Contents

List of Figures

List of Tables

Chapter 14

Nonlinear Oscillators

14.1 Weakly Perturbed Linear Oscillators

Consider a nonlinear oscillator described by the equation of motion

$$\ddot{x} + \Omega_0^2\, x = \epsilon\, h(x) \quad . \tag{14.1}$$

Here, ϵ is a dimensionless parameter, assumed to be small, and $h(x)$ is a nonlinear function of x. In general, we might consider equations of the form

$$\ddot{x} + \Omega_0^2\, x = \epsilon\, h(x, \dot{x}) \quad , \tag{14.2}$$

such as the van der Pol oscillator,

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \Omega_0^2\, x = 0 \quad . \tag{14.3}$$

First, we will focus on nondissipative systems, *i.e.* where we may write $m\ddot{x} = -\partial_x V$, with $V(x)$ some potential.

As an example, consider the simple pendulum, which obeys

$$\ddot{\theta} + \Omega_0^2 \sin\theta = 0 \quad , \tag{14.4}$$

where $\Omega_0^2 = g/\ell$, with ℓ the length of the pendulum. We may rewrite his equation as

$$\begin{aligned} \ddot{\theta} + \Omega_0^2\, \theta &= \Omega_0^2\, (\theta - \sin\theta) \\ &= \tfrac{1}{6}\, \Omega_0^2\, \theta^3 - \tfrac{1}{120}\, \Omega_0^2\, \theta^5 + \ldots \end{aligned} \tag{14.5}$$

The RHS above is a nonlinear function of θ. We can define this to be $h(\theta)$, and take $\epsilon = 1$.

14.1.1 Naïve Perturbation theory and its failure

Let's assume though that ϵ is small, and write a formal power series expansion of the solution $x(t)$ to equation 14.1 as

$$x = x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots \quad . \tag{14.6}$$

We now plug this into 14.1. We need to use Taylor's theorem,

$$h(x_0 + \eta) = h(x_0) + h'(x_0)\,\eta + \tfrac{1}{2}\,h''(x_0)\,\eta^2 + \ldots \tag{14.7}$$

with

$$\eta = \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots \quad . \tag{14.8}$$

Working out the resulting expansion in powers of ϵ is tedious. One finds

$$h(x) = h(x_0) + \epsilon\, h'(x_0)\, x_1 + \epsilon^2 \Big\{ h'(x_0)\, x_2 + \tfrac{1}{2}\, h''(x_0)\, x_1^2 \Big\} + \ldots \quad . \tag{14.9}$$

Equating terms of the same order in ϵ, we obtain a hierarchical set of equations,

$$\begin{aligned}
\ddot{x}_0 + \Omega_0^2\, x_0 &= 0 \\
\ddot{x}_1 + \Omega_0^2\, x_1 &= h(x_0) \\
\ddot{x}_2 + \Omega_0^2\, x_2 &= h'(x_0)\, x_1 \\
\ddot{x}_3 + \Omega_0^2\, x_3 &= h'(x_0)\, x_2 + \tfrac{1}{2}\, h''(x_0)\, x_1^2
\end{aligned} \tag{14.10}$$

et cetera, where prime denotes differentiation with respect to argument. The first of these is easily solved: $x_0(t) = A\,\cos(\Omega_0 t + \varphi)$, where A and φ are constants. This solution then is plugged in at the next order, to obtain an inhomogeneous equation for $x_1(t)$. Solve for $x_1(t)$ and insert into the following equation for $x_2(t)$, *etc.* It looks straightforward enough.

The problem is that resonant forcing terms generally appear in the RHS of each equation of the hierarchy past the first. Define $\theta \equiv \Omega_0 t + \varphi$. Then $x_0(\theta)$ is an even periodic function of θ with period 2π, hence so is $h(x_0)$. We may then expand $h\big(x_0(\theta)\big)$ in a Fourier series:

$$h\big(A\cos\theta\big) = \sum_{n=0}^{\infty} h_n(A)\, \cos(n\theta) \quad . \tag{14.11}$$

The $n = 1$ term leads to resonant forcing. Thus, the solution for $x_1(t)$ is

$$x_1(t) = \frac{1}{\Omega_0^2} \sum_{\substack{n=0 \\ (n\neq 1)}}^{\infty} \frac{h_n(A)}{1-n^2}\, \cos(n\Omega_0 t + n\varphi) + \frac{h_1(A)}{2\Omega_0}\, t\, \sin(\Omega_0 t + \varphi) \quad , \tag{14.12}$$

which increases linearly with time. As an example, consider a cubic nonlinearity with $h(x) = r\,x^3$, where r is a constant. Then using

$$\cos^3\theta = \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos(3\theta) \quad , \tag{14.13}$$

we have $h_1 = \frac{3}{4}\, r A^3$ and $h_3 = \frac{1}{4}\, r A^3$.

14.1.2 Poincaré-Lindstedt method

The problem here is that the nonlinear oscillator has a different frequency than its linear counterpart. Indeed, if we assume the frequency Ω is a function of ϵ, with

$$\Omega(\epsilon) = \Omega_0 + \epsilon\,\Omega_1 + \epsilon^2\Omega_2 + \ldots \quad , \tag{14.14}$$

then subtracting the unperturbed solution from the perturbed one and expanding in ϵ yields

$$\begin{aligned} \cos(\Omega t) - \cos(\Omega_0 t) &= -\sin(\Omega_0 t)\,(\Omega - \Omega_0)\,t - \tfrac{1}{2}\cos(\Omega_0 t)\,(\Omega - \Omega_0)^2\,t^2 + \ldots \\ &= -\epsilon\,\sin(\Omega_0 t)\,\Omega_1 t - \epsilon^2\Big\{\sin(\Omega_0 t)\,\Omega_2 t + \tfrac{1}{2}\cos(\Omega_0 t)\,\Omega_1^2 t^2\Big\} + \mathcal{O}(\epsilon^3) \quad . \end{aligned} \tag{14.15}$$

What perturbation theory can do for us is to provide a good solution *up to a given time*, provided that ϵ is *sufficiently small*. It *will not* give us a solution that is close to the true answer for *all* time. We see above that in order to do that, and to recover the shifted frequency $\Omega(\epsilon)$, we would have to resum perturbation theory to all orders, which is a daunting task.

The Poincaré-Lindstedt method obviates this difficulty by assuming $\Omega = \Omega(\epsilon)$ from the outset. Define a dimensionless time $s \equiv \Omega t$ and write 14.1 as

$$\Omega^2\,\frac{d^2x}{ds^2} + \Omega_0^2\,x = \epsilon\,h(x) \quad , \tag{14.16}$$

where

$$\begin{aligned} x &= x_0 + \epsilon\,x_1 + \epsilon^2\,x_2 + \ldots \\ \Omega^2 &= a_0 + \epsilon\,a_1 + \epsilon^2\,a_2 + \ldots \quad . \end{aligned} \tag{14.17}$$

We now plug the above expansions into 14.16:

$$\begin{aligned} &\big(a_0 + \epsilon\,a_1 + \epsilon^2\,a_2 + \ldots\big)\bigg(\frac{d^2x_0}{ds^2} + \epsilon\,\frac{d^2x_1}{ds^2} + \epsilon^2\,\frac{d^2x_2}{ds^2} + \ldots\bigg) \\ &\qquad + \Omega_0^2\,\big(x_0 + \epsilon\,x_1 + \epsilon^2\,x_2 + \ldots\big) \\ &= \epsilon\,h(x_0) + \epsilon^2\,h'(x_0)\,x_1 + \epsilon^3\,\Big\{h'(x_0)\,x_2 + \tfrac{1}{2}\,h''(x_0)\,x_1^2\Big\} + \ldots \end{aligned} \tag{14.18}$$

Now let's write down equalities at each order in ϵ:

$$a_0\,\frac{d^2x_0}{ds^2} + \Omega_0^2\,x_0 = 0 \tag{14.19}$$

$$a_0\,\frac{d^2x_1}{ds^2} + \Omega_0^2\,x_1 = h(x_0) - a_1\,\frac{d^2x_0}{ds^2} \tag{14.20}$$

$$a_0\,\frac{d^2x_2}{ds^2} + \Omega_0^2\,x_2 = h'(x_0)\,x_1 - a_2\,\frac{d^2x_0}{ds^2} - a_1\,\frac{d^2x_1}{ds^2} \quad , \tag{14.21}$$

et cetera.

The first equation of the hierarchy is immediately solved by

$$a_0 = \Omega_0^2 \qquad , \qquad x_0(s) = A\,\cos(s+\varphi) \quad . \tag{14.22}$$

At $\mathcal{O}(\epsilon)$, then, we have

$$\frac{d^2x_1}{ds^2} + x_1 = \Omega_0^{-2}\, h\big(A\cos(s+\varphi)\big) + \Omega_0^{-2}\, a_1\, A\cos(s+\varphi) \quad . \tag{14.23}$$

The LHS of the above equation has a natural frequency of unity (in terms of the dimensionless time s). We expect $h(x_0)$ to contain resonant forcing terms, per 14.11. However, we now have the freedom to adjust the undetermined coefficient a_1 to *cancel* any such resonant term. Clearly we must choose

$$a_1 = -\frac{h_1(A)}{A} \quad . \tag{14.24}$$

The solution for $x_1(s)$ is then

$$x_1(s) = \frac{1}{\Omega_0^2} \sum_{\substack{n=0 \\ (n\neq 1)}}^{\infty} \frac{h_n(A)}{1-n^2}\, \cos(ns + n\varphi) \quad , \tag{14.25}$$

which is periodic and hence does not increase in magnitude without bound, as does 14.12. The perturbed frequency is then obtained from

$$\Omega^2 = \Omega_0^2 - \frac{h_1(A)}{A}\,\epsilon + \mathcal{O}(\epsilon^2) \quad \Longrightarrow \quad \Omega(\epsilon) = \Omega_0 - \frac{h_1(A)}{2A\Omega_0}\,\epsilon + \mathcal{O}(\epsilon^2) \quad . \tag{14.26}$$

Note that Ω depends on the amplitude of the oscillations.

As an example, consider an oscillator with a quartic nonlinearity in the potential, *i.e.* $h(x) = r\,x^3$. Then

$$h\big(A\cos\theta\big) = \tfrac{3}{4} r A^3\, \cos\theta + \tfrac{1}{4} r A^3\, \cos(3\theta) \quad . \tag{14.27}$$

We then obtain, setting $\epsilon = 1$ at the end of the calculation,

$$\Omega = \Omega_0 - \frac{3\,r A^2}{8\,\Omega_0} + \ldots \tag{14.28}$$

where the remainder is higher order in the amplitude A. In the case of the pendulum,

$$\ddot\theta + \Omega_0^2\,\theta = \tfrac{1}{6}\Omega_0^2\,\theta^3 + \mathcal{O}\big(\theta^5\big) \quad , \tag{14.29}$$

and with $r = \frac{1}{6}\,\Omega_0^2$ and $\theta_0(t) = \theta_0 \sin(\Omega t)$, we find

$$T(\theta_0) = \frac{2\pi}{\Omega} = \frac{2\pi}{\Omega_0} \cdot \Big\{ 1 + \tfrac{1}{16}\,\theta_0^2 + \ldots \Big\} \quad . \tag{14.30}$$

One can check that this is correct to lowest nontrivial order in the amplitude, using the exact result for the period,

$$T(\theta_0) = \frac{4}{\Omega_0}\,\mathbb{K}\big(\sin^2 \tfrac{1}{2}\theta_0\big) \quad , \tag{14.31}$$

where $\mathbb{K}(x)$ is the complete elliptic integral.

The procedure can be continued to the next order, where the free parameter a_2 is used to eliminate resonant forcing terms on the RHS.

A good exercise to test one's command of the method is to work out the lowest order nontrivial corrections to the frequency of an oscillator with a quadratic nonlinearity, such as $h(x) = rx^2$. One finds that there are no resonant forcing terms at first order in ϵ, hence one must proceed to second order to find the first nontrivial corrections to the frequency.

14.2 Multiple Time Scale Method

Another method of eliminating secular terms (*i.e.* driving terms which oscillate at the resonant frequency of the unperturbed oscillator), and one which has applicability beyond periodic motion alone, is that of multiple time scale analysis. Consider the equation

$$\ddot{x} + x = \epsilon\, h(x, \dot{x}) \quad , \tag{14.32}$$

where ϵ is presumed small, and $h(x, \dot{x})$ is a nonlinear function of position and/or velocity. We define a hierarchy of time scales: $T_n \equiv \epsilon^n\, t$. There is a normal time scale $T_0 = t$, slow time scale $T_1 = \epsilon t$, a 'superslow' time scale $T_2 = \epsilon^2 t$, *etc.* Thus,

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\,\frac{\partial}{\partial T_1} + \epsilon^2\,\frac{\partial}{\partial T_2} + \ldots = \sum_{n=0}^{\infty} \epsilon^n\,\frac{\partial}{\partial T_n} \quad . \tag{14.33}$$

Next, we expand

$$x(t) = \sum_{n=0}^{\infty} \epsilon^n\, x_n(T_0\, , \, T_1, \ldots) \quad . \tag{14.34}$$

Thus, we have

$$\bigg(\sum_{n=0}^{\infty} \epsilon^n \frac{\partial}{\partial T_n}\bigg)^2 \bigg(\sum_{k=0}^{\infty} \epsilon^k\, x_k\bigg) + \sum_{k=0}^{\infty} \epsilon^k\, x_k = \epsilon\, h\Bigg(\sum_{k=0}^{\infty} \epsilon^k\, x_k \quad , \sum_{n=0}^{\infty} \epsilon^n\,\frac{\partial}{\partial T_n}\bigg(\sum_{k=0}^{\infty} \epsilon^k\, x_k\bigg)\Bigg) \quad . \tag{14.35}$$

We now evaluate this order by order in ϵ:

$$\mathcal{O}(\epsilon^0) \ : \ \left(\frac{\partial^2}{\partial T_0^2} + 1\right) x_0 = 0 \tag{14.36}$$

$$\mathcal{O}(\epsilon^1) \ : \ \left(\frac{\partial^2}{\partial T_0^2} + 1\right) x_1 = -2\,\frac{\partial^2 x_0}{\partial T_0\,\partial T_1} + h\bigg(x_0\,,\,\frac{\partial x_0}{\partial T_0}\bigg) \tag{14.37}$$

$$\begin{aligned}\mathcal{O}(\epsilon^2) \ : \ \left(\frac{\partial^2}{\partial T_0^2} + 1\right) x_2 &= -2\,\frac{\partial^2 x_1}{\partial T_0\,\partial T_1} - 2\,\frac{\partial^2 x_0}{\partial T_0\,\partial T_2} - \frac{\partial^2 x_0}{\partial T_1^2} \\ &\quad + \frac{\partial h}{\partial x}\bigg|_{\{x_0,\dot{x}_0\}} x_1 + \frac{\partial h}{\partial \dot{x}}\bigg|_{\{x_0,\dot{x}_0\}} \left(\frac{\partial x_1}{\partial T_0} + \frac{\partial x_0}{\partial T_1}\right) \quad ,\end{aligned} \tag{14.38}$$

et cetera. The expansion gets more and more tedious with increasing order in ϵ.

Let's carry this procedure out to first order in ϵ. To order ϵ^0,

$$x_0 = A\,\cos\big(T_0 + \phi\big) \quad , \tag{14.39}$$

where A and ϕ are arbitrary (at this point) functions of $\big\{T_1\,,\,T_2\,,\,\ldots\big\}$. Now we solve the next equation in the hierarchy, for x_1. Let $\theta \equiv T_0 + \phi$. Then $\frac{\partial}{\partial T_0} = \frac{\partial}{\partial \theta}$ and we have

$$\left(\frac{\partial^2}{\partial \theta^2} + 1\right) x_1 = 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2\,A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + h\big(A\cos\theta, -A\sin\theta\big) \quad . \tag{14.40}$$

Since the arguments of h are periodic under $\theta \to \theta + 2\pi$, we may expand h in a Fourier series:

$$h(\theta) \equiv h\big(A\cos\theta, -A\sin\theta\big) = \sum_{k=1}^{\infty} \alpha_k(A)\sin(k\theta) + \sum_{k=0}^{\infty} \beta_k(A)\cos(k\theta) \quad . \tag{14.41}$$

The inverse of this relation is

$$\beta_0(A) = \int_0^{2\pi} \frac{d\theta}{2\pi}\,h(\theta) \tag{14.42}$$

and, for $k > 0$,

$$\alpha_k(A) = \int_0^{2\pi} \frac{d\theta}{\pi}\,h(\theta)\,\sin(k\theta) \qquad , \qquad \beta_k(A) = \int_0^{2\pi} \frac{d\theta}{\pi}\,h(\theta)\,\cos(k\theta) \quad . \tag{14.43}$$

We now demand that the secular terms on the RHS – those terms proportional to $\cos\theta$ and $\sin\theta$ – must vanish. This means

$$\begin{aligned} 2\,\frac{\partial A}{\partial T_1} + \alpha_1(A) &= 0 \\ 2A\,\frac{\partial \phi}{\partial T_1} + \beta_1(A) &= 0 \quad . \end{aligned} \tag{14.44}$$

These two first order equations require two initial conditions, which is sensible since our initial equation $\ddot{x} + x = \epsilon\, h(x, \dot{x})$ is second order in time.

With the secular terms eliminated, we may solve for x_1:

$$x_1 = \sum_{k\neq 1}^{\infty} \left\{ \frac{\alpha_k(A)}{1-k^2}\,\sin(k\theta) + \frac{\beta_k(A)}{1-k^2}\,\cos(k\theta) \right\} + C_0 \cos\theta + D_0 \sin\theta \quad . \tag{14.45}$$

Note: (i) the $k = 1$ terms are excluded from the sum, and (ii) an arbitrary solution to the homogeneous equation, *i.e.* eqn. 14.40 with the right hand side set to zero, is included. The constants C_0 and D_0 are arbitrary functions of T_1, T_2, *etc.* .

The equations for A and ϕ are both first order in T_1. They will therefore involve two constants of integration – call them A_0 and ϕ_0. At second order, these constants are taken as dependent upon the superslow time scale T_2. *The method itself may break down at this order.* (See if you can find out why.)

Let's apply this to the nonlinear oscillator $\ddot{x} + \sin x = 0$, also known as the simple pendulum. We'll expand the sine function to include only the lowest order nonlinear term, and consider

$$\ddot{x} + x = \tfrac{1}{6}\,\epsilon\, x^3 \quad . \tag{14.46}$$

We'll assume ϵ is small and take $\epsilon = 1$ at the end of the calculation. This will work provided the amplitude of the oscillation is itself small. To zeroth order, we have $x_0 = A\,\cos(t+\phi)$, as always. At first order, we must solve

$$\begin{aligned}\left(\frac{\partial^2}{\partial\theta^2} + 1\right) x_1 &= 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2\,A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + \tfrac{1}{6}\,A^3\cos^3\theta \\ &= 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2\,A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + \tfrac{1}{24}\,A^3\,\cos(3\theta) + \tfrac{1}{8}\,A^3\,\cos\theta \quad .\end{aligned} \tag{14.47}$$

We eliminate the secular terms by demanding

$$\frac{\partial A}{\partial T_1} = 0 \qquad , \qquad \frac{\partial \phi}{\partial T_1} = -\tfrac{1}{16}\,A^2 \quad , \tag{14.48}$$

hence $A = A_0$ and $\phi = -\frac{1}{16}\,A_0^2\,T_1 + \phi_0$, and

$$\begin{aligned} x(t) &= A_0 \cos\big(t - \tfrac{1}{16}\,\epsilon\,A_0^2\,t + \phi_0\big) \\ &\qquad - \tfrac{1}{192}\,\epsilon\,A_0^3\,\cos\big(3t - \tfrac{3}{16}\,\epsilon A_0^2\,t + 3\phi_0\big) + \ldots \quad ,\end{aligned} \tag{14.49}$$

which reproduces the result obtained from the Poincaré-Lindstedt method.

14.2.1 Duffing oscillator

Consider the equation

$$\ddot{x} + 2\epsilon\mu\dot{x} + x + \epsilon x^3 = 0 \quad . \tag{14.50}$$

This describes a damped nonlinear oscillator. Here we assume both the damping coefficient $\tilde{\mu} \equiv \epsilon\mu$ as well as the nonlinearity both depend linearly on the small parameter ϵ. We may write this equation in our standard form $\ddot{x} + x = \epsilon\, h(x, \dot{x})$, with $h(x, \dot{x}) = -2\mu\dot{x} - x^3$.

For $\epsilon > 0$, which we henceforth assume, it is easy to see that the only fixed point is $(x, \dot{x}) = (0, 0)$. The linearized flow in the vicinity of the fixed point is given by

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\epsilon\mu \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \mathcal{O}(x^3) \quad . \tag{14.51}$$

The determinant is $D = 1$ and the trace is $T = -2\epsilon\mu$. Thus, provided $\epsilon\mu < 1$, the fixed point is a stable spiral; for $\epsilon\mu > 1$ the fixed point becomes a stable node.

We employ the multiple time scale method to order ϵ. We have $x_0 = A\cos(T_0 + \phi)$ to zeroth order, as usual. The nonlinearity is expanded in a Fourier series in $\theta = T_0 + \phi$:

$$\begin{aligned} h\Big(x_0\,, \tfrac{\partial x_0}{\partial T_0}\Big) &= 2\mu A \sin\theta - A^3\cos^3\theta \\ &= 2\mu A \sin\theta - \tfrac{3}{4}A^3\cos\theta - \tfrac{1}{4}A^3\cos 3\theta \quad . \end{aligned} \tag{14.52}$$

Thus, $\alpha_1(A) = 2\mu A$ and $\beta_1(A) = -\frac{3}{4}A^3$. We now solve the first order equations,

$$\frac{\partial A}{\partial T_1} = -\tfrac{1}{2}\,\alpha_1(A) = -\mu A \qquad \Longrightarrow \qquad A(T) = A_0\, e^{-\mu T_1} \tag{14.53}$$

as well as

$$\frac{\partial \phi}{\partial T_1} = -\frac{\beta_1(A)}{2A} = \tfrac{3}{8}A_0^2\, e^{-2\mu T_1} \qquad \Longrightarrow \qquad \phi(T_1) = \phi_0 + \frac{3A_0^2}{16\mu}\big(1 - e^{-2\mu T_1}\big) \quad . \tag{14.54}$$

After elimination of the secular terms, we may read off

$$x_1(T_0\,, T_1) = \tfrac{1}{32}A^3(T_1)\cos\big(3T_0 + 3\phi(T_1)\big) \quad . \tag{14.55}$$

Finally, we have

$$\begin{aligned} x(t) = A_0\, e^{-\epsilon\mu t}\, &\cos\Big(t + \frac{3A_0^2}{16\mu}\big(1 - e^{-2\epsilon\mu t}\big) + \phi_0\Big) \\ &+ \tfrac{1}{32}\epsilon A_0^3\, e^{-3\epsilon\mu t}\, \cos\Big(3t + \frac{9A_0^2}{16\mu}\big(1 - e^{-2\epsilon\mu t}\big) + 3\phi_0\Big) \quad . \end{aligned} \tag{14.56}$$

14.2.2 Van der Pol oscillator

Let's apply this method to another problem, that of the van der Pol oscillator,

$$\ddot{x} + \epsilon\,(x^2 - 1)\,\dot{x} + x = 0 \quad , \tag{14.57}$$

with $\epsilon > 0$. The nonlinear term acts as a frictional drag for $x > 1$, and as a 'negative friction' (*i.e.* increasing the amplitude) for $x < 1$. Note that the linearized equation at the fixed point $(x = 0, \dot{x} = 0)$ corresponds to an unstable spiral for $\epsilon < 2$.

For the van der Pol oscillator, we have $h(x, \dot{x}) = (1 - x^2)\,\dot{x}$, and plugging in the zeroth order solution $x_0 = A\,\cos(t + \phi)$ gives

$$\begin{aligned} h\bigg(x_0\,,\,\frac{\partial x_0}{\partial T_0}\bigg) &= \big(1 - A^2\cos^2\theta\big)\,\big(-A\sin\theta\big) \\ &= \big(-A + \tfrac{1}{4}A^3\big)\,\sin\theta + \tfrac{1}{4}\,A^3\,\sin(3\theta) \quad , \end{aligned} \tag{14.58}$$

with $\theta \equiv t + \phi$. Thus, $\alpha_1 = -A + \frac{1}{4}A^3$ and $\beta_1 = 0$, which gives $\phi = \phi_0$ and

$$2\,\frac{\partial A}{\partial T_1} = A - \tfrac{1}{4}A^3 \quad . \tag{14.59}$$

The equation for A is easily integrated:

$$\begin{aligned} dT_1 &= -\frac{8\,dA}{A\,(A^2-4)} = \bigg(\frac{2}{A} - \frac{1}{A-2} - \frac{1}{A+2}\bigg)\,dA = d\ln\bigg(\frac{A}{A^2-4}\bigg) \\ \Longrightarrow \qquad A(T_1) &= \frac{2}{\sqrt{1 - \big(1 - \frac{4}{A_0^2}\big)\exp(-T_1)}} \quad . \end{aligned} \tag{14.60}$$

Thus,

$$x_0(t) = \frac{2\cos(t + \phi_0)}{\sqrt{1 - \big(1 - \frac{4}{A_0^2}\big)\exp(-\epsilon t)}} \quad . \tag{14.61}$$

This behavior describes the approach to the limit cycle $2\cos(t + \phi_0)$. With the elimination of the secular terms, we have

$$x_1(t) = -\tfrac{1}{32}A^3\sin(3\theta) = -\frac{\frac{1}{4}\sin\big(3t + 3\phi_0\big)}{\Big[1 - \big(1 - \frac{4}{A_0^2}\big)\exp(-\epsilon t)\Big]^{3/2}} \quad . \tag{14.62}$$

14.3 Forced Nonlinear Oscillations

The forced, damped linear oscillator,

$$\ddot{x} + 2\mu\dot{x} + x = f_0\cos\Omega t \tag{14.63}$$

has the solution

$$x(t) = x_{\rm h}(t) + C(\Omega)\,f_0\,\cos\big(\Omega t + \delta(\Omega)\big) \quad , \tag{14.64}$$

where $x_{\rm h}(t) = A_+\,e^{\lambda_+ t} + A_-\,e^{\lambda_- t}$ is the solution to the homogeneous equation (*i.e.* with $f_0 = 0$), with $\lambda_\pm = -\mu \pm \sqrt{\mu^2 - 1}$ the roots of $\lambda^2 + 2\mu\lambda + 1 = 0$. The 'susceptibility' C and phase shift δ are given by

$$C(\Omega) = \frac{1}{\sqrt{(\Omega^2 - 1)^2 + 4\mu^2\Omega^2}} \qquad , \qquad \delta(\Omega) = \tan^{-1}\bigg(\frac{2\mu\Omega}{1 - \Omega^2}\bigg) \quad . \tag{14.65}$$

The homogeneous solution, $x_{\rm h}(t)$, is a transient and decays exponentially with time, since $\mathrm{Re}(\lambda_\pm) < 0$. The asymptotic behavior is a phase-shifted oscillation at the driving frequency Ω.

Now let's add a nonlinearity. We study the equation

$$\ddot{x} + x = \epsilon\, h(x, \dot{x}) + \epsilon\, f_0\, \cos(t + \epsilon\nu t) \quad . \tag{14.66}$$

Note that amplitude of the driving term, $\epsilon f_0 \cos(\Omega t)$, is assumed to be small, *i.e.* proportional to ϵ, and the driving frequency $\Omega = 1 + \epsilon\nu$ is assumed to be close to resonance. (The resonance frequency of the unperturbed oscillator is $\omega_{\rm res} = 1$.) Were the driving frequency far from resonance, it could be dealt with in the same manner as the non-secular terms encountered thus far. The situation when Ω is close to resonance deserves our special attention.

At order ϵ^0, we still have $x_0 = A\cos(T_0 + \phi)$. We write

$$\Omega t = t + \epsilon\nu t = T_0 + \nu T_1 \equiv \theta - \psi \quad , \tag{14.67}$$

where $\theta = T_0 + \phi(T_1)$ as before, and $\psi(T_1) \equiv \phi(T_1) - \nu T_1$. At order ϵ^1, we must then solve

$$\begin{aligned}
\bigg(\frac{\partial^2}{\partial\theta^2} + 1\bigg) x_1 &= 2A'\sin\theta + 2A\phi'\cos\theta + h\big(A\cos\theta\,,\, -A\sin\theta\big) + f_0\cos(\theta - \psi) \\
&= \sum_{k\neq 1}\Big(\alpha_k \sin(k\theta) + \beta_k\cos(k\theta)\Big) + \Big(2A' + \alpha_1 + f_0\sin\psi\Big)\sin\theta \\
&\qquad + \Big(2A\psi' + 2A\nu + \beta_1 + f_0\cos\psi\Big)\cos\theta \quad ,
\end{aligned} \tag{14.68}$$

where the prime denotes differentiation with respect to T_1. We thus have the $N = 2$ dynamical system

$$\begin{aligned}
\frac{dA}{dT_1} &= -\tfrac{1}{2}\alpha_1(A) - \tfrac{1}{2} f_0 \sin\psi \\
\frac{d\psi}{dT_1} &= -\nu - \frac{\beta_1(A)}{2A} - \frac{f_0}{2A}\cos\psi \quad .
\end{aligned} \tag{14.69}$$

If we assume that $\{A, \psi\}$ approaches a fixed point of these dynamics, then at the fixed point these equations provide a relation between the amplitude A, the 'detuning' parameter ν, and the drive f_0:

$$F(A) \equiv \Big[\alpha_1(A)\Big]^2 + \Big[2\nu A + \beta_1(A)\Big]^2 = f_0^2 \quad . \tag{14.70}$$

In general this is a nonlinear equation for $A(f_0, \nu)$. The linearized (A, ψ) dynamics in the vicinity of a fixed point is governed by the matrix

$$M = \begin{pmatrix} \partial\dot{A}/\partial A & \partial\dot{A}/\partial\psi \\ \partial\dot{\psi}/\partial A & \partial\dot{\psi}/\partial\psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\alpha_1'(A) & \nu A + \frac{1}{2}\beta_1(A) \\ -\frac{\beta_1'(A)}{2A} - \frac{\nu}{A} & -\frac{\alpha_1(A)}{2A} \end{pmatrix} \quad . \tag{14.71}$$

If the (A, ψ) dynamics exhibits a stable fixed point (A^*, ψ^*), then one has

$$x_0(t) = A^*\cos(T_0 + \nu T_1 + \psi^*) = A^*\cos(\Omega t + \psi^*) \quad . \tag{14.72}$$

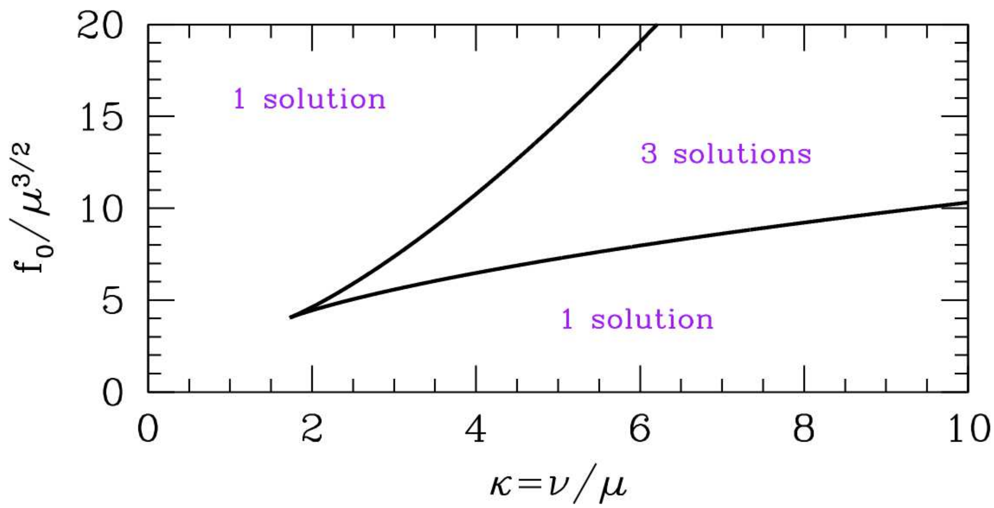

Figure 14.1: Phase diagram for the forced Duffing oscillator.

The oscillator's frequency is then the forcing frequency $\Omega = 1 + \epsilon\nu$, in which case the oscillator is said to be *entrained*, or *synchronized*, with the forcing. Note that

$$\det M = \frac{F'(A^*)}{8A^*} \quad .$$

14.3.1 Forced Duffing oscillator

Thus far our approach has been completely general. We now restrict our attention to the Duffing equation, for which

$$\alpha_1(A) = 2\mu A \qquad , \qquad \beta_1(A) = -\tfrac{3}{4}A^3 \quad , \tag{14.73}$$

which yields the cubic equation

$$A^6 - \tfrac{16}{3}\nu A^4 + \tfrac{64}{9}(\mu^2 + \nu^2)A^2 - \tfrac{16}{9}f_0^2 = 0 \quad . \tag{14.74}$$

Analyzing the cubic is a good exercise. Setting $y = A^2$, we define

$$G(y) \equiv y^3 - \tfrac{16}{3}\nu\, y^2 + \tfrac{64}{9}(\mu^2 + \nu^2)\, y \quad , \tag{14.75}$$

and we seek a solution to $G(y) = \frac{16}{9}f_0^2$. Setting $G'(y) = 0$, we find roots at

$$y_\pm = \tfrac{16}{9}\nu \pm \tfrac{8}{9}\sqrt{\nu^2 - 3\mu^2} \quad . \tag{14.76}$$

If $\nu^2 < 3\mu^2$ the roots are imaginary, which tells us that $G(y)$ is monotonically increasing for real y. There is then a unique solution to $G(y) = \frac{16}{9}f_0^2$.

If $\nu^2 > 3\mu^2$, then the cubic $G(y)$ has a local maximum at $y = y_-$ and a local minimum at $y = y_+$. For $\nu < -\sqrt{3}\,\mu$, we have $y_- < y_+ < 0$, and since $y = A^2$ must be positive, this means that once more there is a unique solution to $G(y) = \frac{16}{9}f_0^2$.

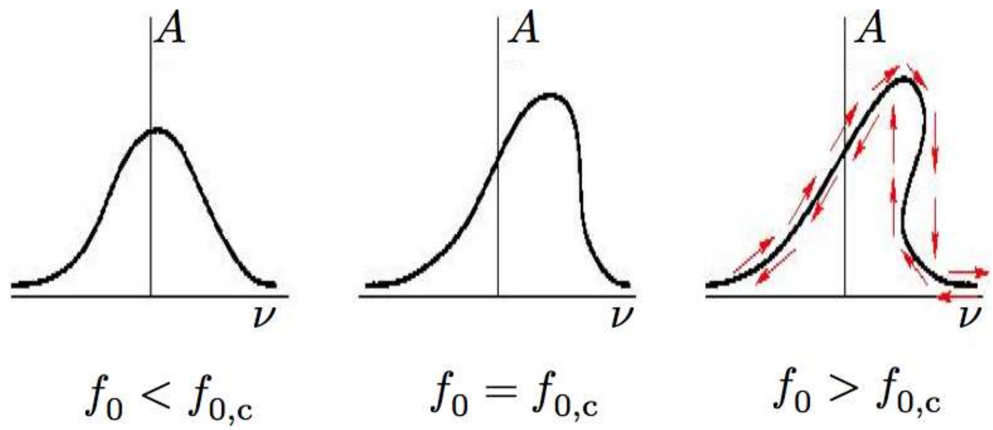

Figure 14.2: Amplitude A *versus* detuning ν for the forced Duffing oscillator for three values of the drive f_0. The critical drive is $f_{0,c} = \frac{16}{3^{5/4}}\,\mu^{3/2}$. For $f_0 > f_{0,c}$, there is hysteresis as a function of the detuning.

For $\nu > \sqrt{3}\,\mu$, we have $y_+ > y_- > 0$. There are then three solutions for $y(\nu)$ for $f_0 \in \big[f_0^-, f_0^+\big]$, where $f_0^\pm = \frac{3}{4}\sqrt{G(y_\mp)}$. If we define $\kappa \equiv \nu/\mu$, then

$$f_0^\pm = \frac{8}{9}\,\mu^{3/2}\sqrt{\kappa^3 + 9\kappa \pm \sqrt{\kappa^2 - 3}\,} \quad . \tag{14.77}$$

The phase diagram is shown in fig. 14.1. The minimum value for f_0 is $f_{0,c} = \frac{16}{3^{5/4}}\,\mu^{3/2}$, which occurs at $\kappa = \sqrt{3}$.

Thus far we have assumed that the (A, ψ) dynamics evolves to a fixed point. We should check to make sure that this fixed point is in fact stable. To do so, we evaluate the linearized dynamics at the fixed point. Writing $A = A^* + \delta A$ and $\psi = \psi^* + \delta\psi$, we have

$$\frac{d}{dT_1}\begin{pmatrix}\delta A\\ \delta\psi\end{pmatrix} = M\begin{pmatrix}\delta A\\ \delta\psi\end{pmatrix} \quad , \tag{14.78}$$

with

$$M = \begin{pmatrix} \frac{\partial \dot A}{\partial A} & \frac{\partial \dot A}{\partial \psi} \\[2ex] \frac{\partial \dot \psi}{\partial A} & \frac{\partial \dot \psi}{\partial \psi}\end{pmatrix} = \begin{pmatrix} -\mu & -\frac{1}{2}f_0\,\cos\psi \\[2ex] \frac{3}{4}A + \frac{f_0}{2A^2}\,\cos\psi & \frac{f_0}{2A}\,\sin\psi \end{pmatrix} = \begin{pmatrix} -\mu & \nu A - \frac{3}{8}A^3 \\[2ex] \frac{9}{8}A - \frac{\nu}{A} & -\mu \end{pmatrix} \quad . \tag{14.79}$$

One then has $T = -2\mu$ and

$$D = \mu^2 + \big(\nu - \tfrac{3}{8}A^2\big)\big(\nu - \tfrac{9}{8}A^2\big) \quad . \tag{14.80}$$

Setting $D = \frac{1}{4}T^2 = \mu^2$ sets the boundary between stable spiral and stable node. Setting $D = 0$ sets the boundary between stable node and saddle. The fixed point structure is as shown in fig. 14.3. Though the amplitude exhibits hysteresis, the oscillator frequency is always synchronized with the forcing as one varies the detuning.

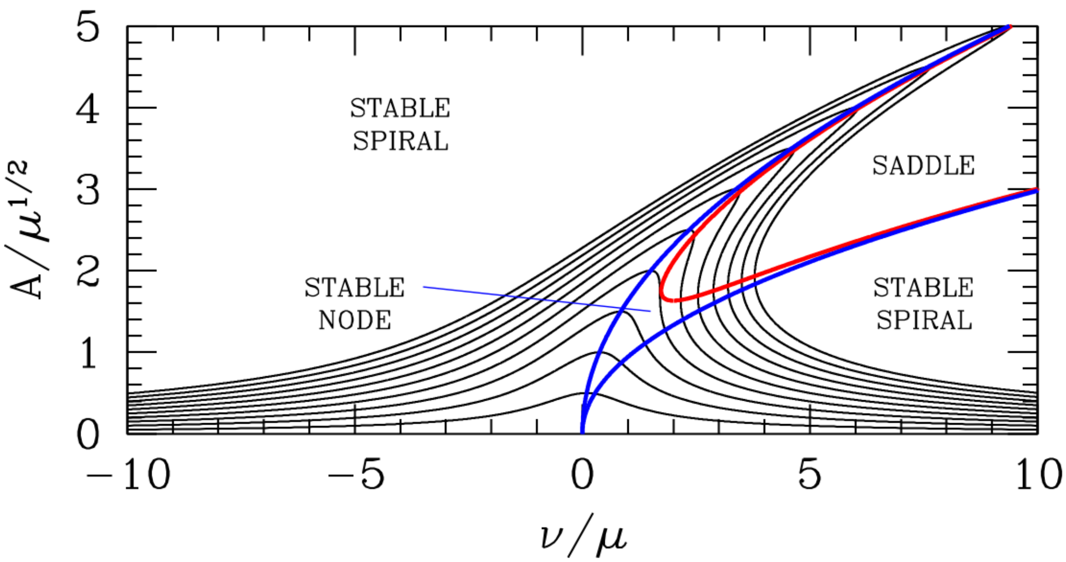

Figure 14.3: Amplitude *versus* detuning for the forced Duffing oscillator for ten equally spaced values of f_0 between $\mu^{3/2}$ and $10\,\mu^{3/2}$. The critical value is $f_{0,\mathrm{c}} = 4.0525\,\mu^{3/2}$. The red and blue curves are boundaries for the fixed point classification.

14.3.2 Forced van der Pol oscillator

Consider now a weakly dissipative, weakly forced van der Pol oscillator, governed by the equation

$$\ddot{x} + \epsilon\,(x^2 - 1)\,\dot{x} + x = \epsilon\, f_0\, \cos(t + \epsilon\nu t) \quad , \tag{14.81}$$

where the forcing frequency is $\Omega = 1 + \epsilon\nu$, which is close to the natural frequency $\omega_0 = 1$. We apply the multiple time scale method, with $h(x,\dot{x}) = (1 - x^2)\,\dot{x}$. As usual, the lowest order solution is $x_0 = A(T_1)\cos\big(T_0 + \phi(T_1)\big)$, where $T_0 = t$ and $T_1 = \epsilon t$. Again, we define $\theta \equiv T_0 + \phi(T_1)$ and $\psi(T_1) \equiv \phi(T_1) - \nu T_1$. From

$$h(A\cos\theta, -A\sin\theta) = \big(\tfrac{1}{4}A^3 - A\big)\,\sin\theta + \tfrac{1}{4}A^3\sin(3\theta) \quad , \tag{14.82}$$

we arrive at

$$\begin{aligned}\bigg(\frac{\partial^2}{\partial\theta^2} + 1\bigg)x_1 &= -2\frac{\partial^2 x_0}{\partial T_0\,\partial T_1} + h\bigg(x_0\,,\,\frac{\partial x_0}{\partial T_0}\bigg)\\ &= \big(\tfrac{1}{4}A^3 - A + 2A' + f_0\sin\psi\big)\sin\theta \;+\big(2A\,\psi' + 2\nu A + f_0\cos\psi\big)\cos\theta + \tfrac{1}{4}A^3\sin(3\theta) \quad .\end{aligned} \tag{14.83}$$

We eliminate the secular terms, proportional to $\sin\theta$ and $\cos\theta$, by demanding

$$\frac{dA}{dT_1} = \tfrac{1}{2}A - \tfrac{1}{8}A^3 - \tfrac{1}{2}f_0\sin\psi \tag{14.84}$$

$$\frac{d\psi}{dT_1} = -\nu - \frac{f_0}{2A}\cos\psi \quad . \tag{14.85}$$

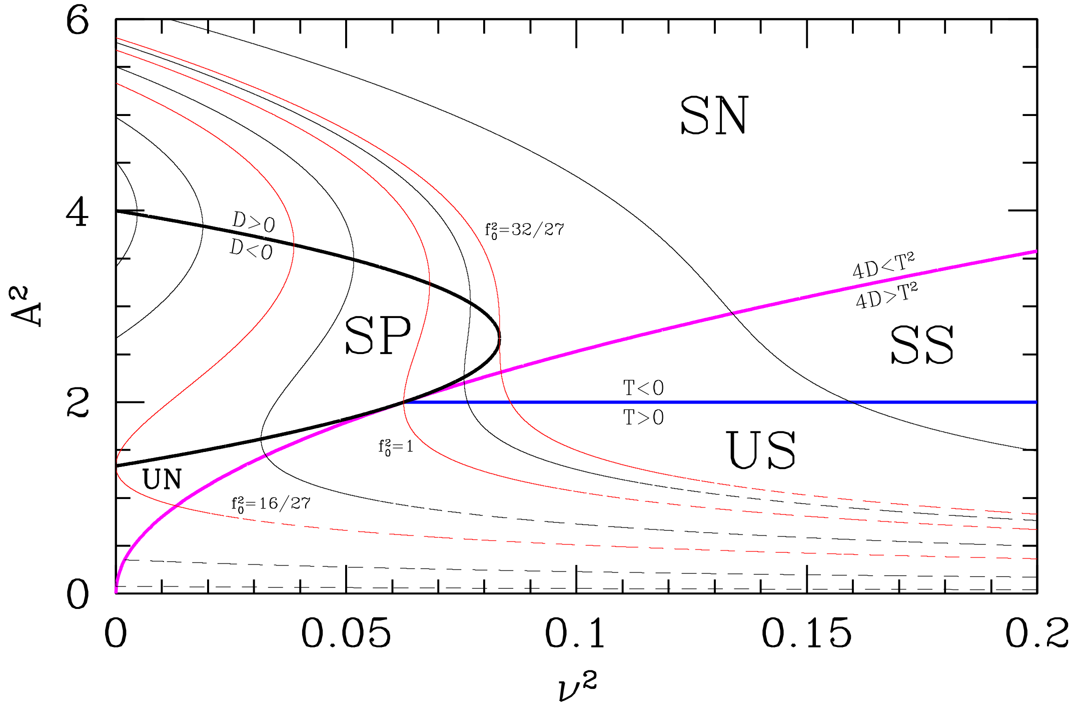

Figure 14.4: Amplitude *versus* detuning for the forced van der Pol oscillator. Fixed point classifications are abbreviated SN (stable node), SS (stable spiral), UN (unstable node), US (unstable spiral), and SP (saddle point).

Stationary solutions have $A' = \psi' = 0$, hence $\cos\psi = -2\nu A/f_0$, and hence

$$\begin{aligned} f_0^2 &= 4\nu^2 A^2 + \big(1 - \tfrac{1}{4}A^2\big)^2 A^2 \\ &= \tfrac{1}{16}A^6 - \tfrac{1}{2}A^4 + (1 + 4\nu^2)A^2 \quad . \end{aligned} \tag{14.86}$$

For this solution, we have

$$x_0 = A^* \cos(T_0 + \nu T_1 + \psi^*) \quad , \tag{14.87}$$

and the oscillator's frequency is the forcing frequency $\varOmega = 1 + \varepsilon\nu$.

To proceed further, let $y = A^2$, and consider the cubic equation

$$G(y) = \tfrac{1}{16}y^3 - \tfrac{1}{2}y^2 + (1 + 4\nu^2)\, y = f_0^2 \quad . \tag{14.88}$$

Setting $G'(y) = 0$, we find the roots of $G'(y)$ lie at $y_\pm = \frac{4}{3}\,(2 \pm u)$, where $u = (1 - 12\,\nu^2)^{1/2}$. Thus, the roots are complex for $\nu^2 > \frac{1}{12}$, in which case $G(y)$ is monotonically increasing, and there is a unique solution to $G(y) = f_0^2$. Since $G(0) = 0 < f_0^2$, that solution satisfies $y > 0$. For $\nu^2 < \frac{1}{12}$, there are two local extrema at $y = y_\pm$. When $G_{\rm min} = G(y_+) < f_0^2 < G(y_-) = G_{\rm max}$, the cubic equation $G(y) = f_0^2$ has three real, positive roots. This is equivalent to the condition

$$-\tfrac{8}{27}\,u^3 + \tfrac{8}{9}\,u^2 < \tfrac{32}{27} - f_0^2 < \tfrac{8}{27}\,u^3 + \tfrac{8}{9}\,u^2 \quad . \tag{14.89}$$

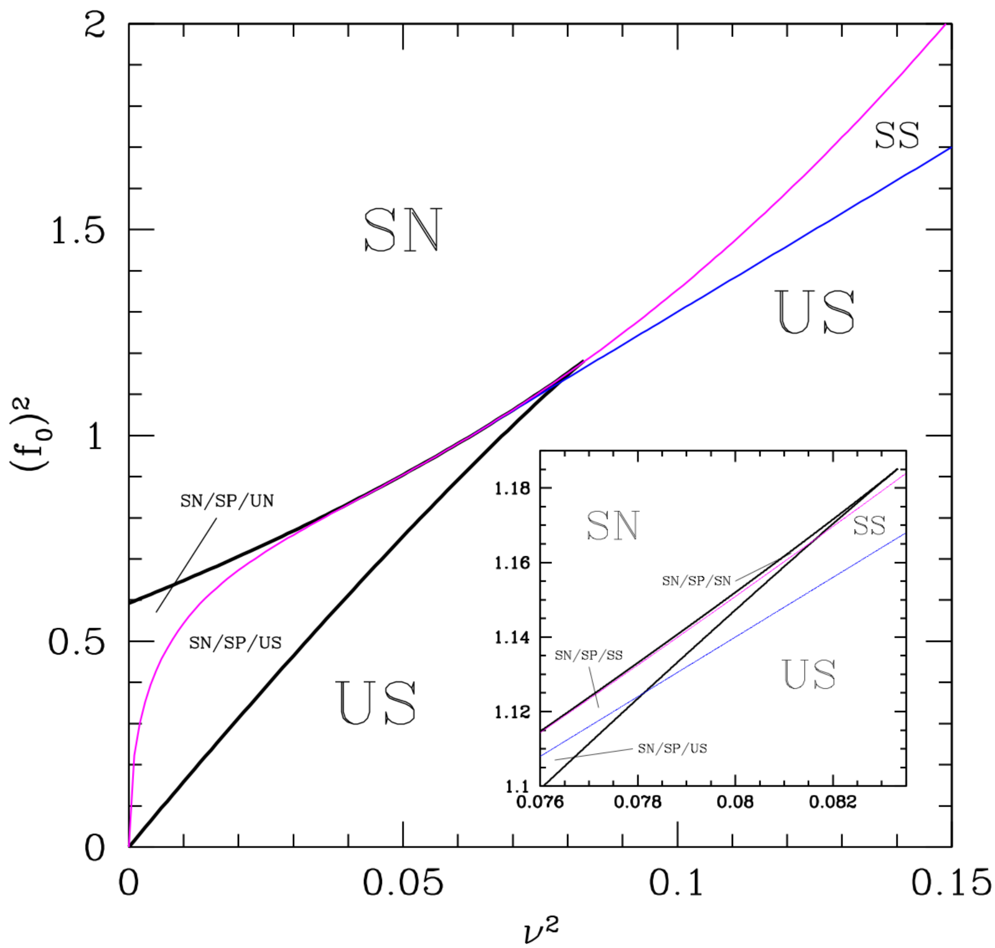

Figure 14.5: Phase diagram for the weakly forced van der Pol oscillator in the (ν^2, f_0^2) plane. Inset shows detail. Abbreviations for fixed point classifications are as in fig. 14.4.

We can say even more by exploring the behavior of eqs. (14.84) and (14.85) in the vicinity of the fixed points. Writing $A = A^* + \delta A$ and $\psi = \psi^* + \delta\psi$, we have

$$\frac{d}{dT_1}\begin{pmatrix}\delta A \\ \delta\psi\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\big(1 - \frac{3}{4}A^{*2}\big) & \nu A^* \\ -\nu/A^* & \frac{1}{2}\big(1 - \frac{1}{4}A^{*2}\big)\end{pmatrix}\begin{pmatrix}\delta A \\ \delta\psi\end{pmatrix} \quad . \tag{14.90}$$

The eigenvalues of the linearized dynamics at the fixed point are given by $\lambda_\pm = \frac{1}{2}\big(T \pm \sqrt{T^2 - 4D}\big)$, where T and D are the trace and determinant of the linearized equation. Recall now the classification scheme for fixed points of two-dimensional phase flows. When $D < 0$, we have $\lambda_- < 0 < \lambda_+$ and the fixed point is a saddle. For $0 < 4D < T^2$, both eigenvalues have the same sign, so the fixed point is a node. For $4D > T^2$, the eigenvalues form a complex conjugate pair, and the fixed point is a spiral. A node/spiral fixed point is stable if $T < 0$ and unstable if $T > 0$. For our forced van der Pol oscillator, we

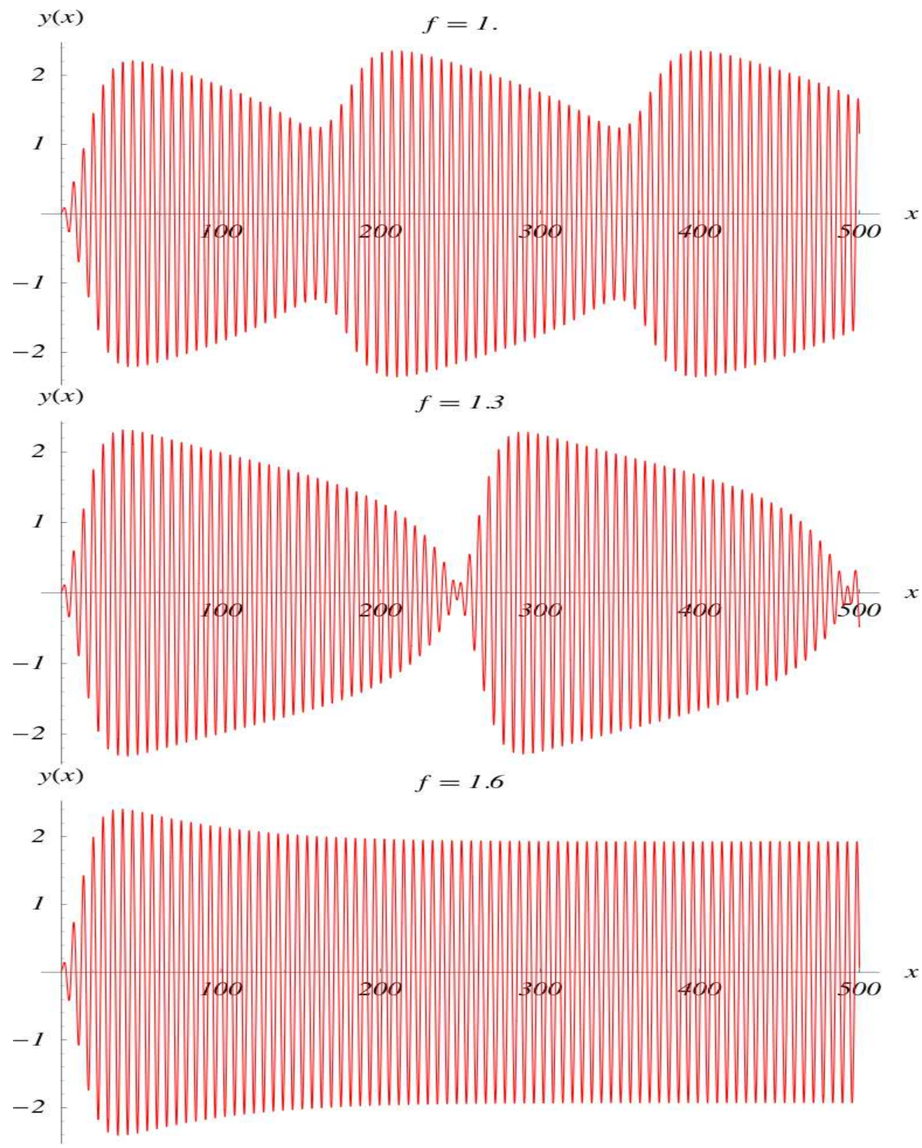

Figure 14.6: Forced van der Pol system with $\epsilon = 0.1$, $\nu = 0.4$ for three values of f_0. The limit entrained solution becomes unstable at $f_0 = 1.334$.

have

$$\begin{aligned} T &= 1 - \tfrac{1}{2}A^{*2} \\ D &= \tfrac{1}{4}\big(1 - A^{*2} + \tfrac{3}{16}A^{*4}\big) + \nu^2 \quad . \end{aligned} \tag{14.91}$$

From these results we can obtain the plot of fig. 14.4, where amplitude is shown *versus* detuning. We now ask: for what values of f_0^2 is there hysteretic behavior over a range $\nu \in \big[\nu_-, \nu_+\big]$? Suppose, following the curves of constant f_0^2 in fig. 14.4, we start somewhere in the upper left corner of the diagram, in the region $D > 0$ and $f_0^2 < \frac{32}{27}$. Now ramp up ν^2 while keeping f_0^2 constant until we arrive on the upper branch of the $D = 0$ curve. An infinitesimal further increase in ν^2 will cause a discontinuous drop in $y = A^2$ to a value below the saddle point region. Clearly if $f_0^2 = 1$, we will wind up on a branch of this curve for which $A^2 < 2$, which is unstable, and so in order to end up on a stable branch, we must start with $f_0^2 > 1$. To find the minimum such value of f_0^2 for which this is possible, we first demand $G(y) = 0$ as well as $D = 0$. The second of these conditions is equivalent to $G'(y) = 0$. Eliminating y, we obtain the equation $\frac{8}{27}(2+u)^2(1-u) = f_0^2$, where $u = \sqrt{1 - 12\nu^2}$ as above. Next, we demand that $G(y) = 0$ at $y = 2$ (*i.e.* the blue line in fig. 14.4) for the *same values* of f_0^2 and ν^2. Thus says $f_0^2 = \frac{1}{6}(7 - 4u^2)$. Eliminating f_0^2, we obtain the equation

$$\tfrac{8}{27}(2 + u)^2\,(1 - u) = \tfrac{1}{6}(7 - 4u^2) \quad , \tag{14.92}$$

which is equivalent to the factorized cubic $(4u-1)(2u+1)^2 = 0$. The root we seek is $u = \frac{1}{4}$, corresponding to $\nu^2 = \frac{15}{16} \cdot \frac{1}{12}$ and $f_0^2 = \frac{27}{24}$. Thus, hysteretic behavior is possible only in the narrow regime $f_0^2 \in \big[\frac{27}{24}, \frac{32}{27}\big]$. The phase diagram in the (ν^2, f_0^2) plane is shown in fig. 14.5. Hysteresis requires two among the three fixed points be stable, so the system can jump from one stable branch to another as ν is varied. These regions are so small they are only discernible in the inset.

Finally, we can make the following statement about the *global* dynamics (*i.e.* not simply in the vicinity of a fixed point). For large A, we have

$$\frac{dA}{dT_1} = -\tfrac{1}{8}A^3 + \ldots \qquad , \qquad \frac{d\psi}{dT_1} = -\nu + \ldots \quad . \tag{14.93}$$

This flow is inward, hence if the flow is not to a stable fixed point, it must be attracted to a limit cycle. The limit cycle necessarily involves several frequencies. This result – the generation of new frequencies by nonlinearities – is called *heterodyning*.

We can see heterodyning in action in the van der Pol system. In fig. 14.5, the blue line which separates stable and unstable spiral solutions is given by $f_0^2 = 8\nu^2 + \frac{1}{2}$. For example, if we take $\nu = 0.40$ then the boundary lies at $f_0 = 1.334$. For $f_0 < 1.334$, we expect heterodyning, as the entrained solution is unstable. For $f > 1.334$ the solution is entrained and oscillates at a fixed frequency. This behavior is exhibited in fig. 14.6.

14.4 Synchronization

Thus far we have assumed both the nonlinearity as well as the perturbation are weak. In many systems, we are confronted with a strong nonlinearity which we can perturb weakly. How does an attractive limit cycle in a strongly nonlinear system respond to weak periodic forcing? Here we shall follow the nice discussion in the book of Pikovsky *et al.*

Consider a forced dynamical system,

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi}) + \varepsilon\, \boldsymbol{f}(\boldsymbol{\varphi}, t) \quad . \tag{14.94}$$

When $\varepsilon = 0$, we assume that the system has at least one attractive limit cycle $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(t + T_0)$. All points on the limit cycle are fixed under the T_0-advance map g_{T_0}, where $g_\tau \boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(t + \tau)$. The idea is now to parameterize the points along the limit cycle by a phase angle ϕ which runs from 0 to 2π such that $\phi(t)$ increases by 2π with each orbit of the limit cycle, with ϕ increasing uniformly with time, so that $\dot{\phi} = \omega_0 = 2\pi/T_0$. Now consider the action of the T_0-advance map g_{T_0} on points in the vicinity of the limit cycle. Since each point $\boldsymbol{\gamma}(\phi)$ on the limit cycle is a fixed point, and since the limit cycle is presumed to be attractive, we can define the *ϕ-isochrone* as the set of points $\{\boldsymbol{\varphi}\}$ in phase space which flow to the fixed point $\boldsymbol{\gamma}(\phi)$ under repeated application of g_{T_0}. The isochrones are $(N-1)$-dimensional hypersurfaces.

Equivalently, consider a point $\boldsymbol{\varphi}_0 \in \Omega_\gamma$ lying within the basin of attraction Ω_γ of the limit cycle $\boldsymbol{\gamma}(t)$. We say that $\boldsymbol{\varphi}_0$ lies along the ϕ-isochrone if

$$\lim_{t\to\infty} \left|\boldsymbol{\varphi}(t) - \boldsymbol{\gamma}\Big(t + \frac{\phi}{2\pi}\, T_0\Big)\right| = 0 \quad , \tag{14.95}$$

where $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0$. For each $\boldsymbol{\varphi}_0 \in \Omega_\gamma$, there exists a unique corresponding value of $\phi(\boldsymbol{\varphi}_0) \in [0, 2\pi]$. This is called the *asymptotic* (or *latent*) *phase* of $\boldsymbol{\varphi}_0$.

To illustrate this, we analyze the example in Pikovsky *et al.* of the complex amplitude equation (CAE),

$$\frac{dA}{dt} = (1 + i\alpha)\, A - (1 + i\beta)\, |A|^2 A \quad , \tag{14.96}$$

where $A \in \mathbb{C}$ is a complex number. It is convenient to work in polar coordinates, writing $A = R\, e^{i\Theta}$, in which case the real and complex parts of the CAE become

$$\begin{aligned} \dot{R} &= (1 - R^2)\, R \\ \dot{\Theta} &= \alpha - \beta R^2 \quad . \end{aligned} \tag{14.97}$$

These equations can be integrated to yield the solution

$$\begin{aligned} R(t) &= \frac{R_0}{\sqrt{R_0^2 + (1 - R_0^2)\, e^{-2t}}} \\ \Theta(t) &= \Theta_0 + (\alpha - \beta)t - \tfrac{1}{2}\beta \ln\!\big[R_0^2 + (1 - R_0^2)\, e^{-2t}\big] \\ &= \Theta_0 + (\alpha - \beta)\, t + \beta\, \ln(R/R_0) \quad . \end{aligned} \tag{14.98}$$

As $t \to \infty$, we have $R(t) \to 1$ and $\dot{\Theta}(t) \to \omega_0$. Thus the limit cycle is the circle $R = 1$, and its frequency is $\omega_0 = \alpha - \beta$.

Since all points on each isochrone share the same phase, we can evaluate $\dot{\phi}$ along the limit cycle, and thus we have $\dot{\phi} = \omega_0$. The functional form of the isochrones is dictated by the rotational symmetry of the vector field, which requires $\phi(R, \Theta) = \Theta - f(R)$, where $f(R)$ is an as-yet undetermined function. Taking the derivative, we immediately find $f(R) = \beta \ln R$, *i.e.*

$$\phi(R, \Theta) = \Theta - \beta \ln R + c \quad , \tag{14.99}$$

where c is a constant. We can now check that

$$\dot{\phi} = \dot{\Theta} - \beta\, \frac{\dot{R}}{R} = \alpha - \beta = \omega_0 \quad . \tag{14.100}$$

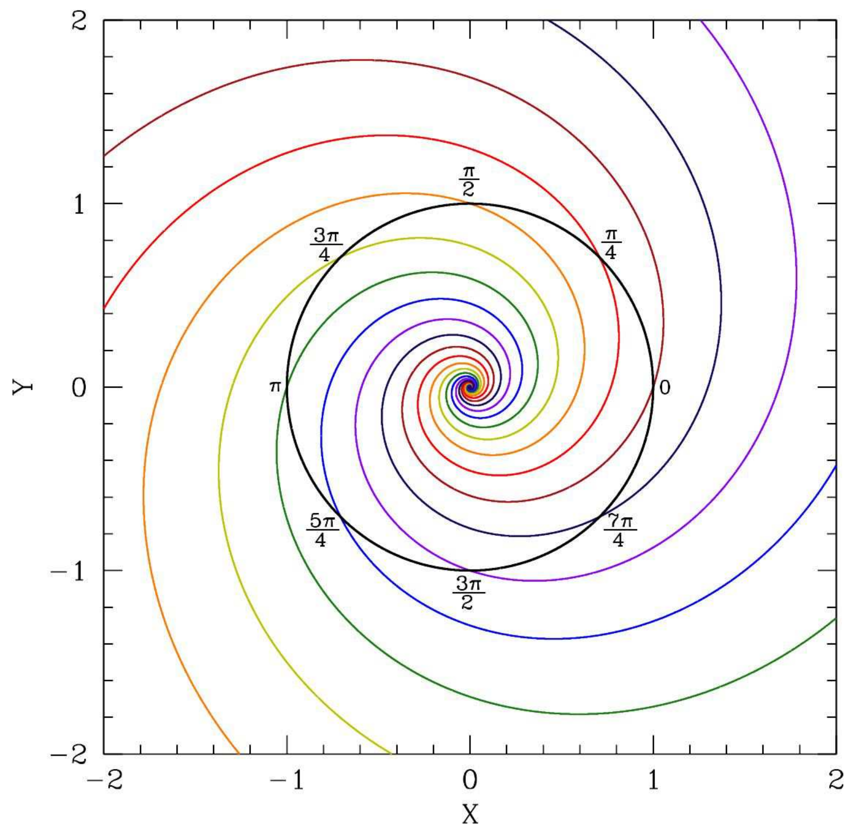

Figure 14.7: Isochrones of the complex amplitude equation $\dot{A} = (1+i\alpha)A - (1+i\beta)|A|^2 A$, where $A = X + iY$.

Without loss of generality we may take $c = 0$. Thus the ϕ-isochrone is given by the curve $\Theta(R) = \phi + \beta \ln R$, which is a logarithmic spiral. These isochrones are depicted in fig. 14.7.

At this point we have defined a phase function $\phi(\boldsymbol{\varphi})$ as the phase of the fixed point along the limit cycle to which $\boldsymbol{\varphi}$ flows under repeated application of the T_0-advance map g_{T_0}. Now let us examine the dynamics of ϕ for the weakly perturbed system of eqn. 14.94. We have

$$\begin{aligned}\frac{d\phi}{dt} &= \sum_{j=1}^{N} \frac{\partial\phi}{\partial\varphi_j}\,\frac{d\varphi_j}{dt} \\ &= \omega_0 + \varepsilon \sum_{j=1}^{N} \frac{\partial\phi}{\partial\varphi_j}\, f_j(\boldsymbol{\varphi},t) \quad .\end{aligned} \tag{14.101}$$

We will assume that $\boldsymbol{\varphi}$ is close to the limit cycle, so that $\boldsymbol{\varphi} - \boldsymbol{\gamma}(\phi)$ is small. As an example, consider once more the complex amplitude equation (14.96), but now adding in a periodic forcing term.

$$\frac{dA}{dt} = (1+i\alpha)\,A - (1+i\beta)\,|A|^2 A + \varepsilon\,\cos\omega t \quad . \tag{14.102}$$

Writing $A = X + iY$, we have

$$\begin{aligned}\dot{X} &= X - \alpha Y - (X - \beta Y)(X^2 + Y^2) + \varepsilon \cos \omega t \\ \dot{Y} &= Y + \alpha X - (\beta X + Y)(X^2 + Y^2) \quad .\end{aligned} \tag{14.103}$$

In Cartesian coordinates, the isochrones for the $\varepsilon = 0$ system are

$$\phi = \tan^{-1}(Y/X) - \tfrac{1}{2}\beta \ln(X^2 + Y^2) \quad , \tag{14.104}$$

hence

$$\begin{aligned}\frac{d\phi}{dt} &= \omega_0 + \varepsilon \, \frac{\partial \phi}{\partial X} \, \cos \omega t \\ &= \alpha - \beta - \varepsilon \left(\frac{\beta X + Y}{X^2 + Y^2} \right) \cos \omega t \\ &\approx \omega_0 - \varepsilon \, (\beta \cos \phi + \sin \phi) \, \cos \omega t \\ &= \omega_0 - \varepsilon \, \sqrt{1 + \beta^2} \, \cos(\phi - \phi_\beta) \cos \omega t \quad .\end{aligned} \tag{14.105}$$

where $\phi_\beta = \operatorname{ctn}^{-1}\beta$. Note that in the third line above we have invoked $R \approx 1$, *i.e.* we assume that we are close to the limit cycle.

We now define the function

$$F(\phi, t) \equiv \sum_{j=1}^{N} \frac{\partial \phi}{\partial \varphi_j} \bigg|_{\boldsymbol{\gamma}(\phi)} f_j \left(\boldsymbol{\gamma}(\phi), t \right) \quad . \tag{14.106}$$

The phase dynamics for ϕ are now written as

$$\dot{\phi} = \omega_0 + \varepsilon \, F(\phi, t) \quad . \tag{14.107}$$

Now $F(\phi, t)$ is periodic in both its arguments, so we may write

$$F(\phi, t) = \sum_{k,l} F_{kl} \, e^{i(k\phi + l\omega t)} \quad . \tag{14.108}$$

For the unperturbed problem, we have $\dot{\phi} = \omega_0$, hence resonant terms in the above sum are those for which $k\omega_0 + l\omega \approx 0$. This occurs when $\omega \approx \frac{p}{q}\,\omega_0$, where p and q are relatively prime integers. In this case the resonance condition is satisfied for $k = jp$ and $l = -jq$ for all $j \in \mathbb{Z}$. We now separate the resonant from the nonresonant terms in the (k, l) sum, writing

$$\dot{\phi} = \omega_0 + \varepsilon \sum_j F_{jp,-jq} \, e^{ij(p\phi - q\omega t)} + \text{NRT} \quad , \tag{14.109}$$

where NRT denotes nonresonant terms, *i.e.* those for which $(k, l) \neq (jp, -jq)$ for some integer j. We now average over short time scales to eliminate the nonresonant terms, and focus on the dynamics of this averaged phase $\langle \phi \rangle$.

We define the angle $\psi \equiv p\langle\phi\rangle - q\omega t$, which obeys

$$\begin{aligned}\dot{\psi} &= p \, \langle \dot{\phi} \rangle - q\omega \\ &= (p\omega_0 - q\omega) + \varepsilon p \sum_j F_{jp,-jq} \, e^{ij\psi} \equiv -\nu + \varepsilon \, G(\psi) \quad ,\end{aligned} \tag{14.110}$$

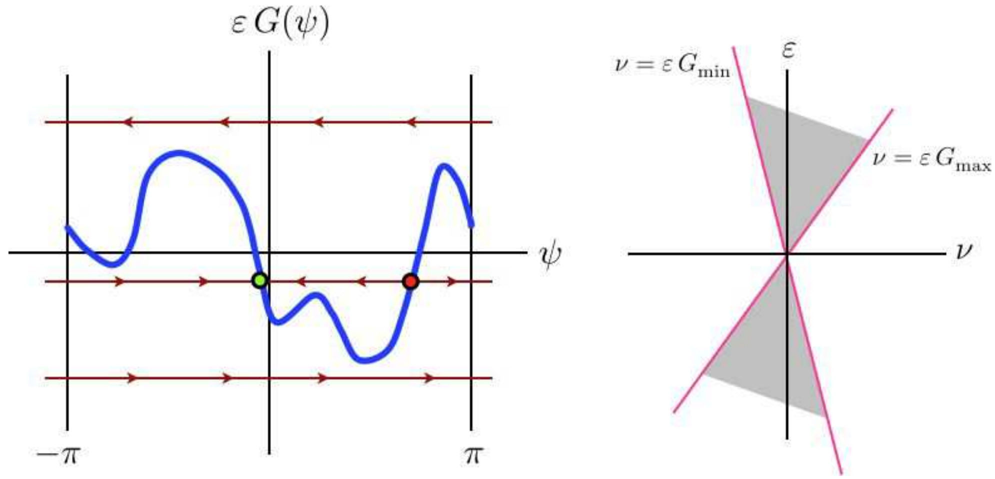

Figure 14.8: Left panel: graphical analysis of the equation $\dot{\psi} = -\nu + \varepsilon\, G(\psi)$. Right panel: Synchronization region (gray) as a function of detuning ν.

where $\nu \equiv q\omega - p\omega_0$ is the detuning and $G(\psi) = p\sum_j F_{jp,-jq}\, e^{ij\psi}$ is the sum over resonant terms. Note that the nonresonant terms have been eliminated by the aforementioned averaging procedure. This last equation is a simple $N = 1$ dynamical system on the circle – a system we have already studied. The dynamics are depicted in fig. 14.8. If the detuning ν falls within the range $[\varepsilon\, G_{\rm min}\, ,\, \varepsilon\, G_{\rm max}]$, then ψ flows to a fixed point, and the nonlinear oscillator is synchronized with the periodic external force, with $\langle\dot{\phi}\rangle \to \frac{q}{p}\,\omega$. If the detuning is too large and lies outside this region, then there is no synchronization. Rather, $\psi(t)$ increases on average linearly with time. In this case we have $\langle\phi(t)\rangle = \phi_0 + \frac{q}{p}\omega t + \frac{1}{p}\,\psi(t)$, where

$$dt = \frac{d\psi}{\varepsilon\, G(\psi) - \nu} \qquad \Longrightarrow \qquad T_\psi = \int\limits_{-\pi}^{\pi} \frac{d\psi}{\varepsilon\, G(\psi) - \nu} \quad . \tag{14.111}$$

Thus, $\psi(t) = \Omega_\psi\, t + \Psi(t)$, where $\Psi(t) = \Psi(t+T)$ is periodic with period $T_\psi = 2\pi/\Omega_\psi$. This leads to heterodyning with a beat frequency $\Omega_\psi(\nu,\varepsilon)$.

Why do we here find the general resonance condition $\omega = \frac{p}{q}\,\omega_0$, whereas for weakly forced, weakly nonlinear oscillators resonance could only occur for $\omega = \omega_0$? There are two reasons. The main reason is that in the latter case, the limit cycle is harmonic to zeroth order, with $x_0(t) = A\,\cos(t+\phi)$. There are only two frequencies, then, in the Fourier decomposition of the limit cycle: $\omega_0 = \pm 1$. In the strongly nonlinear case, the limit cycle is decomposed into what is in general a countably infinite set of frequencies which are all multiples of a fundamental ω_0. In addition, if the forcing $\boldsymbol{f}(\boldsymbol{\varphi}, t)$ is periodic in t, its Fourier decomposition in t will involve all integer multiples of some fundamental ω. Thus, the most general resonance condition is $k\omega_0 + l\omega = 0$.

Our analysis has been limited to the lowest order in ε, and we have averaged out the nonresonant terms. When one systematically accounts for both these features, there are two main effects. One is that the boundaries of the synchronous region are no longer straight lines as depicted in the right panel of fig. 14.8. The boundaries themselves can be curved. Moreover, even if there are no resonant terms in the (k,l) sum to lowest order, they can be generated by going to higher order in ε. In such a case, the width of the synchronization region $\Delta\nu$ will be proportional to a higher power of ε: $\Delta\nu \propto \varepsilon^n$, where n is the order of ε where resonant forcing terms first appear in the analysis.

14.5 Relaxation Oscillations

We saw how to use multiple time scale analysis to identify the limit cycle of the van der Pol oscillator when ϵ is small. Consider now the opposite limit, where the coefficient of the damping term is very large. We generalize the van der Pol equation to

$$\ddot{x} + \mu\,\Phi(x)\,\dot{x} + x = 0 \quad , \tag{14.112}$$

and suppose $\mu \gg 1$. Define now the variable

$$\begin{aligned} y &\equiv \frac{\dot{x}}{\mu} + \int\limits_0^x \! dx'\,\Phi(x') \\ &= \frac{\dot{x}}{\mu} + F(x) \quad , \end{aligned} \tag{14.113}$$

where $F'(x) = \Phi(x)$. (y is sometimes called the *Lièǹard variable*, and (x, y) the *Lièǹard plane*.) Then the original second order equation may be written as two coupled first order equations:

$$\dot{x} = \mu\Big(y - F(x)\Big) \tag{14.114}$$

$$\dot{y} = -\frac{x}{\mu} \quad . \tag{14.115}$$

Since $\mu \gg 1$, the first of these equations is *fast* and the second one *slow*. The dynamics rapidly achieves $y \approx F(x)$, and then slowly evolves along the curve $y = F(x)$, until it is forced to make a large, fast excursion.

A concrete example is useful. Consider $F(x)$ of the form sketched in fig. 14.9. This is what one finds for the van der Pol oscillator, where $\Phi(x) = x^2 - 1$ and $F(x) = \frac{1}{3}x^3 - x$. The limit cycle behavior $x_{\rm LC}(t)$ is sketched in fig. 14.10. We assume $\Phi(x) = \Phi(-x)$ for simplicity.

Assuming $\Phi(x) = \Phi(-x)$ is symmetric, $F(x)$ is antisymmetric. For the van der Pol oscillator and other similar cases, $F(x)$ resembles the sketch in fig. 14.9. There are two local extrema: a local maximum at $x = -a$ and a local minimum at $x = +a$. We define b such that $F(b) = F(-a)$, as shown in the figure; antisymmetry then entails $F(-b) = F(+a)$. Starting from an arbitrary initial condition, the y dynamics are slow, since $\dot{y} = -\mu^{-1}x$ (we assume $\mu \gg x(0)$). So y can be regarded as essentially constant for the fast dynamics of eqn. 14.115, according to which $x(t)$ flows rapidly to the right if $y > F(x)$ and rapidly to the left if $y < F(x)$. This fast motion stops when $x(t)$ reaches a point where $y = F(x)$. At this point, the slow dynamics takes over. Assuming $y \approx F(x)$, we have

$$y \approx F(x) \qquad \Rightarrow \qquad \dot{y} = -\frac{x}{\mu} \approx F'(x)\,\dot{x} \quad , \tag{14.116}$$

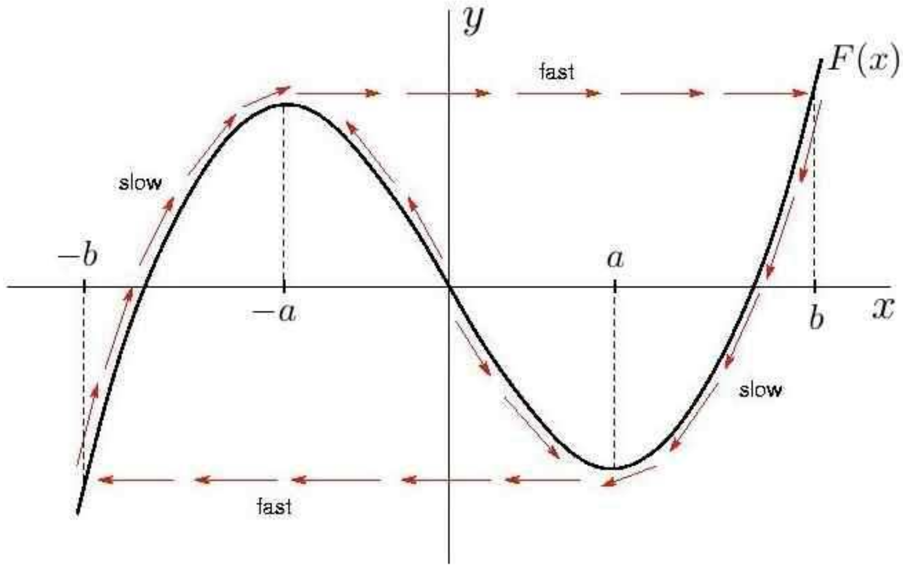

Figure 14.9: Relaxation oscillations in the so-called Liènard plane (x, y). The system rapidly flows to a point on the curve $y = F(x)$, and then crawls slowly along this curve. The slow motion takes x from $-b$ to $-a$, after which the system executes a rapid jump to $x = +b$, then a slow retreat to $x = +a$, followed by a rapid drop to $x = -b$.

which says that

$$\dot{x} \approx -\frac{x}{\mu\, F'(x)} \qquad \text{if } y \approx F(x) \tag{14.117}$$

over the slow segments of the motion, which are the regions $x \in [-b, -a]$ and $x \in [a, b]$. The relaxation oscillation is then as follows. Starting at $x = -b$, $x(t)$ increases slowly according to eqn. 14.117. At $x = -a$, the motion can no longer follow the curve $y = F(x)$, since $\dot{y} = -\mu^{-1}x$ is still positive. The motion thus proceeds quickly to $x = +b$, with

$$\dot{x} \approx \mu\Big(F(b) - F(x)\Big) \qquad x \in \big[-a, +b\big] \quad . \tag{14.118}$$

After reaching $x = +b$, the motion once again is slow, and again follows eqn. 14.117, according to which $x(t)$ now decreases slowly until it reaches $x = +a$, at which point the motion is again fast, with

$$\dot{x} \approx \mu\Big(F(a) - F(x)\Big) \qquad x \in \big[-b, +a\big] \quad . \tag{14.119}$$

The cycle then repeats.

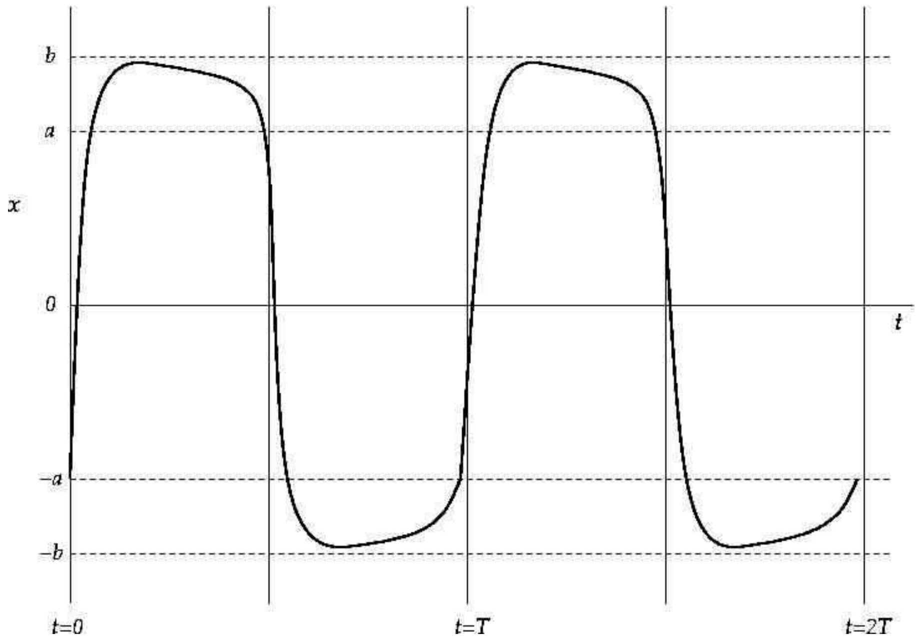

Figure 14.10: A sketch of the limit cycle for the relaxation oscillation studied in this section.

Thus, the limit cycle is given by the following segments:

$$x \in [-b,-a] \qquad (\dot{x} > 0) \ : \qquad \dot{x} \approx -\frac{x}{\mu\, F'(x)} \tag{14.120}$$

$$x \in [-a,b] \qquad (\dot{x} > 0) \ : \qquad \dot{x} \approx \mu\,\big[F(b) - F(x)\big] \tag{14.121}$$

$$x \in [a,b] \qquad (\dot{x} < 0) \ : \qquad \dot{x} \approx -\frac{x}{\mu\, F'(x)} \tag{14.122}$$

$$x \in [-b,a] \qquad (\dot{x} < 0) \ : \qquad \dot{x} \approx \mu\,\big[F(a) - F(x)\big] \quad . \tag{14.123}$$

A sketch of the limit cycle is given in fig. 14.11, showing the slow and fast portions.

When $\mu \gg 1$ we can determine approximately the period of the limit cycle. Clearly the period is twice the time for either of the slow portions, hence

$$T \approx 2\mu \int\limits_a^b \! dx\, \frac{\Phi(x)}{x} \quad , \tag{14.124}$$

where $F'(\pm a) = \Phi(\pm a) = 0$ and $F(\pm b) = F(\mp a)$. For the van der Pol oscillator, with $\Phi(x) = x^2 - 1$, we have $a = 1$, $b = 2$, and $T \simeq (3 - 2\ln 2)\,\mu$.

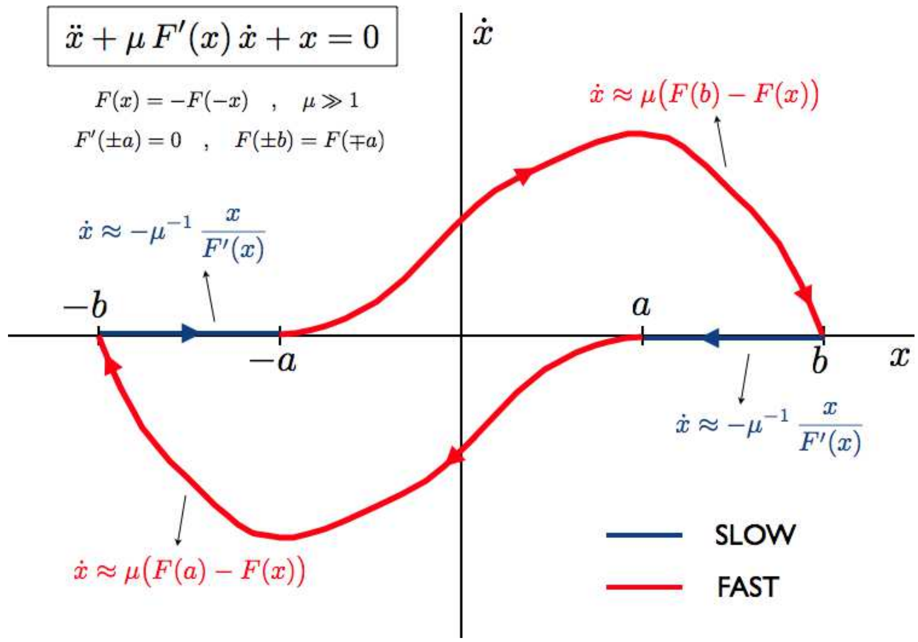

Figure 14.11: Limit cycle for large μ relaxation oscillations, shown in the phase plane $(x, \dot{x})$.

14.5.1 Example problem

Consider the equation

$$\ddot{x} + \mu\big(|x| - 1\big)\dot{x} + x = 0 \quad . \tag{14.125}$$

Sketch the trajectory in the Liènard plane, and find the approximate period of the limit cycle for $\mu \gg 1$.

Solution : We define

$$F'(x) = |x| - 1 \quad \Rightarrow \quad F(x) = \begin{cases} +\frac{1}{2}x^2 - x & \text{if } x > 0 \\ \\ -\frac{1}{2}x^2 - x & \text{if } x < 0 \quad . \end{cases} \tag{14.126}$$

We therefore have

$$\dot{x} = \mu\big\{y - F(x)\big\} \quad , \quad \dot{y} = -\frac{x}{\mu} \quad , \tag{14.127}$$

with $y \equiv \mu^{-1}\, \dot{x} + F(x)$.

Setting $F'(x) = 0$ we find $x = \pm a$, where $a = 1$ and $F(\pm a) = \mp\frac{1}{2}$. We also find $F(\pm b) = F(\mp a)$, where $b = 1 + \sqrt{2}$. Thus, the limit cycle is as follows: (i) fast motion from $x = -a$ to $x = +b$, (ii) slow relaxation from $x = +b$ to $x = +a$, (iii) fast motion from $x = +a$ to $x = -b$, and (iv) slow relaxation from $x = -b$ to $x = -a$. The period is approximately the time it takes for the slow portions of the cycle. Along these portions, we have $y \simeq F(x)$, and hence $\dot{y} \simeq F'(x)\, \dot{x}$. But $\dot{y} = -x/\mu$, so

$$F'(x)\, \dot{x} \simeq -\frac{x}{\mu} \quad \Rightarrow \quad dt = -\mu\, \frac{F'(x)}{x}\, dx \quad , \tag{14.128}$$

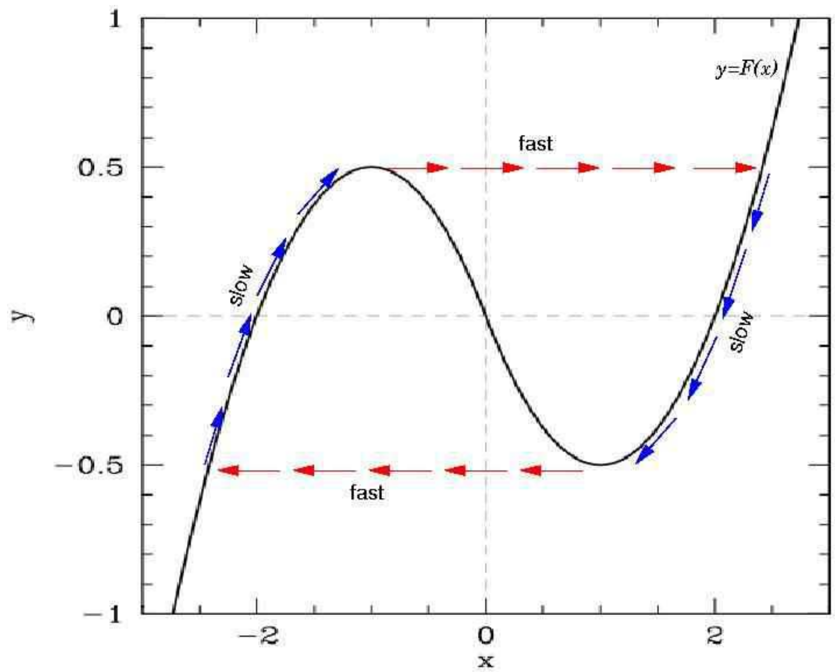

Figure 14.12: Relaxation oscillations for $\ddot{x} + \mu\big(|x| - 1\big)\,\dot{x} + x = 0$ plotted in the Liénard plane. The solid black curve is $y = F(x) = \frac{1}{2}x^2\,\text{sgn}(x) - x$. The variable y is defined to be $y = \mu^{-1}\,\dot{x} + F(x)$. Along slow portions of the limit cycle, $y \simeq F(x)$.

which we integrate to obtain

$$\begin{aligned} T &\simeq -2\mu\int\limits_b^a dx\,\frac{F'(x)}{x} = 2\mu\int\limits_1^{1+\sqrt{2}} dx\left(1 - \frac{1}{x}\right) \\ &= 2\mu\Big[\sqrt{2} - \ln\big(1+\sqrt{2}\big)\Big] \simeq 1.066\,\mu \quad . \end{aligned} \tag{14.129}$$

14.5.2 Multiple limit cycles

For the equation

$$\ddot{x} + \mu\,F'(x)\,\dot{x} + x = 0 \quad , \tag{14.130}$$

it is illustrative to consider what sort of $F(x)$ would yield more than one limit cycle. Such an example is shown in fig. 14.13.

In polar coordinates, it is very easy to construct such examples. Consider, for example, the system

$$\begin{aligned} \dot{r} &= \sin(\pi r) + \epsilon\cos\theta \\ \dot{\theta} &= b\,r \quad , \end{aligned} \tag{14.131}$$

with $|\epsilon| < 1$. First consider the case $\epsilon = 0$. Clearly the radial flow is outward for $\sin(\pi r) > 0$ and inward for $\sin(\pi r) < 0$. Thus, we have stable limit cycles at $r = 2n + 1$ and unstable limit cycles at $r = 2n$, for

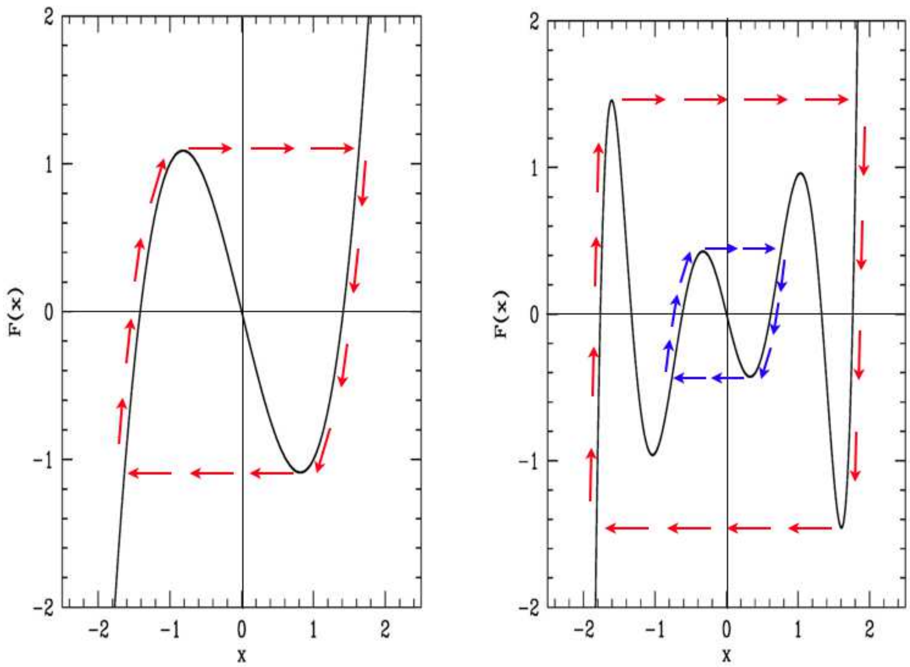

Figure 14.13: Liénard plots for systems with one (left) and two (right) relaxation oscillations.

all $n \in \mathbb{Z}$. With $0 < |\epsilon| < 1$, we have

$$\dot{r} > 0 \qquad \text{for} \qquad r \in \big[2n + \tfrac{1}{\pi}\sin^{-1}\epsilon\,,\, 2n + 1 - \tfrac{1}{\pi}\sin^{-1}\epsilon\big] \tag{14.132}$$

$$\dot{r} < 0 \qquad \text{for} \qquad r \in \big[2n + 1 + \tfrac{1}{\pi}\sin^{-1}\epsilon\,,\, 2n + 2 - \tfrac{1}{\pi}\sin^{-1}\epsilon\big] \tag{14.133}$$

The Poincaré-Bendixson theorem then guarantees the existence of stable and unstable limit cycles. We can put bounds on the radial extent of these limit cycles.

$$\text{stable limit cycle} \quad : \quad r \in \big[2n + 1 - \tfrac{1}{\pi}\sin^{-1}\epsilon\,,\, 2n + 1 + \tfrac{1}{\pi}\sin^{-1}\epsilon\big] \tag{14.134}$$

$$\text{unstable limit cycle} \quad : \quad r \in \big[2n - \tfrac{1}{\pi}\sin^{-1}\epsilon\,,\, 2n + \tfrac{1}{\pi}\sin^{-1}\epsilon\big] \tag{14.135}$$

Note that an unstable limit cycle is a repeller, which is to say that it is stable (an attractor) if we run the dynamics backwards, sending $t \to -t$.

14.5.3 Example problem

Consider the nonlinear oscillator,

$$\ddot{x} + \mu\,\Phi(x)\,\dot{x} + x = 0 \quad , \tag{14.136}$$

with $\mu \gg 1$. For each case in fig. 14.14, sketch the flow in the Liènard plane, starting with a few different initial conditions. For which case(s) do relaxation oscillations occur?

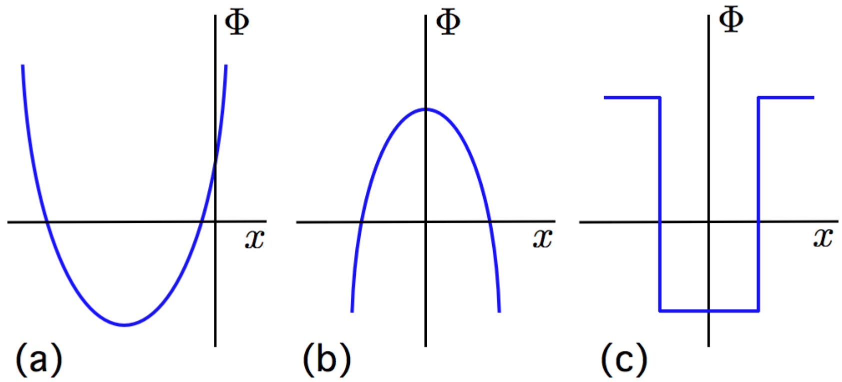

Figure 14.14: Three instances of $\Phi(x)$.

Solution : Recall the general theory of relaxation oscillations. We define

$$y \equiv \frac{\dot{x}}{\mu} + \int\limits_0^x dx' \, \Phi(x') = \frac{\dot{x}}{\mu} + F(x) \quad , \tag{14.137}$$

in which case the second order ODE for the oscillator may be written as two coupled first order ODEs:

$$\dot{y} = -\frac{x}{\mu} \quad , \quad \dot{x} = \mu\Big(y - F(x)\Big) \quad . \tag{14.138}$$

Since $\mu \gg 1$, the first of these equations is *slow* and the second one *fast*. The dynamics rapidly achieves $y \approx F(x)$, and then slowly evolves along the curve $y = F(x)$, until it is forced to make a large, fast excursion.

To explore the dynamics in the Liènard plane, we plot $F(x)$ *versus* x, which means we must integrate $\Phi(x)$. This is done for each of the three cases in fig. 14.14.

Note that a fixed point corresponds to $x = 0$ and $\dot{x} = 0$. In the Liènard plane, this means $x = 0$ and $y = F(0)$. Linearizing by setting $x = \delta x$ and $y = F(0) + \delta y$, we have[1]

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \mu\,\delta y - \mu F'(0)\,\delta x \\ -\mu^{-1}\,\delta x \end{pmatrix} = \begin{pmatrix} -\mu F'(0) & \mu \\ -\mu^{-1} & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \quad . \tag{14.139}$$

The linearized map has trace $T = -\mu\, F'(0)$ and determinant $D = 1$. Since $\mu \gg 1$ we have $0 < D < \frac{1}{4}T^2$, which means the fixed point is either a stable node, for $F'(0) > 0$, or an unstable node, for $F'(0) < 0$. In cases (a) and (b) the fixed point is a stable node, while in case (c) it is unstable. The flow in case (a) always collapses to the stable node. In case (b) the flow either is unbounded or else it collapses to the stable node. In case (c), all initial conditions eventually flow to a unique limit cycle exhibiting relaxation oscillations.

[1]We could, of course, linearize about the fixed point in $(x, \dot{x})$ space and obtain the same results.

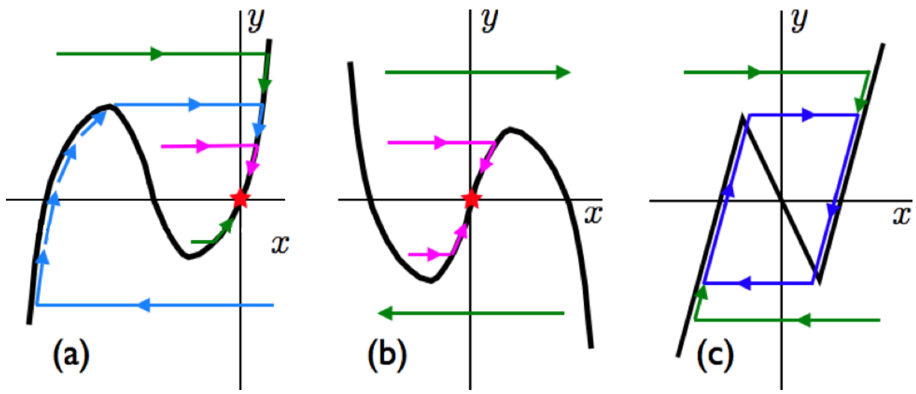

Figure 14.15: Phase flows in the Liénard plane for the three examples in fig. 14.14.

14.6 Appendix I : Multiple Time Scale Analysis to $\mathcal{O}(\epsilon^2)$

Problem : A particle of mass m moves in one dimension subject to the potential

$$U(x) = \tfrac{1}{2}m\,\omega_0^2\,x^2 + \tfrac{1}{3}\epsilon\,m\,\omega_0^2\,\frac{x^3}{a} \quad , \tag{14.140}$$

where ϵ is a dimensionless parameter.

(a) Find the equation of motion for x. Show that by rescaling x and t you can write this equation in dimensionless form as

$$\frac{d^2u}{ds^2} + u = -\epsilon u^2 \quad . \tag{14.141}$$

Solution : The equation of motion is

$$\begin{aligned} m\ddot{x} &= -U'(x) \\ &= -m\omega_0^2 x - \epsilon m\omega_0^2\,\frac{x^2}{a} \quad . \end{aligned} \tag{14.142}$$

We now define $s \equiv \omega_0 t$ and $u \equiv x/a$, yielding

$$\frac{d^2u}{ds^2} + u = -\epsilon u^2 \quad . \tag{14.143}$$

(b) You are now asked to perform an $\mathcal{O}(\epsilon^2)$ multiple time scale analysis of this problem, writing

$$T_0 = s \quad , \quad T_1 = \epsilon s \quad , \quad T_2 = \epsilon^2 s \quad ,$$

and

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots \quad .$$

This results in a hierarchy of coupled equations for the functions $\{u_n\}$. Derive the first three equations in the hierarchy.

Solution : We have

$$\frac{d}{ds} = \frac{\partial}{\partial T_0} + \epsilon\, \frac{\partial}{\partial T_1} + \epsilon^2\, \frac{\partial}{\partial T_2} + \ldots \quad . \tag{14.144}$$

Therefore

$$\bigg(\frac{\partial}{\partial T_0} + \epsilon\, \frac{\partial}{\partial T_1} + \epsilon^2\, \frac{\partial}{\partial T_2} + \ldots\bigg)^2 \Big(u_0 + \epsilon\, u_1 + \epsilon^2\, u_2 + \ldots\Big) + \Big(u_0 + \epsilon\, u_1 + \epsilon^2\, u_2 + \ldots\Big) \\ = -\epsilon\,\Big(u_0 + \epsilon\, u_1 + \epsilon^2\, u_2 + \ldots\Big)^2 \quad . \tag{14.145}$$

Expanding and then collecting terms order by order in ϵ, we derive the hierarchy. The first three levels are

$$\frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \tag{14.146}$$

$$\frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\,\frac{\partial^2 u_0}{\partial T_0\,\partial T_1} - u_0^2 \tag{14.147}$$

$$\frac{\partial^2 u_2}{\partial T_0^2} + u_2 = -2\,\frac{\partial^2 u_0}{\partial T_0\,\partial T_2} - \frac{\partial^2 u_0}{\partial T_1^2} - 2\,\frac{\partial^2 u_1}{\partial T_0\,\partial T_1} - 2\,u_0\,u_1 \quad . \tag{14.148}$$

(c) Show that there is no frequency shift to first order in ϵ.

Solution : At the lowest (first) level of the hierarchy, the solution is

$$u_0 = A(T_1,T_2)\,\cos\big(T_0 + \phi(T_1,T_2)\big) \quad . \tag{14.149}$$

At the second level, then,

$$\frac{\partial^2 u_1}{\partial T_0^2} + u_1 = 2\,\frac{\partial A}{\partial T_1}\,\sin(T_0+\phi) + 2A\,\frac{\partial \phi}{\partial T_1}\,\cos(T_0+\phi) - A^2\cos^2(T_0+\phi) \quad . \tag{14.150}$$

We eliminate the resonant forcing terms on the RHS by demanding

$$\frac{\partial A}{\partial T_1} = 0 \qquad \text{and} \qquad \frac{\partial \phi}{\partial T_1} = 0 \quad . \tag{14.151}$$

Thus, we must have $A = A(T_2)$ and $\phi = \phi(T_2)$. To $\mathcal{O}(\epsilon)$, then, ϕ is a constant, which means there is no frequency shift at this level of the hierarchy.

(d) Find $u_0(s)$ and $u_1(s)$.

Solution :The equation for u_1 is that of a non-resonantly forced harmonic oscillator. The solution is easily found to be

$$u_1 = -\tfrac{1}{2}A^2 + \tfrac{1}{6}A^2\cos(2T_0 + 2\phi) \quad . \tag{14.152}$$

We now insert this into the RHS of the third equation in the hierarchy:

$$\begin{aligned}\frac{\partial^2 u_2}{\partial T_0^2} + u_2 &= -2\frac{\partial^2 u_0}{\partial T_0\,\partial T_2} - 2\,u_0\,u_1 \\ &= 2\,\frac{\partial A}{\partial T_2}\,\sin(T_0+\phi) + 2A\,\frac{\partial \phi}{\partial T_2}\,\cos(T_0+\phi) - 2A\cos(T_0+\phi)\,\Big\{ -\tfrac{1}{2}A^2 + \tfrac{1}{6}A^2\cos(2T_0+2\phi)\Big\} \\ &= 2\,\frac{\partial A}{\partial T_2}\,\sin(T_0+\phi) + \Big(2A\,\frac{\partial \phi}{\partial T_2} + \tfrac{5}{6}A^3\Big)\cos(T_0+\phi) - \tfrac{1}{6}A^3\,\cos(3T_0+3\phi) \quad .\end{aligned} \tag{14.153}$$

Setting the coefficients of the resonant terms on the RHS to zero yields

$$\begin{aligned}\frac{\partial A}{\partial T_2} = 0 \quad &\Rightarrow \quad A = A_0 \\ 2A\,\frac{\partial \phi}{\partial T_2} + \tfrac{5}{6}A^3 = 0 \quad &\Rightarrow \quad \phi = -\tfrac{5}{12}\,A_0^2\,T_2 \quad .\end{aligned} \tag{14.154}$$

Therefore,

$$u(s) \;= \overbrace{A_0\,\cos\big(s - \tfrac{5}{12}\,\epsilon^2 A_0^2\,s\big)}^{u_0(s)} \;+\; \overbrace{\tfrac{1}{6}\,\epsilon\,A_0^2\,\cos\big(2s - \tfrac{5}{6}\,\epsilon^2 A_0^2\,s\big) - \tfrac{1}{2}\,\epsilon\,A_0^2}^{\epsilon\,u_1(s)} + \mathcal{O}\big(\epsilon^2\big) \tag{14.155}$$

14.7 Appendix II : MSA and Poincaré-Lindstedt Methods

14.7.1 Problem using multiple time scale analysis

Consider the central force law $F(r) = -k\,r^{\beta^2-3}$.

(a) Show that a stable circular orbit exists at radius $r_0 = (\ell^2/\mu k)^{1/\beta^2}$.

Solution : For a circular orbit, the effective radial force must vanish:

$$F_{\rm eff}(r) = \frac{\ell^2}{\mu r^3} + F(r) = \frac{\ell^2}{\mu r^3} - \frac{k}{r^{3-\beta^2}} = 0 \quad . \tag{14.156}$$

Solving for $r = r_0$, we have $r_0 = (\ell^2/\mu k)^{1/\beta^2}$. The second derivative of $U_{\rm eff}(r)$ at this point is

$$U''_{\rm eff}(r_0) = -F'_{\rm eff}(r_0) = \frac{3\ell^2}{\mu r_0^4} + (\beta^2 - 3)\,\frac{k}{r_0^{4-\beta^2}} = \frac{\beta^2\ell^2}{\mu r_0^4} \quad , \tag{14.157}$$

which is manifestly positive. Thus, the circular orbit at $r = r_0$ is stable.

(b) Show that the geometric equation for the shape of the orbit may be written

$$\frac{d^2 s}{d\phi^2} + s = K(s) \tag{14.158}$$

where $s = 1/r$, and

$$K(s) = s_0 \left(\frac{s}{s_0}\right)^{1-\beta^2} \quad , \tag{14.159}$$

with $s_0 = 1/r_0$.

Solution : We have previously derived (*e.g.* in the notes) the equation

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\, F(s^{-1}) \quad . \tag{14.160}$$

From the given $F(r)$, we then have

$$\frac{d^2 s}{d\phi^2} + s = \frac{\mu k}{\ell^2}\, s^{1-\beta^2} \equiv K(s) \quad , \tag{14.161}$$

where $s_0 \equiv (\mu k/\ell^2)^{1/\beta^2} = 1/r_0$, and where

$$K(s) = s_0 \left(\frac{s}{s_0}\right)^{1-\beta^2} \quad . \tag{14.162}$$

(c) Writing $s \equiv (1+u)\, s_0$, show that u satisfies

$$\frac{1}{\beta^2}\,\frac{d^2 u}{d\phi^2} + u = a_1\, u^2 + a_2\, u^3 + \ldots \quad . \tag{14.163}$$

Find a_1 and a_2.

Solution : Writing $s \equiv s_0\,(1+u)$, we have

$$\begin{aligned} \frac{d^2 u}{d\phi^2} + 1 + u &= (1+u)^{1-\beta^2} \\ &= 1 + (1-\beta^2)\, u + \tfrac{1}{2}(-\beta^2)(1-\beta^2)\, u^2 \\ &\qquad + \tfrac{1}{6}(-1-\beta^2)(-\beta^2)(1-\beta^2)\, u^3 + \ldots \quad . \end{aligned} \tag{14.164}$$

Thus,

$$\frac{1}{\beta^2}\,\frac{d^2 u}{d\phi^2} + u = a_1\, u^2 + a_2\, u^3 + \ldots \quad , \tag{14.165}$$

where

$$a_1 = -\tfrac{1}{2}(1-\beta^2) \quad , \quad a_2 = \tfrac{1}{6}(1-\beta^4) \quad . \tag{14.166}$$

(d) Now let us associate a power of ε with each power of the deviation u and write

$$\frac{1}{\beta^2}\,\frac{d^2 u}{d\phi^2} + u = \varepsilon\, a_1\, u^2 + \varepsilon^2\, a_2\, u^3 + \ldots \quad , \tag{14.167}$$

Solve this equation using the method of multiple scale analysis (MSA). You will have to go to second order in the multiple scale expansion, writing

$$X \equiv \beta\phi \quad , \quad Y \equiv \varepsilon\, \beta\phi \quad , \quad Z \equiv \varepsilon^2\, \beta\phi \tag{14.168}$$

and hence

$$\frac{1}{\beta}\,\frac{d}{d\phi} = \frac{\partial}{\partial X} + \varepsilon\,\frac{\partial}{\partial Y} + \varepsilon^2\,\frac{\partial}{\partial Z} + \ldots \quad . \tag{14.169}$$

Further writing

$$u = u_0 + \varepsilon\, u_1 + \varepsilon^2\, u_2 + \ldots \quad , \tag{14.170}$$

derive the equations for the multiple scale analysis, up to second order in ε.

Solution : We now associate one power of ε with each additional power of u beyond order u^1. In this way, a uniform expansion in terms of ε will turn out to be an expansion in powers of the amplitude of the oscillations. We'll see how this works below. We then have

$$\frac{1}{\beta^2}\,\frac{d^2u}{d\phi^2} + u = a_1\,\varepsilon\, u^2 + a_2\,\varepsilon^2\, u^3 + \ldots \quad , \tag{14.171}$$

with $\varepsilon = 1$. We now perform a multiple scale analysis, writing

$$X \equiv \beta\phi \quad , \quad Y \equiv \varepsilon\, \beta\phi \quad , \quad Z \equiv \varepsilon^2\, \beta\phi \quad . \tag{14.172}$$

This entails

$$\frac{1}{\beta}\,\frac{d}{d\phi} = \frac{\partial}{\partial X} + \varepsilon\,\frac{\partial}{\partial Y} + \varepsilon^2\,\frac{\partial}{\partial Z} + \ldots \quad . \tag{14.173}$$

We also expand u in powers of ε, as

$$u = u_0 + \varepsilon\, u_1 + \varepsilon^2\, u_2 + \ldots \quad . \tag{14.174}$$

Thus, we obtain

$$\begin{aligned} \big(\partial_X + \varepsilon\,\partial_Y + \varepsilon^2\,\partial_Z + \ldots\big)^2 \,(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) + (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) \\ = \varepsilon\, a_1\,(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)^2 + \varepsilon^2\, a_2\,(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)^3 + \ldots \quad . \end{aligned} \tag{14.175}$$

We now extract a hierarchy of equations, order by order in powers of ε.

We find, out to order ε^2,

$$\mathcal{O}(\varepsilon^0): \quad \frac{\partial^2 u_0}{\partial X^2} + u_0 = 0 \tag{14.176}$$

$$\mathcal{O}(\varepsilon^1): \quad \frac{\partial^2 u_1}{\partial X^2} + u_1 = -2\frac{\partial^2 u_0}{\partial Y\,\partial X} + a_1\, u_0^2 \tag{14.177}$$

$$\mathcal{O}(\varepsilon^2): \quad \frac{\partial^2 u_2}{\partial X^2} + u_2 = -2\frac{\partial^2 u_0}{\partial Z\,\partial X} - \frac{\partial^2 u_0}{\partial Y^2} - 2\frac{\partial^2 u_1}{\partial Z\,\partial X} + 2a_1\, u_0\, u_1 + a_2\, u_0^3 \quad . \tag{14.178}$$

(e) Show that there is no shift of the angular period $\Delta\phi = 2\pi/\beta$ if one works only to leading order in ε.

Solution : The $\mathcal{O}(\varepsilon^0)$ equation in the hierarchy is solved by writing

$$u_0 = A\,\cos(X + \psi) \quad , \tag{14.179}$$

where

$$A = A(Y, Z) \quad , \quad \psi = \psi(Y, Z) \quad . \tag{14.180}$$

We define $\theta \equiv X + \psi(Y, Z)$, so we may write $u_0 = A\cos\theta$. At the next order, we obtain

$$\begin{aligned}\frac{\partial^2 u_1}{\partial\theta^2} + u_1 &= 2\frac{\partial A}{\partial Y}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Y}\,\cos\theta + a_1\,A^2\cos\theta \\ &= 2\frac{\partial A}{\partial Y}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Y}\,\cos\theta + \tfrac{1}{2}a_1A^2 + \tfrac{1}{2}a_1A^2\,\cos 2\theta \quad .\end{aligned} \tag{14.181}$$

In order that there be no resonantly forcing terms on the RHS of eqn. 14.181, we demand

$$\frac{\partial A}{\partial Y} = 0 \quad , \quad \frac{\partial\psi}{\partial Y} = 0 \quad \Rightarrow \quad A = A(Z) \quad , \quad \psi = \psi(Z) \quad . \tag{14.182}$$

The solution for u_1 is then

$$u_1(\theta) = \tfrac{1}{2}a_1A^2 - \tfrac{1}{6}a_1A^2\cos 2\theta \quad . \tag{14.183}$$

Were we to stop at this order, we could ignore $Z = \varepsilon^2\beta\phi$ entirely, since it is of order ε^2, and the solution would be

$$u(\phi) = A_0\cos(\beta\phi + \psi_0) + \tfrac{1}{2}\varepsilon a_1A_0^2 - \tfrac{1}{6}\varepsilon a_1A_0^2\cos(2\beta\phi + 2\psi_0) \quad . \tag{14.184}$$

The angular period is still $\Delta\phi = 2\pi/\beta$, and, starting from a small amplitude solution at order ε^0 we find that to order ε we must add a constant shift proportional to A_0^2, as well as a second harmonic term, also proportional to A_0^2.

(f) Carrying out the MSA to second order in ε, show that the shift of the angular period vanishes only if $\beta^2 = 1$ or $\beta^2 = 4$.

Solution : Carrying out the MSA to the next order, $\mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{aligned}\frac{\partial^2 u_2}{\partial\theta^2} + u_2 &= 2\frac{\partial A}{\partial Z}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Z}\,\cos\theta + 2a_1A\cos\theta\,\big(\tfrac{1}{2}a_1A^2 - \tfrac{1}{6}a_1A^2\cos 2\theta\big) + a_2A^3\cos^3\!\theta \\ &= 2\frac{\partial A}{\partial Z}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Z}\,\cos\theta + \big(\tfrac{5}{6}a_1^2 + \tfrac{3}{4}a_2\big)\,A^3\cos\theta + \big(-\tfrac{1}{6}a_1^2 + \tfrac{1}{4}a_2\big)\,A^3\cos 3\theta \quad .\end{aligned} \tag{14.185}$$

Now in order to make the resonant forcing terms on the RHS vanish, we must choose

$$\frac{\partial A}{\partial Z} = 0 \tag{14.186}$$

as well as

$$\begin{aligned}\frac{\partial\psi}{\partial Z} &= -\big(\tfrac{5}{12}a_1^2 + \tfrac{3}{8}a_2\big)\,A^2 \\ &= -\tfrac{1}{24}(\beta^2 - 4)(\beta^2 - 1) \quad .\end{aligned} \tag{14.187}$$

The solutions to these equations are trivial:

$$A(Z) = A_0 \quad , \quad \psi(Z) = \psi_0 - \tfrac{1}{24}(\beta^2-1)(\beta^2-4)A_0^2\, Z \quad . \tag{14.188}$$

With the resonant forcing terms eliminated, we may write

$$\frac{\partial^2 u_2}{\partial\theta^2} + u_2 = \big(-\tfrac{1}{6}a_1^2 + \tfrac{1}{4}a_2\big)\, A^3 \cos 3\theta \quad , \tag{14.189}$$

with solution

$$\begin{aligned} u_2 &= \tfrac{1}{96}(2a_1^2 - 3a_2)\, A^3\, \cos 3\theta \\ &= \tfrac{1}{96}\, \beta^2\, (\beta^2-1)\, A_0^2\, \cos\big(3X + 3\psi(Z)\big) \quad . \end{aligned} \tag{14.190}$$

The full solution to second order in this analysis is then

$$\begin{aligned} u(\phi) = A_0 \cos(\beta'\phi + \psi_0) &+ \tfrac{1}{2}\varepsilon a_1 A_0^2 - \tfrac{1}{6}\varepsilon a_1 A_0^2 \cos(2\beta'\phi + 2\psi_0) \\ &+ \tfrac{1}{96}\varepsilon^2\, (2a_1^2 - 3a_2)\, A_0^3\, \cos(3\beta'\phi + 3\psi_0) \quad . \end{aligned} \tag{14.191}$$

with

$$\beta' = \beta \cdot \Big\{ 1 - \tfrac{1}{24}\, \varepsilon^2\, (\beta^2-1)(\beta^2-4)A_0^2 \Big\} \quad . \tag{14.192}$$

The angular period shifts:

$$\Delta\phi = \frac{2\pi}{\beta'} = \frac{2\pi}{\beta} \cdot \Big\{ 1 + \tfrac{1}{24}\, \varepsilon^2\, (\beta^2-1)(\beta^2-4)A_0^2 \Big\} + \mathcal{O}(\varepsilon^3) \quad . \tag{14.193}$$

Note that there is no shift in the period, for any amplitude, if $\beta^2 = 1$ (*i.e.* Kepler potential) or $\beta^2 = 4$ (*i.e.* harmonic oscillator).

14.7.2 Solution using Poincaré-Lindstedt method

Recall that geometric equation for the shape of the (relative coordinate) orbit for the two body central force problem is

$$\begin{aligned} \frac{d^2 s}{d\phi^2} + s &= K(s) \\ K(s) &= s_0 \bigg(\frac{s}{s_0}\bigg)^{1-\beta^2} \end{aligned} \tag{14.194}$$

where $s = 1/r$, $s_0 = (l^2/\mu k)^{1/\beta^2}$ is the inverse radius of the stable circular orbit, and $f(r) = -kr^{\beta^2-3}$ is the central force. Expanding about the stable circular orbit, one has

$$\frac{d^2 y}{d\phi^2} + \beta^2\, y = \tfrac{1}{2}K''(s_0)\, y^2 + \tfrac{1}{6}K'''(s_0)\, y^3 + \ldots \quad , \tag{14.195}$$

where $s = s_0(1+y)$, with

$$\begin{aligned} K'(s) &= (1-\beta^2)\left(\frac{s_0}{s}\right)^{\beta^2} \\ K''(s) &= -\beta^2\,(1-\beta^2)\left(\frac{s_0}{s}\right)^{1+\beta^2} \\ K'''(s) &= \beta^2\,(1-\beta^2)\,(1+\beta^2)\left(\frac{s_0}{s}\right)^{2+\beta^2} \quad . \end{aligned} \tag{14.196}$$

Thus,

$$\frac{d^2y}{d\phi^2} + \beta^2\,y = \epsilon\,a_1\,y^2 + \epsilon^2\,a_2\,y^3 \quad , \tag{14.197}$$

with $\epsilon = 1$ and

$$\begin{aligned} a_1 &= -\tfrac{1}{2}\,\beta^2\,(1-\beta^2) \\ a_2 &= +\tfrac{1}{6}\,\beta^2\,(1-\beta^2)\,(1+\beta^2) \quad . \end{aligned} \tag{14.198}$$

Note that we assign one factor of ϵ for each order of nonlinearity beyond order y^1. Note also that while y here corresponds to u in eqn. 14.165, the constants $a_{1,2}$ here are a factor of β^2 larger than those defined in eqn. 14.166.

We now apply the Poincaré-Lindstedt method, by defining $\theta = \Omega\,\phi$, with

$$\Omega^2 = \Omega_0^2 + \epsilon\,\Omega_1^2 + \epsilon^2\,\Omega_2^2 + \ldots \tag{14.199}$$

and

$$y(\theta) = y_0(\theta) + \epsilon\,y_1(\theta) + \epsilon^2\,y_2(\theta) + \ldots \quad . \tag{14.200}$$

We therefore have

$$\frac{d}{d\phi} = \Omega\,\frac{d}{d\theta} \tag{14.201}$$

and

$$\begin{aligned} \big(\Omega_0^2 + \epsilon\,\Omega_1^2 + \epsilon^2\,\Omega_2^2 + \ldots\big)\big(y_0'' + \epsilon\,y_1'' + \epsilon^2\,y_2'' + \ldots\big) + \beta^2\,\big(y_0 + \epsilon\,y_1 + \epsilon^2\,y_2 + \ldots\big) \\ = \epsilon\,a_1\,\big(y_0 + \epsilon\,y_1 + \epsilon^2\,y_2 + \ldots\big)^2 + \epsilon^2\,a_2\,\big(y_0 + \epsilon\,y_1 + \epsilon^2\,y_2 + \ldots\big)^3 \quad . \end{aligned} \tag{14.202}$$

We now extract equations at successive orders of ϵ. The first three in the hierarchy are

$$\begin{aligned} \Omega_0^2\,y_0'' + \beta^2\,y_0 &= 0 \\ \Omega_1^2\,y_0'' + \Omega_0^2\,y_1'' + \beta^2\,y_1 &= a_1 y_0^2 \\ \Omega_2^2\,y_0'' + \Omega_1^2\,y_1'' + \Omega_0^2\,y_2'' + \beta^2\,y_2 &= 2\,a_1\,y_0\,y_1 + a_2\,y_0^3 \quad , \end{aligned} \tag{14.203}$$

where prime denotes differentiation with respect to θ.

To order ϵ^0, the solution is $\Omega_0^2 = \beta^2$ and

$$y_0(\theta) = A\,\cos(\theta+\delta) \quad , \tag{14.204}$$

where A and δ are constants.

At order ϵ^1, we have

$$\begin{aligned} \beta^2\big(y_1''+y_1\big) &= -\Omega_1^2\,y_0'' + a_1\,y_0^2 \\ &= \Omega_1^2\,A\,\cos(\theta+\delta) + a_1 A^2\cos^2(\theta+\delta) \\ &= \Omega_1^2\,A\,\cos(\theta+\delta) + \tfrac{1}{2}\,a_1 A^2 + \tfrac{1}{2}\,a_1 A^2\cos(2\theta+2\delta) \quad . \end{aligned} \tag{14.205}$$

The secular forcing terms on the RHS are eliminated by the choice $\Omega_1^2 = 0$. The solution is then

$$y_1(\theta) = \frac{a_1 A^2}{2\,\beta^2}\Big\{1 - \tfrac{1}{3}\cos(2\theta+2\delta)\Big\} \quad . \tag{14.206}$$

At order ϵ^2, then, we have

$$\begin{aligned} \beta^2\big(y_2''+y_2\big) &= -\Omega_2^2\,y_0'' - \Omega_1^2\,y_1'' + 2\,a_1\,y_1\,y_1 + a_2\,y_0^3 \\ &= \Omega_2^2\,A\cos(\theta+\delta) + \frac{a_1^2 A^3}{\beta^2}\Big\{1-\tfrac{1}{3}\cos(2\theta+2\delta)\Big\}\cos(\theta+\delta) + a_2 A^3\cos^2(\theta+\delta) \\ &= \bigg\{\Omega_2^2 + \frac{5\,a_1^2 A^3}{6\beta^2} + \tfrac{3}{4}\,a_2 A^3\bigg\}A\cos(\theta+\delta) + \bigg\{-\frac{a_1^2 A^3}{6\beta^2} + \tfrac{1}{4}\,a_2 A^3\bigg\}\cos(3\theta+3\delta) \quad . \end{aligned} \tag{14.207}$$

The resonant forcing terms on the RHS are eliminated by the choice

$$\begin{aligned} \Omega_2^2 &= -\Big(\tfrac{5}{6}\,\beta^{-2}\,a_1^2 + \tfrac{3}{4}\,a_2\Big)A^3 \\ &= -\tfrac{1}{24}\,\beta^2\,(1-\beta^2)\,\Big[5\,(1-\beta^2) + 3\,(1+\beta^2)\Big] \\ &= -\tfrac{1}{12}\,\beta^2\,(1-\beta^2)\,(4-\beta^2) \quad . \end{aligned} \tag{14.208}$$

Thus, the frequency shift to this order vanishes whenever $\beta^2 = 0$, $\beta^2 = 1$, or $\beta^2 = 4$. Recall the force law is $F(r) = -C\,r^{\beta^2-3}$, so we see that there is no shift – hence no precession – for inverse cube, inverse square, or linear forces.

14.8 Appendix III : Modified van der Pol Oscillator

Consider the nonlinear oscillator

$$\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0 \quad . \tag{14.209}$$

Analyze this using the same approach we apply to the van der Pol oscillator.

(a) Sketch the vector field $\dot{\varphi}$ for this problem. It may prove convenient to first identify the *nullclines*, which are the curves along which $\dot{x} = 0$ or $\dot{v} = 0$ (with $v = \dot{x}$). Argue that a limit cycle exists.

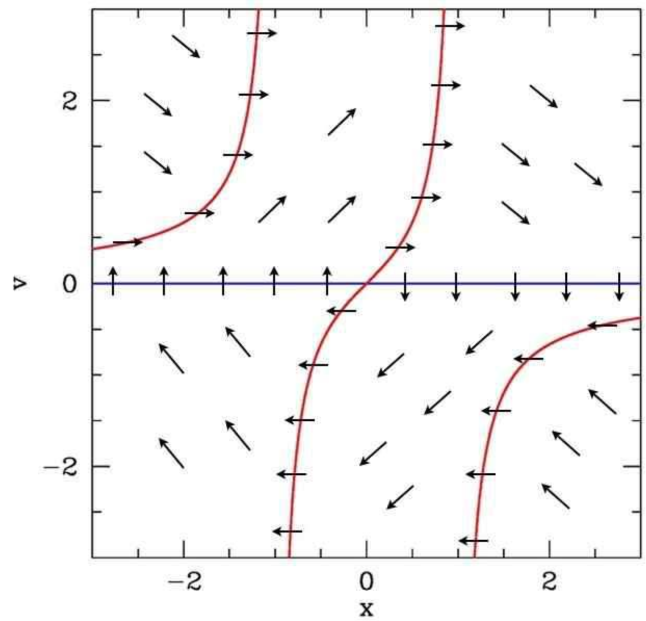

Figure 14.16: Sketch of phase flow and nullclines for the oscillator $\ddot{x}+\epsilon\,(x^4-1)\,\dot{x}+x=0$. Red nullclines: $\dot{v}=0$; blue nullcline: $\dot{x}=0$.

Solution : There is a single fixed point, at the origin $(0,0)$, for which the linearized dynamics obeys

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \mathcal{O}(x^4\,v) \quad . \tag{14.210}$$

One finds $T=\epsilon$ and $D=1$ for the trace and determinant, respectively. The origin is an unstable spiral for $0<\epsilon<2$ and an unstable node for $\epsilon>2$.

The nullclines are sketched in fig. 14.16. One has

$$\dot{x}=0 \;\leftrightarrow\; v=0 \qquad , \qquad \dot{v}=0 \;\leftrightarrow\; v=\frac{1}{\epsilon}\,\frac{x}{1-x^4} \quad . \tag{14.211}$$

The flow at large distances from the origin winds once around the origin and spirals in. The flow close to the origin spirals out ($\epsilon<2$) or flows radially out ($\epsilon>2$). Ultimately the flow must collapse to a limit cycle, as can be seen in the accompanying figures.

(b) In the limit $0<\varepsilon\ll 1$, use multiple time scale analysis to obtain a solution which reveals the approach to the limit cycle.

Solution : We seek to solve the equation

$$\ddot{x}+x=\epsilon\,h(x,\dot{x}) \quad , \tag{14.212}$$

with

$$h(x,\dot{x})=(1-x^4)\,\dot{x} \quad . \tag{14.213}$$

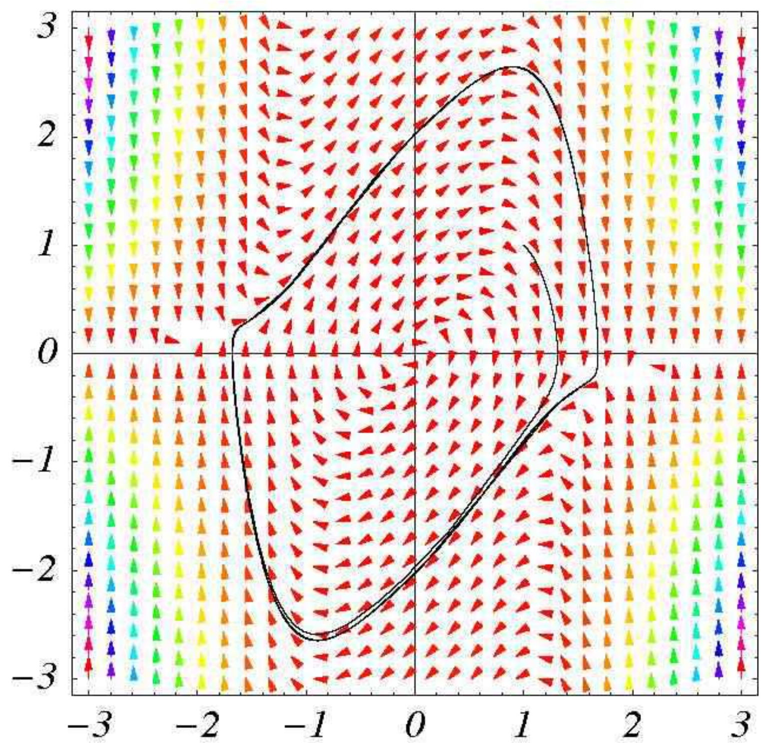

Figure 14.17: Vector field and phase curves for the oscillator $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$, with $\epsilon = 1$ and starting from $(x_0, v_0) = (1, 1)$.

Employing the multiple time scale analysis to lowest nontrivial order, we write $T_0 \equiv t$, $T_1 \equiv \epsilon t$,

$$x = x_0 + \epsilon x_1 + \ldots \tag{14.214}$$

and identify terms order by order in ϵ. At $\mathcal{O}(\epsilon^0)$, this yields

$$\frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \qquad \Rightarrow \qquad x_0 = A\cos(T_0 + \phi) \quad , \tag{14.215}$$

where $A = A(T_1)$ and $\phi = \phi(T_1)$. At $\mathcal{O}(\epsilon^1)$, we have

$$\begin{aligned} \frac{\partial^2 x_1}{\partial T_0^2} + x_1 &= -2\,\frac{\partial^2 x_0}{\partial T_0\,\partial T_1} + h\bigg(x_0\,,\,\frac{\partial x_0}{\partial T_0}\bigg) \\ &= 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + h\big(A\cos\theta, -A\sin\theta\big) \end{aligned} \tag{14.216}$$

with $\theta = T_0 + \phi(T_1)$ as usual. We also have

$$\begin{aligned} h(A\cos\theta, -A\sin\theta\big) &= A^5\sin\theta\cos\theta - A\sin\theta \\ &= \big(\tfrac{1}{8}A^5 - A\big)\,\sin\theta + \tfrac{3}{16}\,A^5\sin 3\theta + \tfrac{1}{16}\,A^5\sin 5\theta \quad . \end{aligned} \tag{14.217}$$

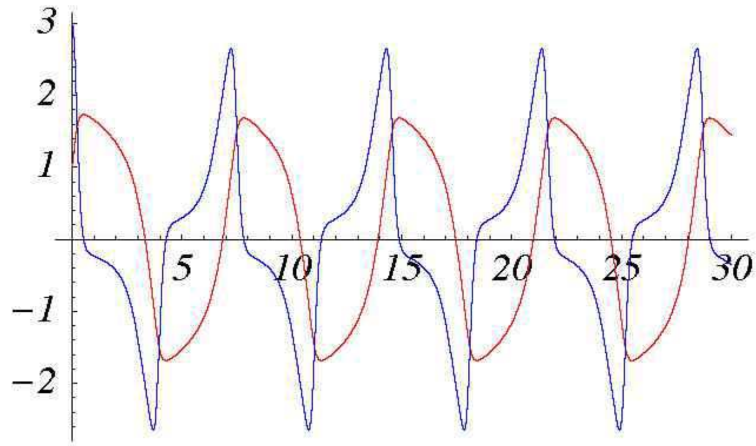

Figure 14.18: Solution to the oscillator equation $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$ with $\epsilon = 1$ and initial conditions $(x_0, v_0) = (1, 3)$. $x(t)$ is shown in red and $v(t)$ in blue. Note that $x(t)$ resembles a relaxation oscillation for this moderate value of ϵ.

To eliminate the resonant terms in eqn. 14.216, we must choose

$$\frac{\partial A}{\partial T_1} = \tfrac{1}{2}A - \tfrac{1}{16}A^5 \quad , \quad \frac{\partial \phi}{\partial T_1} = 0 \quad . \tag{14.218}$$

The A equation is similar to the logistic equation. Clearly $A = 0$ is an unstable fixed point, and $A = 8^{1/4} \approx 1.681793$ is a stable fixed point. Thus, the amplitude of the oscillations will asymptotically approach $A^* = 8^{1/4}$. (Recall the asymptotic amplitude in the van der Pol case was $A^* = 2$.)

To integrate the A equation, substitute $y = \frac{1}{\sqrt{8}}A^2$, and obtain

$$dT_1 = \frac{dy}{y\,(1 - y^2)} = \tfrac{1}{2}d\ln\frac{y^2}{1 - y^2} \quad \Rightarrow \quad y^2(T_1) = \frac{1}{1 + (y_0^{-2} - 1)\,\exp(-2T_1)} \quad . \tag{14.219}$$

We then have

$$A(T_1) = 8^{1/4}\sqrt{y(T_1)} = \left(\frac{8}{1 + (8A_0^{-4} - 1)\,\exp(-2T_1)}\right)^{1/4} \quad . \tag{14.220}$$

(c) In the limit $\epsilon \gg 1$, find the period of relaxation oscillations, using Liénard plane analysis. Sketch the orbit of the relaxation oscillation in the Liénard plane.

Solution : Our nonlinear oscillator may be written in the form

$$\ddot{x} + \epsilon\,\frac{dF(x)}{dt} + x = 0 \quad , \tag{14.221}$$

with

$$F(x) = \tfrac{1}{5}x^5 - x \quad . \tag{14.222}$$

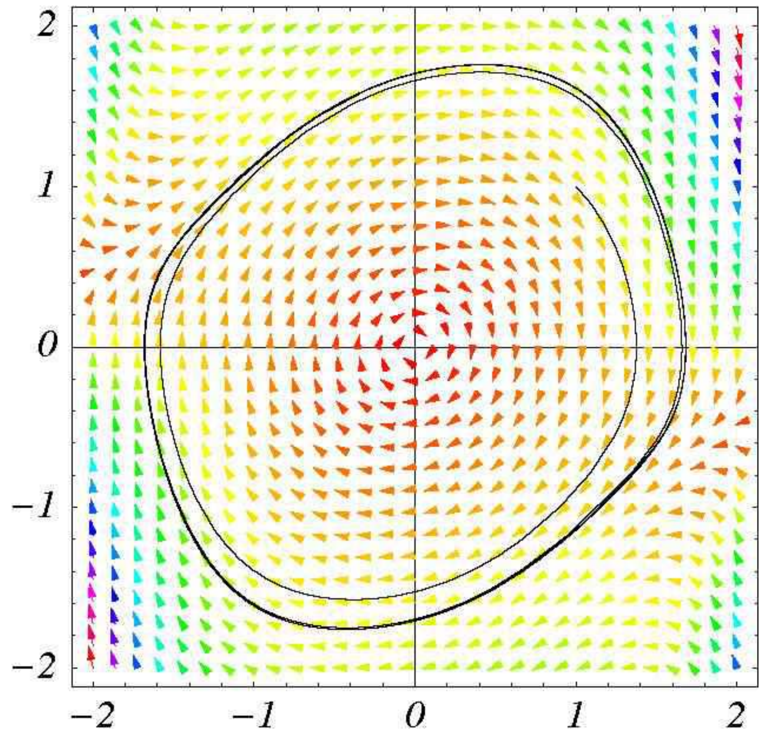

Figure 14.19: Vector field and phase curves for the oscillator $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$, with $\epsilon = 0.25$ and starting from $(x_0, v_0) = (1, 1)$. As $\epsilon \to 0$, the limit cycle is a circle of radius $A^* = 8^{1/4} \approx 1.682$.

Note $\dot{F} = (x^4 - 1)\,\dot{x}$. Now we define the Liénard variable

$$y \equiv \frac{\dot{x}}{\epsilon} + F(x) \quad , \tag{14.223}$$

and in terms of (x, y) we have

$$\dot{x} = \epsilon\Big[y - F(x)\Big] \qquad , \qquad \dot{y} = -\frac{x}{\epsilon} \quad . \tag{14.224}$$

As we have seen in the notes, for large ϵ the motion in the (x, y) plane is easily analyzed. $x(t)$ must move quickly over to the curve $y = F(x)$, at which point the motion slows down and slowly creeps along this curve until it can no longer do so, at which point another big fast jump occurs. The jumps take place between the local extrema of $F(x)$, which occur for $F'(a) = a^4 - 1 = 0$, *i.e.* at $a = \pm 1$, and points on the curve with the same values of $F(a)$. Thus, we solve $F(-1) = \frac{4}{5} = \frac{1}{5}b^5 - b$ and find the desired root at $b^* \approx 1.650629$. The period of the relaxation oscillations, for large ϵ, is

$$T \approx 2\epsilon \int\limits_a^b \! dx\, \frac{F'(x)}{x} = \epsilon \cdot \Big[\tfrac{1}{2}x^4 - 2\ln x\Big]_a^b \approx 2.20935\,\epsilon \quad . \tag{14.225}$$

(d) Numerically integrate the equation (14.209) starting from several different initial conditions.

Solution : The accompanying `Mathematica` plots show $x(t)$ and $v(t)$ for this system for two representative values of ϵ.

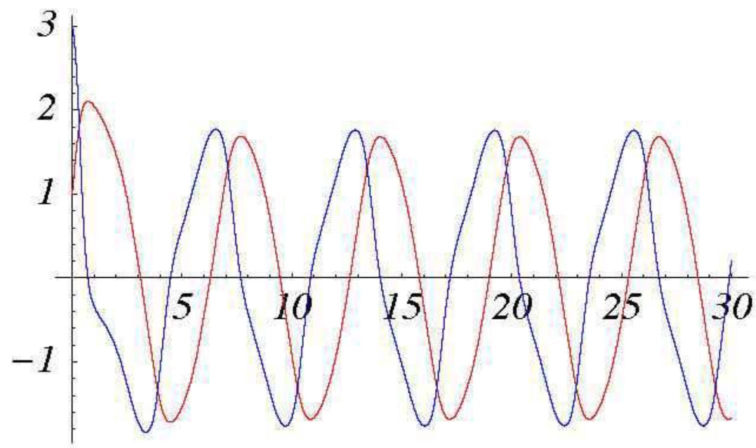

Figure 14.20: Solution to the oscillator equation $\ddot{x}+\epsilon\,(x^4-1)\,\dot{x}+x=0$ with $\epsilon=0.25$ and initial conditions $(x_0,v_0)=(1,3)$. $x(t)$ is shown in red and $v(t)$ in blue. As $\epsilon\to 0$, the amplitude of the oscillations tends to $A^*=8^{1/4}\approx 1.682$.