k}_n\rangle$ for $n = 0, 1, 2, \ldots$ In particular, if the initial state is the vacuum, then the expansion of $S^\dagger|0\rangle$ in the Fock space basis tells us the probability amplitude for emitting any number of particles. More generally

$${}_{\text{out}}\langle\chi|\psi\rangle_{\text{in}} = {}_{\text{in}}\langle\chi|S|\psi\rangle_{\text{in}}$$

is the probability amplitude for starting in the state $|\psi\rangle$ and ending in the state $|\chi\rangle$. Note that the left hand side is not to be taken literally as an inner product in the same Hilbert space of free particles (else it would vanish except when the initial and final states are physically identical, *i.e.*, when nothing happens).

Now, $S^\dagger \phi_{\text{in}}(x) S$ is an operator acting on out states that satisfies the KG equation. Similarly, if $\pi_{\text{in}}(x) = \partial_t \phi_{\text{in}}(x)$, $S^\dagger \pi_{\text{in}}(x) S$ acts on out states. Moreover, the commutator $[S^\dagger \phi_{\text{in}}(x) S, S^\dagger \pi_{\text{in}}(y) S = S^\dagger [\phi_{\text{in}}(x), \pi_{\text{in}}(y)] S = [\phi_{\text{in}}(x), \pi_{\text{in}}(y)]$ since the commutator is a c-number and $S^\dagger S = 1$, and similarly for the other commutators of $S^\dagger \phi_{\text{in}}(x) S$ and $S^\dagger \pi_{\text{in}}(x) S$. This means that up to a canonical transformation,

$$\phi_{\text{out}}(x) = S^\dagger \phi_{\text{in}}(x) S\,.$$

Since we are free to choose the out fields in the class of canonical equivalent fields, we take the above relation to be our choice.

There is a simple way to determine S. We will present this now, but the method works only for the case of an external source and cannot be generalized to the case of interacting fields. So after presenting this method we will present a more powerful technique that can be generalized. From Eq. (4.7) we have

$$\begin{aligned} S^\dagger \phi_{\text{in}}(x) S &= \phi_{\text{in}}(x) + \int d^4y\, G^{(-)}(x-y) J(y) \\ &= \phi_{\text{in}}(x) + i \int d^4y\, [\phi_{\text{in}}(x), \phi_{\text{in}}(y) J(y)]\,. \end{aligned}$$

Recall

$$\begin{aligned} e^A B e^{-A} &= B + [A,B] + \tfrac{1}{2}[A,[A,B]] + \cdots \\ &= B + [A,B] \qquad \text{if } [A,[A,B]] = 0. \end{aligned}$$

It follows that

$$\boxed{S = \exp\left[i \int d^4y\, \phi_{\text{in}}(y) J(y)\right]} \tag{4.8}$$

It is convenient to normal-order S. Since $[\phi^{(+)}(x), \phi^{(-)}(y)]$ is a c-number we can use

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

which is valid provided $[A,B]$ commutes with both A and B. So using $i\int d^4x\, \phi^{(-)}(x) J(x)$ and $i\int d^4x\, \phi^{(+)}(x) J(x)$ for A and B,

$$S = e^{i\int d^4x\, \phi^{(-)}(x) J(x)} e^{i\int d^4x\, \phi^{(+)}(x) J(x)} e^{\frac{1}{2}\int d^4x\, d^4y\, [\phi^{(-)}(x), \phi^{(+)}(y)] J(x) J(y)}$$

From (2.10),

$$\begin{aligned}\Delta_+(x_2 - x_1) &\equiv [\hat{\phi}^{(+)}(x_1), \hat{\phi}^{(-)}(x_2)] \\ &= \int (dk)\, e^{-ik\cdot(x_2-x_1)} \\ &= \int \frac{d^4k}{(2\pi)^4}\, \theta(k^0)\delta(k^2 - m^2) e^{-ik\cdot(x_2-x_1)}\end{aligned}$$

and introducing the Fourier transform of the source,

$$J(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x} \widetilde{J}(k)$$

we have

$$\begin{aligned}&\int d^4x\, d^4y\, [\phi_{\rm in}^{(-)}(x), \phi_{\rm in}^{(+)}(y)] J(x) J(y) \\ &= -\int \prod_{i=1}^{3} \frac{d^4k_i}{(2\pi)^4} \theta(k_1^0)\delta(k_1^2 - m^2) \widetilde{J}(k_2)\widetilde{J}(k_3) \int d^4x \int d^4y\, e^{ik_2\cdot x} e^{ik_3\cdot y} e^{ik_1\cdot(y-x)} \\ &= -\int \frac{d^4k}{(2\pi)^4}\, \theta(k^0)\delta(k^2 - m^2) \widetilde{J}(k)\widetilde{J}(-k) = -\int (dk) |\widetilde{J}(k)|^2\,,\end{aligned}$$

where we have used $J^*(x) = J(x) \Rightarrow \widetilde{J}(-k) = \widetilde{J}^*(k)$, and it is understood that $k^0 = E_{\vec{k}}$. Hence,

$$\boxed{S = e^{i\int d^4x\, \phi^{(-)}(x) J(x)} e^{i\int d^4x\, \phi^{(+)}(x) J(x)} e^{-\frac{1}{2}\int (dk) |\widetilde{J}(k)|^2}\,.}$$

As an example of an application, we can compute the probability of finding particles in the final state if we start form no particles initially (emission probability). Start from probability of persistence of the vacuum,

$$|_{\rm out}\langle 0|0\rangle_{\rm in}|^2 = |_{\rm in}\langle 0|S|0\rangle_{\rm in}|^2 = \exp\left(-\int (dk) |\widetilde{J}(k)|^2\right)\,.$$

Next compute the probability that one particle is produced with momentum $\vec{k}$:

$$|_{\rm out}\langle \vec{k}|0\rangle_{\rm in}|^2 = |_{\rm in}\langle \vec{k}|S|0\rangle_{\rm in}|^2 = |_{\rm in}\langle 0|\alpha_{\vec{k}} S|0\rangle_{\rm in}|^2\,.$$

where "in" in $\alpha_{\vec{k}}$ is implicit. To proceed we use

$$[\alpha_{\vec{k}}, \phi^{(-)}(x)] = \int (dk') [\alpha_{\vec{k}}, \alpha^\dagger_{\vec{k}'}] e^{ik'\cdot x} = e^{ik\cdot x} \quad (\text{with } k^0 = E_{\vec{k}})$$

so that

$$e^{-i\int \phi^{(-)} J} \alpha_{\vec{k}} e^{i\int \phi^{(-)} J} = \alpha_{\vec{k}} - i \int [\phi^{(-)}, \alpha_{\vec{k}}] J = \alpha_{\vec{k}} + i \int d^4x e^{ik\cdot x} J(x) = \alpha_{\vec{k}} + i\widetilde{J}(-k)\,.$$

Hence

$$|_{\rm out}\langle \vec{k}|0\rangle_{\rm in}|^2 = |\widetilde{J}(k)|^2 \exp\left(-\int (dk) |\widetilde{J}(k)|^2\right)\,.$$

4.2.1 Phase Space

Suppose we want to find the probability of finding a single particle as $t \to \infty$ regardless of its momentum, that is, in any state. We need to sum over all final states consisting of a single particle, once each. Since we have a continuum of states we have to integrate over all $\vec{k}$ with some measure, $\int d\mu(\vec{k})$. Clearly $d\mu(\vec{k})$ must involve d^3k, possibly weighted by a function $f(\vec{k})$. Presumably this function is rotationally invariant, $f = f(|\vec{k}|)$. But we suspect also $d\mu(\vec{k})$ is Lorentz invariant, so $d\mu(\vec{k}) \propto (dk)$, that is, it equals the invariant measure up to a constant. Let's figure this out by counting. Note that how we normalize states matters.

First we determine this via a shortcut, and later we repeat the calculation via a more physical approach (and obtain, of course the same result). We want

$$\sum_n |{}_{\rm out}\langle n|0\rangle_{\rm in}|^2 = \sum_n {}_{\rm in}\langle 0|n\rangle_{\rm out}\,{}_{\rm out}\langle n|0\rangle_{\rm in} = {}_{\rm out}\langle 0|\left(\sum_n |n\rangle_{\rm out}\,{}_{\rm out}\langle n|_{\rm out}\right)|0\rangle_{\rm in}\,,$$

where the sum is restricted over some states. The operator

$$\sum_n |n\rangle_{\rm out}\,{}_{\rm out}\langle n|$$

is a projection operator onto the space of "some states," the ones we want to sum in the final state. We have already discussed this projection operator for the case of one particle states when $|\vec{k}\rangle$ is relativistically normalized. It is

$$\int (dk)|\vec{k}\rangle\langle\vec{k}|\,.$$

So we have

$$\text{1-particle emission probability } = \int (dk)|\widetilde{J}(k)|^2 \exp\left(-\int (dk)|\widetilde{J}(k)|^2\right)\,.$$

Now we repeat the calculation by counting states. It is easier to count discrete sets of states. We can do so by placing the system in a box of volume $L_xL_yL_z$. Take, say, periodic boundary conditions. Then instead of $\int (dk)(\alpha_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \text{h.c.})$, we have a Fourier sum, $\sum_{\vec{k}}(a_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \text{h.c.})$. Here $a_{\vec{k}}$ are creation operators with some normalization we will have to sort out. For periodic boundary conditions we must have $k_xL_x = 2\pi n_x$, $k_yL_y = 2\pi n_y$, $k_zL_z = 2\pi n_z$, with n_i integers. We label the one particle states by these, $|\vec{n}\rangle = a^\dagger_{\vec{k}}|0\rangle$, where $\vec{k} = 2\pi\left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z}\right)$. The probability of finding $|\psi\rangle$ in state $|\vec{n}\rangle$ is $|\langle\vec{n}|\psi\rangle|^2$ if both $|\psi\rangle$ and $|\vec{n}\rangle$ are normalized to unity. (Note that this means $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{n_xn'_x}\delta_{n_yn'_y}\delta_{n_zn'_z}$). The probability of finding $|\psi\rangle$ in a 1-particle state is then $\sum_{\vec{n}}|\langle\vec{n}|\psi\rangle|^2$.

Let's be more specific. What is the probability of finding $|\psi\rangle$ in a 1-particle state with momentum in a box $(k_x, k_x + \Delta k_x) \times (k_y, k_y + \Delta k_y) \times (k_z, k_z + \Delta k_z)$?

There are $\Delta n_x \Delta n_y \Delta n_z = \frac{L_x L_y L_z}{(2\pi)^3} \Delta k_x \Delta k_y \Delta k_z$ state in the box. For large volume the spacing between values of $\vec{k}$ becomes small, so in the limit of large volume $\Delta k_x \to dk_x$, etc, and the number of particles in the momentum box is

$$\frac{L_x L_y L_z}{(2\pi)^3} \Delta k_x \Delta k_y \Delta k_z \to V \frac{d^3k}{(2\pi)^3}$$

where $V = L_x L_y L_z$ is the volume fo the box. As we switch from discrete to continuum labels for our states we must be careful with their normalization condition. Since on the space of 1-particle states we have

$$\sum_{\vec{n}} |\vec{n}\rangle\langle\vec{n}| = \mathbb{1} \quad \Rightarrow \quad \sum_{\vec{n}} |\vec{n}\rangle\langle\vec{n}|\vec{n}\,'\rangle = |\vec{n}\,'\rangle$$

then in the limit, denoting by $|\vec{k}\rangle$ the continuum normalized states,

$$\sum_{\vec{n}} \frac{V\Delta^3 k}{(2\pi)^3} |\vec{n}\rangle\langle\vec{n}|\vec{n}\,'\rangle = |\vec{n}\,'\rangle \to \int d^3k |\vec{k}\rangle\langle\vec{k}|\vec{k}\,'\rangle = |\vec{k}\,'\rangle$$
$$\Rightarrow \quad \frac{V}{(2\pi)^3} \delta_{n_x n'_x} \delta_{n_y n'_y} \delta_{n_z n'_z} \to \delta^{(3)}(\vec{k} - \vec{k}\,') \, .$$

Putting these elements together, the probability of finding $|\psi\rangle$ in a 1-particle state is

$$\sum_{\vec{n}} |\langle\vec{n}|\psi\rangle|^2 \to \int d^3k |\langle\vec{k}|\psi\rangle|^2 \, .$$

Finally, if we want to change the normalization of states so that $\langle\vec{k}\,'|\vec{k}\rangle = N_{\vec{k}} \delta^{(3)}(\vec{k} - \vec{k}\,')$ then the probability of finding $|\psi\rangle$ in a 1-particle state is

$$\int \frac{d^3k}{N_{\vec{k}}} |\langle\vec{k}|\psi\rangle|^2 \, .$$

For relativistic normalization of states, $N_{\vec{k}} = (2\pi)^3 2E_{\vec{k}}$ and the probability is

$$\int (dk) |\langle\vec{k}|\psi\rangle|^2 \, .$$

This is precisely our earlier result. Denoting the emission probability for n particles in the presence of a localized source $J(x)$ by p_n, we have

$$p_0 = e^{-\xi}, \;\; p_1 = \xi e^{-xi}, \quad \text{where } \xi = \int (dk) |\widetilde{J}(\vec{k})|^2 \, . \tag{4.9}$$

4.2.2 Poisson

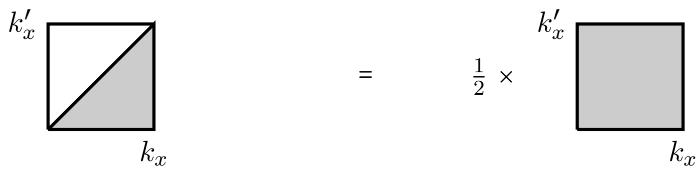
We can now go on to compute p_n for arbitrary n. Start with $n = 2$, that is, emission of two particles.

$$\begin{aligned} e^{-i\int \phi^{(-)}J}\alpha_{\vec{k}}\alpha_{\vec{k}'}e^{i\int \phi^{(-)}J} &= e^{-i\int \phi^{(-)}J}\alpha_{\vec{k}}e^{i\int \phi^{(-)}J}e^{-i\int \phi^{(-)}J}\alpha_{\vec{k}'}e^{i\int \phi^{(-)}J} \\ &= (\alpha_{\vec{k}} + i\widetilde{J}(-k))(\alpha_{\vec{k}'} + i\widetilde{J}(-k')) \end{aligned} \tag{4.10}$$

so that

$$_{\text{out}}\langle \vec{k}\,\vec{k}'|0\rangle_{\text{in}} = -\widetilde{J}(-k)\widetilde{J}(-k')e^{-\frac{1}{2}\xi}\,.$$

To get the emission probability we must sum over all distinguishable 2-particle states. Since $|\vec{k}\,\vec{k}'\rangle = |\vec{k}'\vec{k}\rangle$ we must not double count. When we integrate over a box in momentum space, we count twice the state $|\vec{k}\,\vec{k}'\rangle$ if we sum over k_1 and k_2 with values $k_1 = k$, $k_2 = k'$ and $k_1 = k'$, $k_2 = k$:



Hence

$$p_2 = \tfrac{1}{2}\xi^2 e^{-\xi}\,.$$

The generalization is straightforward:

$$p_n = \frac{1}{n!}\xi^n \exp(-\xi)\,.$$

This is a Poisson distribution! Note that $\sum_n p_n = 1$, that is, there is certainty of finding anything (that is, either no or some particles). The mean of the distribution is $\xi = \int (dk)|\widetilde{J}(\vec{k})|^2$.

4.3 Evolution Operator

We now introduce a more general formalism to derive the same results, but that will be more useful when we consider interacting quantum fields. We want to construct an operator $U(t)$ that gives the connection between the field ϕ and the "in" field ϕ_{in}:

$$\phi(\vec{x},t) = U^{-1}(t)\phi_{\text{in}}(\vec{x},t)U(t)\,. \tag{4.11}$$

Since we are assuming $\phi \to \phi_{\text{in}}$ as $t \to -\infty$, we must have

$$U(t) \to 1 \quad \text{as} \quad t \to -\infty\,. \tag{4.12}$$

The S matrix is then

$$S = \lim_{t\to\infty} U(t)\,.$$

The time evolution of ϕ and $\phi_{\rm in}$ are given by

$$\frac{\partial\phi}{\partial t}(\vec{x},t) = i[H(t),\phi(\vec{x},t)] \qquad \frac{\partial\phi_{\rm in}}{\partial t}(\vec{x},t) = i[H_{0\rm in}(t),\phi_{\rm in}(\vec{x},t)] \tag{4.13}$$

Here $H = H_0 + H'$, where H_0 is the *free Hamiltonian* and H' describes interactions. In the present case,

$$H_0 = \int d^3x\,\left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2\right] \qquad H'(t) = \int d^3x\,J(\vec{x},t)\phi(\vec{x},t)\,.$$

Also the subscript "in" means the argument has "in" fields. So $H = H(\phi,\pi,J)$ with $H_0 = H_0(\phi,\pi)$ while $H_{0\rm in} = H_{0\rm in}(\phi_{\rm in},\pi_{\rm in})$. From Eq. (4.11) we have

$$\begin{aligned} &U^{-1}(t)H(\phi(t),\pi(t),J(t))U(t)\\ &\qquad = H(U^{-1}(t)\phi(t)U(t),U^{-1}(t)\pi(t)U(t),U^{-1}(t)J(t)U(t))\\ &\qquad = H(\phi_{\rm in}(t),\pi_{\rm in}(t),J(t)) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \frac{\partial\phi_{\rm in}}{\partial t}(\vec{x},t) &= \frac{\partial}{\partial t}\left[U\phi(\vec{x},t)U^{-1}(t)\right]\\ &= \frac{dU}{dt}\phi U^{-1} + U\phi\frac{dU^{-1}}{dt} + Ui[H(t),\phi(\vec{x},t)]U^{-1}\\ &= \frac{dU}{dt}U^{-1}\phi_{\rm in} - \phi_{\rm in}\frac{dU}{dt}U^{-1} + i[H_{\rm in}(t),\phi_{\rm in}(\vec{x},t)]\\ &= i[-i\frac{dU}{dt}U^{-1} + H_{\rm in}(t),\phi_{\rm in}(\vec{x},t)] \end{aligned}$$

Comparing with Eq. (4.13) this is the commutator with $H_{0\rm in}$ so we must have

$$-i\frac{dU}{dt}U^{-1} + H_{\rm in}(t) = H_{0\rm in}(t)$$

or

$$\frac{dU}{dt} = -i(H_{\rm in} - H_{0\rm in})U = -iH'_{\rm in}U \tag{4.15}$$

The solution to this equation with the boundary condition (4.12) gives the S matrix, $S = U(\infty)$. Note that the equation contains only "in" fields, which we know how to handle.

We can solve (4.15) by iteration. Integrating (4.15) form $-\infty$ to t we have

$$U(t) - 1 = -i\int_{-\infty}^{t} dt'\,H'_{\rm in}(t')U(t')\,.$$

Now use this again, repeatedly,

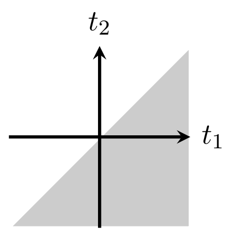
$$
\begin{aligned}
U(t) &= 1 - i\int_{-\infty}^{t} dt'\, H'_{\rm in}(t')\left[1 - i\int_{-\infty}^{t'} dt''\, H'_{\rm in}(t'')U(t'')\right]\\
&= 1 - i\int_{-\infty}^{t} dt_1\, H'_{\rm in}(t_1) + (-i)^2\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2 H'_{\rm in}(t_1)H'_{\rm in}(t_2)U(t_2)\\
&= \cdots = 1 + \sum_{n=1}^{\infty}(-i)^n\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2\cdots\int_{-\infty}^{t_{n-1}} dt_n\, H'_{\rm in}(t_1)H'_{\rm in}(t_2)\cdots H'_{\rm in}(t_n)\,.
\end{aligned}
$$

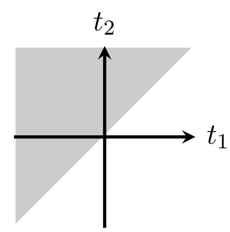
Note that the product $H'_{\rm in}(t_1)H'_{\rm in}(t_2)\cdots H'_{\rm in}(t_n)$ is *time-ordered*, that is, the Hamiltonians appear ordered by $t \geq t_1 \geq t_2 \cdots \geq t_n$. This is the main result of this section.

We can write the result more compactly. Note that

$$
\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2 H'_{\rm in}(t_1)H'_{\rm in}(t_2) = \int_{-\infty}^{t} dt_2\int_{-\infty}^{t_2} dt_1 H'_{\rm in}(t_2)H'_{\rm in}(t_1)
$$

The two integrals cover the whole t_1 *vs* t_2 plane, as can be seen from the followin figures in which the shaded regions correspond to the region of integration of teh first and second integrals, respectively:





For any two time dependent operators, $A(t)$ and $B(t)$, we define the *time-ordered product*

$$
T(A(t_1)B(t_2)) = \theta(t_1-t_2)A(t_1)B(t_2)+\theta(t_2-t_1)B(t_2)A(t_1) = \begin{cases} A(t_1)B(t_2) & t_1 > t_2\\ B(t_2)A(t_1) & t_2 > t_1\end{cases}
$$

and similarly when there are more than two operators in the product. Then

$$
\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2 H'_{\rm in}(t_1)H'_{\rm in}(t_2) = \frac{1}{2}\int_{-\infty}^{t} dt_1\int_{-\infty}^{t} dt_2 T\left(H'_{\rm in}(t_1)H'_{\rm in}(t_2)\right)\,.
$$

For the term with n integrals there are $n!$ orderings of $t_1,\dots,t_n$ so we obtain

$$
U(t) = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{t} dt\int_{-\infty}^{t} dt_2\cdots\int_{-\infty}^{t} dt_n\, T(H'_{\rm in}(t_1)H'_{\rm in}(t_2)\cdots H'_{\rm in}(t_n))\,. \quad (4.16)
$$

or, comparing with $e^z = \sum_{n=0}^{\infty}\frac{1}{n!}z^n$ we write

$$
U(t) = T\left[\exp\left(-i\int_{-\infty}^{t} dt'\, H'_{\rm in}(t')\right)\right]\,.
$$

You should keep in mind that the meaning of this expression is the explicit expansion in (4.16). Finally, taking $t \to \infty$,

$$S = T\left[\exp\left(-i\int_{-\infty}^{\infty} dt\, H'_{\text{in}}(t)\right)\right] = T\left[\exp\left(-i\int d^4x\, \mathcal{H}'_{\text{in}}\right)\right]$$

or, using $\mathcal{L}' = -\mathcal{H}'$,

$$\boxed{S = T\left[\exp\left(i\int d^4x\, \mathcal{L}'_{\text{in}}\right)\right].}$$

Let's use this general result for the specific example we have been discussing, and compare with previous results. Use $\mathcal{L}'_{\text{in}} = J(x)\phi_{\text{in}}(x)$. Then

$$S = T\left[\exp\left(i\int d^4x\, J(x)\phi_{\text{in}}(x)\right)\right].$$

But we had obtained

$$S = e^{-\frac{1}{2}\xi} :\!\exp\left(i\int d^4x\, J(x)\phi_{\text{in}}(x)\right)\!: \tag{4.17}$$

with $\xi = \frac{1}{2}\int (dk)|\widetilde{J}(k)|^2$, as in (4.9). Are these two expression for S the same? To show they are we need some additional machinery, some relation between T-ordered and normal-ordered products.

4.4 Wick's Theorem

In order to answer the question, what is the difference between T-ordered and normal-ordered products, we consider

$$T(\phi(x_1)\phi(x_2)) - :\!\phi(x_1)\phi(x_2)\!:$$

Here and below ϕ stands for an "in" field, that is, a free field satisfying the KG equation. For notational conciseness we write ϕ_i for $\phi(x_1$, etc. Letting $\phi = \phi^{(+)} + \phi^{(-)}$ and taking $x_1^0 > x_2^0$ the difference is

$$(\phi_1^{(+)} + \phi_1^{(-)})(\phi_2^{(+)} + \phi_2^{(-)}) - (\phi_1^{(+)}\phi_2^{(+)} + \phi_1^{(-)}\phi_2^{(+)} + \phi_2^{(-)}\phi_1^{(+)} + \phi_1^{(-)} + \phi_2^{(-)}) = [\phi_1^{(+)}, \phi_2^{(-)}]$$

This is a c-number (equals $\Delta_+(x_2 - x_1)$). Taling the expectation value in the vacuum we obtain, restoring full notation momentarily,

$$T(\phi(x_1)\phi(x_2)) = :\!\phi(x_1)\phi(x_2)\!: + \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle\,.$$

Clearly this holds for the more general case of several fields,

$$T(\phi_n(x_1)\phi_m(x_2)) =: :\!\phi_n(x_1)\phi_m(x_2)\!: + \langle 0|T(\phi_n(x_1)\phi_m(x_2))|0\rangle\,.$$

Consider next the case of three fields, $T(\phi_1\phi_2\phi_3)$. Without loss of generality take $x_1^0 \geq x_2^0 \geq x_3^0$:

$$T(\phi_1\phi_2\phi_3) = \phi_1\phi_2\phi_3 = \phi_1(:\!\phi_2\phi_3\!: + \langle 0|T(\phi_2\phi_3)|0\rangle)$$

Now, $\phi_1 :\!\phi_2\phi_3\!: = (\phi_1^{(+)} + \phi_1^{(-)}) :\!\phi_2\phi_3\!:$. We need to move $\phi_1^{(+)}$ to the right of all of the $\phi^{(-)}$ operators:

$$\phi_1^{(+)} :\!\phi_2\phi_3\!: = :\!\phi_2\phi_3\!: \phi_1^{(+)} + [\phi_1^{(+)}, \phi_2^{(-)}] :\!\phi_3\!: + [\phi_1^{(+)}, \phi_3^{(-)}] :\!\phi_2\!:$$

where we have used the fact that $[\phi_1^{(+)}, \phi_n^{(-)}]$ is a c-number. In fact, it equals $\langle 0|T(\phi_1\phi_n)|0\rangle$. So we have

$$T(\phi_1\phi_2\phi_3) = :\!\phi_1\phi_2\phi_3\!: + :\!\phi_1\!:\langle 0|T(\phi_2\phi_3)|0\rangle + :\!\phi_2\!:\langle 0|T(\phi_1\phi_3)|0\rangle + :\!\phi_3\!:\langle 0|T(\phi_1\phi_2)|0\rangle\,.$$

Of course, $:\!\phi\!: = \phi$, but the notation will more easily generalize below.

More notation, rewrite the above as

$$T(\phi_1\phi_2\phi_3) = :\!\phi_1\phi_2\phi_3\!: + :\!\phi_1\overbracket{\phi_2\phi_3}\!: + :\!\overbracket{\phi_1\phi_2}\phi_3\!: + :\!\overbracket{\phi_1\phi_2\phi_3}\!:\,.$$

where

$$\overbracket{\phi_n\phi_m} = \langle 0|T(\phi_n\phi_m)|0\rangle$$

is called a *contraction*.

Wick's theorem states that

$$\begin{aligned} T(\phi_1\cdots\phi_n) = {} & :\!\phi_1\cdots\phi_n\!: + \sum_{\substack{\text{pairs}\\(i,j)}} :\!\phi_1\cdots\overbracket{\phi_i\cdots\phi_j}\cdots\phi_n\!: \\ & + \sum_{\substack{\text{2-pairs}\\(i,j),(k,l)}} :\!\phi_1\cdots\overbracket{\phi_i\cdots\phi_j}\cdots\overbracket{\phi_k\cdots\phi_l}\cdots\phi_n\!: + \cdots \\ & + \begin{cases} :\!\overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4}\cdots\overbracket{\phi_{n-1}\phi_n}\!: + \text{all pairings} & n = \text{even} \\ :\!\overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4}\cdots\overbracket{\phi_{n-2}\phi_{n-1}}\phi_n\!: + \text{all pairings} & n = \text{odd} \end{cases} \end{aligned} \tag{4.18}$$

In words, the right hand side is the sum over all possible contractions in the normal ordered product $:\!\phi_1\cdots\phi_n\!:$ (including the term with no contractions). The proof is by induction. We have already demonstrated this for $n = 2, 3$. Let $W(\phi_1, \ldots, \phi_n)$ stand for the right hand side of (4.18), and assume the theorem is valid for $n-1$ fields. Then assuming $x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$,

$$\begin{aligned} T(\phi_1\cdots\phi_n) &= \phi_1 T(\phi_1\cdots\phi_n) \\ &= \phi_1 W(\phi_2, \ldots, \phi_n) \\ &= (\phi_1^{(+)} + \phi_1^{(-)}) W(\phi_2, \ldots, \phi_n) \\ &= \phi_1^{(-)} W(\phi_2, \ldots, \phi_n) + W(\phi_2, \ldots, \phi_n)\phi_1^{(+)} + [\phi_1^{(+)}, W(\phi_2, \ldots, \phi_n)] \end{aligned}$$

The first two terms are normal ordered and contain all contractions that do not involve ϕ_1. The last term involves the contractions of ϕ_1 with every field in every term in $W(\phi_2, \ldots, \phi_n)$, therefore all possible contractions. Hence the right hand side contains all possible contractions, which is $W(\phi_1, \ldots, \phi_n)$.

4.4.1 Combinatorics

Let's go back to our computation of S. We'd like to use Wick's theorem to relate $T[\exp(i\int d^4x\, \phi_{\text{in}}(x)J(x))]$ to $:\exp(i\int d^4x\, \phi_{\text{in}}(x)J(x)):$, and for this we need a little combinatorics. To make the notation more compact we will continue dropping the "in" label on the "in" fields for the rest of this section. Now, $T[\exp(i\int d^4x\, \phi_{\text{in}}(x)J(x))]$ is a sum of terms of the form

$$\frac{i^n}{n!}\int\prod_{i=1}^{n} d^4x_i\; J_1\cdots J_n T(\phi_1\cdots\phi_n) = \frac{i^n}{n!}\int\prod_{i=1}^{n} d^4x_i\; J_1\cdots J_n W(\phi_1, \ldots, \phi_n) \tag{4.19}$$

Consider the term on the right hand side with one contraction,

$$\begin{aligned}\frac{1}{n!}\int\prod_{i=1}^{n} d^4x_i\; J_1\cdots J_n \sum_{\substack{\text{pairs}\\(i,j)}} &:\phi_1\cdots\overbracket{\phi_i\cdots\phi_j}\cdots\phi_n: \\ &= \frac{N_{\text{pairs}}}{n!}\int\prod_{i=1}^{n-2} d^4x_i\; J_1\cdots J_{n-2} :\phi_1\cdots\phi_{n-2}: \int d^4x\, d^4x' JJ' \overbracket{\phi\phi'}\end{aligned}$$

where N_{pairs} is the number of contractions in the sum, which is the same as the number of pairs (i,j) in the list $1, \ldots, n$. That is, the number of ways of choosing two elements of a list of n objects:

$$N_{\text{pairs}} = \binom{n}{2} = \frac{n!}{2!(n-2)!}$$

Let

$$\zeta = \int d^4x\, d^4y J(x)J(y)\langle 0|T(\phi(x)\phi(y))|0\rangle\,. \tag{4.20}$$

Then

$$1 \text{ pairing terms} = \frac{-\zeta}{2}\frac{i^{n-2}}{(n-2)!}\int\prod_{i=1}^{n-2} d^4x_i\; J_1\cdots J_{n-2} :\phi_1\cdots\phi_{n-2}:\,.$$

We want to repeat this calculation for the rest of the terms on the right hand side of (4.19). For this we need to count the number of terms for a given number of contractions. Now, k contractions involve $2k$ fields. Now, there are $\binom{n}{2k}$ ways of picking $2k$ fields out of n, and we need to determine how many distinct contractions one can make among them. We can figure this out by inspecting a few simple cases. For $k = 1$

$$\overbrace{12} \qquad \to \text{ one contraction}$$

and for $k = 2$

$$\overbrace{12}\,\overbrace{34} \quad 1234 \quad 1234 \qquad \to 3 \text{ contractions.}$$

Instead of continuing in this explicit way, we analyze the $k = 3$ case using inductive reasoning:

$$\overbrace{12} \times (k=2) \quad \overbrace{13} \times (k=2) \quad \cdots \quad \overbrace{16} \times (k=2) \qquad \to 5 \times 3 \text{ contractions.}$$

By induction the arbitrary k case has $(2k-1)!!$ pairings:

$$\left[(2k-1)\text{-pairings:} \overbrace{1j}\right] \times \left[(k-1)\text{-case: } (2k-3)!!\right] = (2k-1)!!$$

So the terms with k contractions give

$$\underbrace{\frac{1}{n!}\frac{n!}{(n-2k)!(2k)!}(2k-1)!!}_{\frac{1}{(n-2k)!}\frac{1}{2^k k!}}\, i^{n-2k} \int \prod_{i=1}^{n-2k} d^4x_i J_1 \cdots J_{n-2k} :\phi_1 \cdots \phi_{n-2k}: \underbrace{\left(i^2 \int d^4x\, d^4x'\, JJ' \overbrace{\phi\phi'}\right)^k}_{(-1)^k \zeta^k}$$

$$= \frac{(-\zeta/2)^k}{k!} \frac{i^m}{m!} \int \prod_{i=1}^{m} d^4x_i\, J_1 \cdots J_m :\phi_1 \cdots \phi_m: \qquad \text{with } m = n - 2k.$$

Aha! We recognize this as a term in an exponential expansion. Considering all the terms in he expansion of $T[\exp(i \int d^4x\, \phi_{\text{in}}(x) J(x))]$, the coefficient of $\frac{i^m}{m!} \int \prod_{i=1}^m d^4x_i\, J_1 \cdots J_m :\phi_1 \cdots \phi_m:$ (fixed m) is $\sum_{k=0}^\infty \frac{1}{k!}(-\zeta/2)^k = \exp(-\frac{1}{2}\zeta)$, and is independent of m, so it factors out. We are left with

$$e^{-\zeta/2} \sum_{m=0}^{\infty} \frac{i^m}{m!} \int \prod_{i=1}^m d^4x_i\, J_1 \cdots J_m :\phi_1 \cdots \phi_m: = e^{-\zeta/2} :\exp\left(i \int d^4x\, J(x)\phi(x)\right):$$

That is

$$S = T\left[\exp\left(i \int d^4x\, \phi_{\text{in}}(x) J(x)\right)\right] = e^{-\zeta/2} :\exp\left(i \int d^4x\, J(x)\phi(x)\right):$$

This will equal our previous expression for S in (4.17) if $\zeta = \xi$. To answer this we need to know more about the T-ordered product in the definition of ζ in (4.20).

4.5 Scalar Field Propagator

Let

$$\Delta_F(x,y) \equiv \langle 0|T(\phi(x)\phi(y))|0\rangle$$

where $\phi(x)$ is a real, scalar, free field satisfying the Klein-Gordon equation. This is the quantity we need for the computation fo ζ, but it is also important for several other reasons, so we spend some time investigating it.

First of all, $i\Delta_F(x,y)$ is a Green's function for the Klein-Gordon equation; see (4.3). To verify this claim compute directly. Taking ∂_μ to be the derivative with respect to x^μ keeping y^μ fixed and using $(\partial^2+m^2)\phi(x)=0$ we have

$$\begin{aligned}(\partial^2+m^2)\Delta_F(x,y) &= (\partial^2+m^2)\left[\theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle+\theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle\right]\\ &= \frac{\partial^2}{\partial x^{02}}\theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle+\frac{\partial^2}{\partial x^{02}}\theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle\\ &\quad+2\frac{\partial}{\partial x^0}\theta(x^0-y^0)\langle 0|\frac{\partial\phi(x)}{\partial x^0}\phi(y)|0\rangle+2\frac{\partial}{\partial x^0}\theta(y^0-x^0)\langle 0|\phi(y)\frac{\partial\phi(x)}{\partial x^0}|0\rangle\end{aligned}$$

Now using $d\theta(x)/dx=\delta(x)$ and $\partial_0\phi(x)=\pi(x)$ the last line above is

$$2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle=-2i\delta^{(4)}(x-y)$$

The line above that gives

$$\frac{\partial}{\partial x^0}\delta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle=-\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle=i\delta^{(4)}(x-y)$$

Combining these we have

$$(\partial^2+m^2)\Delta_F(x,y)=-i\delta^{(4)}(x-y)$$

As we saw in Sec. 4.1 Green functions are not unique, since one can always add solutions to the homogenous Klein-Gordon equation to obtain a new Green's function. We must have

$$i\Delta_F(x,y)=-i\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}\frac{1}{k^2-m^2}$$

with some prescription for the contour of integration. Note that $\Delta_F(x,y)=\Delta_F(x-y)$ depends only on the difference $x-y$, which is as expected from homogeneity pof space-time.

Recall that $[\phi^{(+)}(x),\phi^{(-)}(y)]=\langle 0|T\phi(x)\phi(y)|0\rangle$ for $x^0>y^0$. More generally

$$\langle 0|T\phi(x)\phi(y)|0\rangle=\theta(x^0-y^0)[\phi^{(+)}(x),\phi^{(-)}(y)]+\theta(y^0-x^0)[\phi^{(+)}(y),\phi^{(-)}(x)]$$

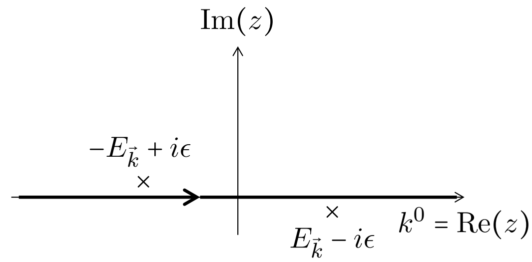
Recall also that

$$[\phi^{(+)}(x),\phi^{(-)}(y)]=\int(dk)e^{-iE_{\vec{k}}(x^0-y^0)+i\vec{k}\cdot(\vec{x}-\vec{y})}$$

Without loss of generality and to simplify notation we set $y^\mu = 0$. We have

$$\langle 0|T\phi(x)\phi(0)|0\rangle = \theta(x^0)\int (dk)e^{-iE_{\vec{k}}x^0+i\vec{k}\cdot\vec{x}} + \theta(-x^0)\int (dk)e^{iE_{\vec{k}}x^0-i\vec{k}\cdot\vec{x}}$$

Comparing with Eq. (4.6) we see this is precisely $-iG_F(x)$, which was obtained by taking a contour that goes below $-E_{\vec{k}}$ and above $E_{\vec{k}}$,



So we have

$$\boxed{\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\frac{i}{k^2-m^2+i\epsilon}} \tag{4.21}$$

This will be of much use later. We will refer to this as the *two-point function* of the KG field, and the Fourier transform, $1/(k^2-m^2+i\epsilon)$, as the KG *propagator*.

We can finally return to the question of relating ζ to ξ:

$$\begin{aligned}\zeta &= \int d^4x\, d^4y\, J(x)J(y)\,\langle 0|T\phi(x)\phi(y)|0\rangle \\ &= \int d^4x\, d^4y \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{ik_1\cdot x}e^{ik_2\cdot y}\widetilde{J}(k_1)\widetilde{J}(k_2)\int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2-m^2+i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4}\widetilde{J}(p)\widetilde{J}(-p)\frac{i}{p^2-m^2+i\epsilon}\end{aligned}$$

Perform the integral over p^0, assuming $\widetilde{J}(p)\widetilde{J}(-p) = |\widetilde{J}(p)|^2$ vanishes as $|p^0| \to \infty$, which is justified by our assumption that the source is localized in space-time. Closing the contour on the upper half of the complex p^0 plane we pick the pole at $-E_{\vec{k}}$ so that

$$\zeta = 2\pi i \int \frac{d^3p}{(2\pi)^4}|\widetilde{J}(p)|^2\frac{i}{-2E_{\vec{k}}} = \int (dk)|\widetilde{J}(p)|^2 = \xi\,.$$