PHYSICS 210B : NONEQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #2 : STOCHASTIC PROCESSES

(1) Show that for time scales sufficiently greater than γ^{-1} that the solution $x(t)$ to the Langevin equation $\ddot{x} + \gamma\dot{x} = \eta(t)$ describes a Markov process. You will have to construct the matrix M defined in Eqn. 2.60 of the lecture notes. You should assume that the random force $\eta(t)$ is distributed as a Gaussian, with $\langle \eta(s) \rangle = 0$ and $\langle \eta(s)\, \eta(s') \rangle = \Gamma\, \delta(s - s')$.

(2) Provide the missing steps in the solution of the Ornstein-Uhlenbeck process described in §2.4.3 of the lecture notes. Show that applying the method of characteristics to Eqn. 2.78 leads to the solution in Eqn. 2.79.

(3) Consider a discrete one-dimensional random walk where the probability to take a step of length 1 in either direction is $\frac{1}{2}p$ and the probability to take a step of length 2 in either direction is $\frac{1}{2}(1-p)$. Define the generating function

$$\hat{P}(k,t) = \sum_{n=-\infty}^{\infty} P_n(t)\, e^{-ikn} \ ,$$

where $P_n(t)$ is the probability to be at position n at time t. Solve for $\hat{P}(k,t)$ and provide an expression for $P_n(t)$. Evaluate $\sum_n n^2\, P_n(t)$.

(4) Numerically simulate the one-dimensional Wiener and Cauchy processes discussed in §2.6.1 of the lecture notes, and produce a figure similar to Fig. 2.3.

(5) Due to quantum coherence effects in the backscattering from impurities, one-dimensional wires don't obey Ohm's law in the limit where the 'inelastic mean free path' is greater than the sample dimensions, which you may assume here. Rather, let $\mathcal{R}(L) = e^2 R(L)/h$ be the dimensionless resistance of a quantum wire of length L, in units of $h/e^2 = 25.813\,\mathrm{k}\Omega$. The dimensionless resistance of a quantum wire of length $L + \delta L$ is then given by

$$\begin{aligned}\mathcal{R}(L+\delta L) = \mathcal{R}(L) + \mathcal{R}(\delta L) + 2\,\mathcal{R}(L)\,\mathcal{R}(\delta L) \\ + 2\cos\alpha\,\sqrt{\mathcal{R}(L)\,\big[1+\mathcal{R}(L)\big]\,\mathcal{R}(\delta L)\,\big[1+\mathcal{R}(\delta L)\big]} \ ,\end{aligned}$$

where α is a *random phase* uniformly distributed over the interval $[0, 2\pi)$. Here,

$$\mathcal{R}(\delta L) = \frac{\delta L}{2\ell} \ ,$$

is the dimensionless resistance of a small segment of wire, of length $\delta L \lesssim \ell$, where ℓ is the 'elastic mean free path'.

(a) Show that the distribution function $P(\mathcal{R}, L)$ for resistances of a quantum wire obeys the equation

$$\frac{\partial P}{\partial L} = \frac{1}{2\ell}\,\frac{\partial}{\partial \mathcal{R}}\bigg\{ \mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}} \bigg\} \ .$$

(b) Show that this equation may be solved in the limits $\mathcal{R} \ll 1$ and $\mathcal{R} \gg 1$, with

$$P(\mathcal{R}, z) = \frac{1}{z}\, e^{-\mathcal{R}/z}$$

for $\mathcal{R} \ll 1$, and

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\, \frac{1}{\mathcal{R}}\, e^{-(\ln \mathcal{R} - z)^2/4z}$$

for $\mathcal{R} \gg 1$, where $z = L/2\ell$ is the dimensionless length of the wire. Compute $\langle \mathcal{R} \rangle$ in the former case, and $\langle \ln \mathcal{R} \rangle$ in the latter case.