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Chapter 4

Central Forces and Orbital Mechanics

4.1 Reduction to a one-body problem

Consider two particles interacting via a potential $U(\boldsymbol{r}_1, \boldsymbol{r}_2) = U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big)$. Such a potential, which depends only on the relative distance between the particles, is called a *central* potential. The Lagrangian of this system is then

$$L = T - U = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \quad . \tag{4.1}$$

4.1.1 Center-of-mass (CM) and relative coordinates

The two-body central force problem may always be reduced to two independent one-body problems, by transforming to center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates (see fig. 4.1), *viz.*

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \qquad\qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{4.2}$$

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\,\boldsymbol{r} \qquad\qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\,\boldsymbol{r} \tag{4.3}$$

We then have

$$\begin{aligned} L &= \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1{}^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2{}^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \\ &= \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(r) \quad . \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} M &= m_1 + m_2 \qquad && \text{(total mass)} \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} \qquad && \text{(reduced mass)} \quad . \end{aligned} \tag{4.5}$$

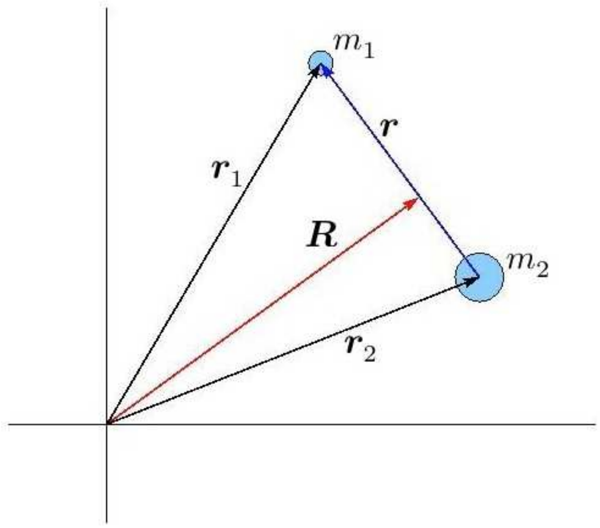

Figure 4.1: Center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates.

4.1.2 Solution to the CM problem

We have $\partial L/\partial \boldsymbol{R} = 0$, which gives $\dot{\boldsymbol{R}} = 0$ and hence

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)\, t \quad . \tag{4.6}$$

Thus, the CM problem is trivial. The center-of-mass moves at constant velocity.

4.1.3 Solution to the relative coordinate problem

`Angular momentum conservation:` We have that $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is a constant of the motion. This means that the motion $\boldsymbol{r}(t)$ is confined to a plane perpendicular to $\boldsymbol{\ell}$. It is convenient to adopt two-dimensional polar coordinates (r, ϕ). The magnitude of $\boldsymbol{\ell}$ is

$$\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}} \tag{4.7}$$

where $d\mathcal{A} = \frac{1}{2} r^2 d\phi$ is the differential element of area subtended relative to the force center. *The relative coordinate vector for a central force problem subtends equal areas in equal times.* This is known as *Kepler's Second Law.*

`Energy conservation:` The equation of motion for the relative coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}} \quad \Rightarrow \quad \mu \ddot{\boldsymbol{r}} = -\frac{\partial U}{\partial \boldsymbol{r}} \quad . \tag{4.8}$$

Taking the dot product with $\dot{\boldsymbol{r}}$, we have

$$\begin{aligned} 0 &= \mu \ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \frac{\partial U}{\partial \boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \\ &= \frac{d}{dt}\Big\{ \tfrac{1}{2}\mu \dot{\boldsymbol{r}}^2 + U(r) \Big\} = \frac{dE}{dt} \quad . \end{aligned} \tag{4.9}$$

Thus, the relative coordinate contribution to the total energy is itself conserved. The total energy is of course $E_{\rm tot} = E + \frac{1}{2}M\dot{\boldsymbol{R}}^2$.

Since $\boldsymbol{\ell}$ is conserved, and since $\boldsymbol{r}\cdot\boldsymbol{\ell} = 0$, all motion is confined to a plane perpendicular to $\boldsymbol{\ell}$. Choosing coordinates such that $\hat{\boldsymbol{z}} = \hat{\boldsymbol{\ell}}$, we have

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2 + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ U_{\rm eff}(r) &= \frac{\ell^2}{2\mu r^2} + U(r) \quad . \end{aligned} \tag{4.10}$$

`Integration of the Equations of Motion, Step I:` The second order equation for $r(t)$ is

$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{dU(r)}{dr} = -\frac{dU_{\rm eff}(r)}{dr} \quad . \tag{4.11}$$

However, conservation of energy reduces this to a first order equation, via

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\Big(E - U_{\rm eff}(r)\Big)} \quad \Rightarrow \quad dt = \pm\frac{\sqrt{\frac{\mu}{2}}\,dr}{\sqrt{E - \frac{\ell^2}{2\mu r^2} - U(r)}} \quad . \tag{4.12}$$

This gives $t(r)$, which must be inverted to obtain $r(t)$. In principle this is possible. Note that a constant of integration also appears at this stage – call it $r_0 = r(t=0)$.

`Integration of the Equations of Motion, Step II:` After finding $r(t)$ one can integrate to find $\phi(t)$ using the conservation of ℓ:

$$\dot{\phi} = \frac{\ell}{\mu r^2} \quad \Rightarrow \quad d\phi = \frac{\ell}{\mu r^2(t)}\,dt \quad . \tag{4.13}$$

This gives $\phi(t)$, and introduces another constant of integration – call it $\phi_0 = \phi(t=0)$.

`Pause to Reflect on the Number of Constants:` Confined to the plane perpendicular to $\boldsymbol{\ell}$, the relative coordinate vector has two degrees of freedom. The equations of motion are second order in time, leading to *four* constants of integration. Our four constants are E, ℓ, r_0, and ϕ_0.

The original problem involves two particles, hence six positions and six velocities, making for 12 initial conditions. Six constants are associated with the CM system: $\boldsymbol{R}(0)$ and $\dot{\boldsymbol{R}}(0)$. The six remaining constants associated with the relative coordinate system are $\boldsymbol{\ell}$ (three components), E, r_0, and ϕ_0.

`Geometric Equation of the Orbit:` From $\ell = \mu r^2\dot{\phi}$, we have

$$\frac{d}{dt} = \frac{\ell}{\mu r^2}\,\frac{d}{d\phi} \quad , \tag{4.14}$$

leading to

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\bigg(\frac{dr}{d\phi}\bigg)^2 = \frac{\mu r^4}{\ell^2}\,F(r) + r \tag{4.15}$$

where $F(r) = -dU(r)/dr$ is the magnitude of the central force. This second order equation may be reduced to a first order one using energy conservation:

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \frac{\ell^2}{2\mu r^4}\bigg(\frac{dr}{d\phi}\bigg)^2 + U_{\rm eff}(r) \quad . \end{aligned} \tag{4.16}$$

Thus,

$$d\phi = \pm\frac{\ell}{\sqrt{2\mu}}\cdot\frac{dr}{r^2\sqrt{E-U_{\rm eff}(r)}} \quad , \tag{4.17}$$

which can be integrated to yield $\phi(r)$, and then inverted to yield $r(\phi)$. Note that only one integration need be performed to obtain the geometric shape of the orbit, while two integrations – one for $r(t)$ and one for $\phi(t)$ – must be performed to obtain the full motion of the system.

It is sometimes convenient to rewrite Eqn. 4.15 in terms of the variable $s = 1/r$:

$$\frac{d^2s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\,F\big(s^{-1}\big) \quad . \tag{4.18}$$

As an example, suppose the geometric orbit is $r(\phi) = k\,e^{\alpha\phi}$, known as a logarithmic spiral. What is the force? We invoke (4.15), with $s''(\phi) = \alpha^2\,s$, yielding

$$F\big(s^{-1}\big) = -(1+\alpha^2)\,\frac{\ell^2}{\mu}\,s^3 \quad \Rightarrow \quad F(r) = -\frac{C}{r^3} \tag{4.19}$$

with

$$\alpha^2 = \frac{\mu C}{\ell^2} - 1 \quad . \tag{4.20}$$

The general solution for $s(\phi)$ for this force law is

$$s(\phi) = \begin{cases} A\cosh(\alpha\phi) + B\sinh(-\alpha\phi) & \text{if } \ell^2 > \mu C \\ \\ A'\,\cos\big(|\alpha|\phi\big) + B'\,\sin\big(|\alpha|\phi\big) & \text{if } \ell^2 < \mu C \quad . \end{cases} \tag{4.21}$$

The logarithmic spiral shape is a special case of the first kind of orbit.

4.2 Almost Circular Orbits

A circular orbit with $r(t) = r_0$ satisfies $\ddot{r} = 0$, which means that $U'_{\rm eff}(r_0) = 0$, which says that $F(r_0) = -\ell^2/\mu r_0^3$. This is negative, indicating that a circular orbit is possible only if the force is attractive over

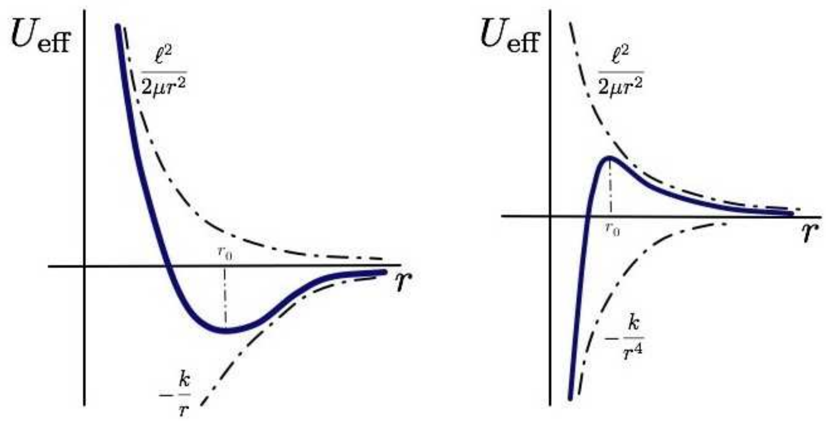

Figure 4.2: Stable and unstable circular orbits. Left panel: $U(r) = -k/r$ produces a stable circular orbit. Right panel: $U(r) = -k/r^4$ produces an unstable circular orbit.

some range of distances. Since $\dot{r} = 0$ as well, we must also have $E = U_{\rm eff}(r_0)$. An almost circular orbit has $r(t) = r_0 + \eta(t)$, where $|\eta/r_0| \ll 1$. To lowest order in η, one derives the equations

$$\frac{d^2\eta}{dt^2} = -\omega^2\,\eta \qquad , \qquad \omega^2 = \frac{1}{\mu}\,U''_{\rm eff}(r_0) \quad . \tag{4.22}$$

If $\omega^2 > 0$, the circular orbit is *stable* and the perturbation oscillates harmonically. If $\omega^2 < 0$, the circular orbit is *unstable* and the perturbation grows exponentially. For the geometric shape of the perturbed orbit, we write $r = r_0 + \eta$, and from (4.15) we obtain

$$\frac{d^2\eta}{d\phi^2} = \bigg(\frac{\mu r_0^4}{\ell^2}\,F'(r_0) - 3\bigg)\,\eta = -\beta^2\,\eta \quad , \tag{4.23}$$

with

$$\beta^2 = 3 + \frac{d\ln F(r)}{d\ln r}\bigg|_{r_0} \quad . \tag{4.24}$$

The solution here is

$$\eta(\phi) = \eta_0\,\cos\beta(\phi - \delta_0) \quad , \tag{4.25}$$

where η_0 and δ_0 are initial conditions. Setting $\eta = \eta_0$, we obtain the sequence of ϕ values

$$\phi_n = \delta_0 + \frac{2\pi n}{\beta} \quad , \tag{4.26}$$

at which $\eta(\phi)$ is a local maximum, *i.e.* at *apoapsis*, where $r = r_0 + \eta_0$. Setting $r = r_0 - \eta_0$ is the condition for closest approach, *i.e. periapsis*. The condition for periapsis is thus $\phi = \phi_n + \pi\beta^{-1}$. The difference,

$$\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi\big(\beta^{-1} - 1\big) \quad , \tag{4.27}$$

is the amount by which the apsides (*i.e.* periapsis and apoapsis) *precess* during each cycle. If $\beta > 1$, the apsides advance, *i.e.* it takes less than a complete revolution $\Delta\phi = 2\pi$ between successive periapses. If

$\beta < 1$, the apsides retreat, and it takes longer than a complete revolution between successive periapses. The situation is depicted in fig. 4.3 for the case $\beta = 1.1$. Below, we will exhibit a soluble model in which the precessing orbit may be determined exactly. Finally, note that if $\beta = p/q$ is a rational number, then the orbit is *closed*, *i.e.* it eventually retraces itself, after every q revolutions.

As an example, let $F(r) = -kr^{-\alpha}$. Solving for a circular orbit, we write

$$U'_{\rm eff}(r) = \frac{k}{r^\alpha} - \frac{\ell^2}{\mu r^3} = 0 \quad , \tag{4.28}$$

which has a solution only for $k > 0$, corresponding to an attractive potential. We then find

$$r_0 = \left(\frac{\ell^2}{\mu k}\right)^{1/(3-\alpha)} \quad , \tag{4.29}$$

and $\beta^2 = 3 - \alpha$. The shape of the perturbed orbits follows from $\eta'' = -\beta^2\,\eta$. Thus, while circular orbits exist whenever $k > 0$, small perturbations about these orbits are stable only for $\beta^2 > 0$, *i.e.* for $\alpha < 3$. One then has $\eta(\phi) = A\cos\beta(\phi - \phi_0)$. The perturbed orbits are closed, at least to lowest order in η, for $\alpha = 3 - (p/q)^2$, *i.e.* for $\beta = p/q$. The situation is depicted in fig. 4.2, for the potentials $U(r) = -k/r$ ($\alpha = 2$) and $U(r) = -k/r^4$ ($\alpha = 5$).

4.3 Precession in a Soluble Model

Let's start with the answer and work backwards. Consider the geometrical orbit,

$$r(\phi) = \frac{r_0}{1 - \epsilon\cos\beta\phi} \quad . \tag{4.30}$$

Our interest is in bound orbits, for which $0 \le \epsilon < 1$ (see fig. 4.3). What sort of potential gives rise to this orbit? Writing $s = 1/r$ as before, we have

$$s(\phi) = s_0\,(1 - \varepsilon\cos\beta\phi) \quad . \tag{4.31}$$

Substituting into (4.18), we have

$$\begin{aligned} -\frac{\mu}{\ell^2 s^2}\,F\big(s^{-1}\big) &= \frac{d^2 s}{d\phi^2} + s \\ &= \beta^2\,s_0\,\epsilon\cos\beta\phi + s \\ &= (1-\beta^2)\,s + \beta^2\,s_0 \quad , \end{aligned} \tag{4.32}$$

from which we conclude

$$F(r) = -\frac{k}{r^2} + \frac{C}{r^3} \quad , \tag{4.33}$$

with

$$k = \beta^2 s_0\,\frac{\ell^2}{\mu} \quad , \quad C = (\beta^2 - 1)\,\frac{\ell^2}{\mu} \quad . \tag{4.34}$$

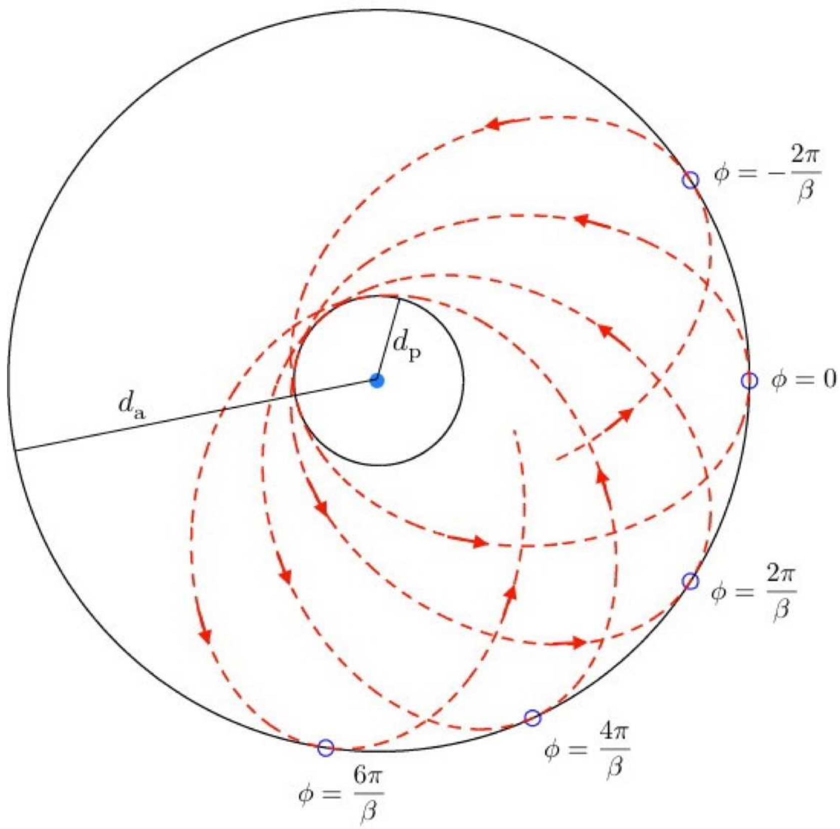

Figure 4.3: Precession in a soluble model, with geometric orbit $r(\phi) = r_0/(1 - \varepsilon \cos \beta\phi)$, shown here with $\beta = 1.1$. Periapsis and apoapsis advance by $\Delta\phi = 2\pi(1 - \beta^{-1})$ per cycle.

The corresponding potential is

$$U(r) = -\frac{k}{r} + \frac{C}{2r^2} + U_\infty \quad , \tag{4.35}$$

where U_∞ is an arbitrary constant, conveniently set to zero. If μ and C are given, we have

$$r_0 = \frac{\ell^2}{\mu k} + \frac{C}{k} \quad , \quad \beta = \sqrt{1 + \frac{\mu C}{\ell^2}} \quad . \tag{4.36}$$

When $C = 0$, these expressions recapitulate those from the Kepler problem. Note that when $\ell^2 + \mu C < 0$ that the effective potential is monotonically increasing as a function of r. In this case, the angular momentum barrier is overwhelmed by the (attractive, $C < 0$) inverse square part of the potential, and $U_{\text{eff}}(r)$ is monotonically increasing. The orbit then passes through the force center. It is a useful exercise to derive the total energy for the orbit,

$$E = (\varepsilon^2 - 1)\,\frac{\mu k^2}{2(\ell^2 + \mu C)} \quad \Longleftrightarrow \quad \varepsilon^2 = 1 + \frac{2E(\ell^2 + \mu C)}{\mu k^2} \quad . \tag{4.37}$$

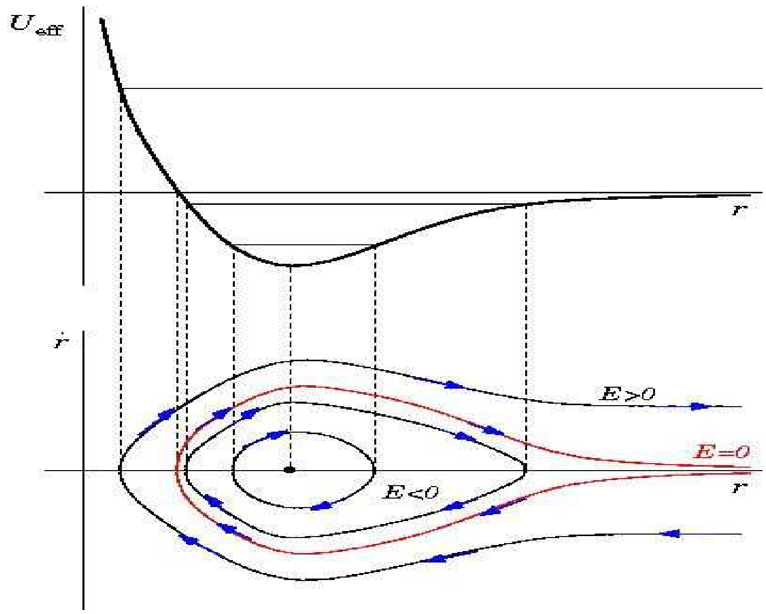

Figure 4.4: The effective potential for the Kepler problem, and associated phase curves. The orbits are geometrically described as conic sections: hyperbolae ($E > 0$), parabolae ($E = 0$), ellipses ($E_{\rm min} < E < 0$), and circles ($E = E_{\rm min}$).

4.4 The Kepler Problem: $U(r) = -k\,r^{-1}$

4.4.1 Geometric shape of orbits

The force is $F(r) = -kr^{-2}$, hence the equation for the geometric shape of the orbit is

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\,F(s^{-1}) = \frac{\mu k}{\ell^2} \quad , \tag{4.38}$$

with $s = 1/r$. Thus, the most general solution is

$$s(\phi) = s_0 - C\cos(\phi - \phi_0) \quad , \tag{4.39}$$

where C and ϕ_0 are constants. Thus,

$$r(\phi) = \frac{r_0}{1 - \varepsilon\cos(\phi - \phi_0)} \quad , \tag{4.40}$$

where $r_0 = \ell^2/\mu k$ and where we have defined a new constant $\varepsilon \equiv C r_0$.

4.4.2 Laplace-Runge-Lenz vector

Consider the vector

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{\ell} - \mu k\,\hat{\boldsymbol{r}} \quad , \tag{4.41}$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is the unit vector pointing in the direction of $\boldsymbol{r}$. We may now show that $\boldsymbol{A}$ is conserved:

$$\begin{aligned}\frac{d\boldsymbol{A}}{dt} &= \frac{d}{dt}\Big\{\boldsymbol{p}\times\boldsymbol{\ell} - \mu k\frac{\boldsymbol{r}}{r}\Big\}\\ &= \dot{\boldsymbol{p}}\times\boldsymbol{\ell} + \boldsymbol{p}\times\dot{\boldsymbol{\ell}} - \mu k\,\frac{r\dot{\boldsymbol{r}} - \boldsymbol{r}\dot{r}}{r^2}\\ &= -\frac{k\boldsymbol{r}}{r^3}\times(\mu\boldsymbol{r}\times\dot{\boldsymbol{r}}) - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot{r}\boldsymbol{r}}{r^2}\\ &= -\mu k\,\frac{\boldsymbol{r}(\boldsymbol{r}\cdot\dot{\boldsymbol{r}})}{r^3} + \mu k\,\frac{\dot{\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{r})}{r^3} - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot{r}\boldsymbol{r}}{r^2} = 0 \quad .\end{aligned} \tag{4.42}$$

So $\boldsymbol{A}$ is a conserved vector which clearly lies in the plane of the motion. $\boldsymbol{A}$ points toward periapsis, *i.e.* toward the point of closest approach to the force center.

Let's assume *apo*apsis occurs at $\phi = \phi_0$. Then

$$\boldsymbol{A}\cdot\boldsymbol{r} = -Ar\cos(\phi-\phi_0) = \ell^2 - \mu k r \tag{4.43}$$

giving

$$r(\phi) = \frac{\ell^2}{\mu k - A\cos(\phi-\phi_0)} = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos(\phi-\phi_0)} \quad , \tag{4.44}$$

where

$$\varepsilon = \frac{A}{\mu k} \qquad , \qquad a(1-\varepsilon^2) = \frac{\ell^2}{\mu k} \quad . \tag{4.45}$$

The orbit is a *conic section* with eccentricity ε. Squaring $\boldsymbol{A}$, one finds

$$\begin{aligned}\boldsymbol{A}^2 &= (\boldsymbol{p}\times\boldsymbol{\ell})^2 - 2\mu k\,\hat{\boldsymbol{r}}\cdot\boldsymbol{p}\times\boldsymbol{\ell} + \mu^2k^2\\ &= p^2\ell^2 - 2\mu\ell^2\,\frac{k}{r} + \mu^2k^2\\ &= 2\mu\ell^2\bigg(\frac{p^2}{2\mu} - \frac{k}{r} + \frac{\mu k^2}{2\ell^2}\bigg) = 2\mu\ell^2\bigg(E + \frac{\mu k^2}{2\ell^2}\bigg)\end{aligned} \tag{4.46}$$

and thus

$$a = -\frac{k}{2E} \quad , \quad \varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} \quad . \tag{4.47}$$

Note that for circular orbits $\boldsymbol{A} = 0$.

4.4.3 Kepler orbits are conic sections

There are four classes of conic sections:

- *Circle:* $\varepsilon = 0$, $E = -\mu k^2/2\ell^2$, radius $a = \ell^2/\mu k$. The force center lies at the center of circle.
- *Ellipse:* $0 < \varepsilon < 1$, $-\mu k^2/2\ell^2 < E < 0$, semimajor axis $a = -k/2E$, semiminor axis $b = a\sqrt{1-\varepsilon^2}$. The force center is at one of the foci.

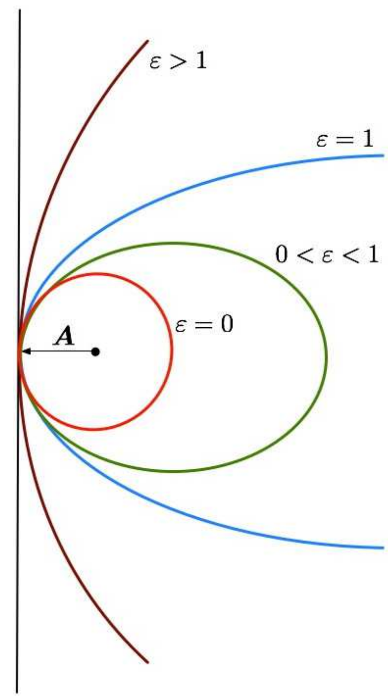

Figure 4.5: Keplerian orbits are conic sections, classified according to eccentricity: hyperbola ($\varepsilon > 1$), parabola ($\varepsilon = 1$), ellipse ($0 < \varepsilon < 1$), and circle ($\varepsilon = 0$). The Laplace-Runge-Lenz vector, $\boldsymbol{A}$, points toward periapsis, but its length $A = \mu k \varepsilon$ vanishes for circular orbits.

- *Parabola:* $\varepsilon = 1$, $E = 0$, force center is the focus.
- *Hyperbola:* $\varepsilon > 1$, $E > 0$, force center is closest focus (attractive) or farthest focus (repulsive).

To see that the Keplerian orbits are indeed conic sections, consider the ellipse of fig. 4.6. The law of cosines gives

$$\rho^2 = r^2 + 4f^2 - 4rf\cos\phi \quad , \tag{4.48}$$

where $f = \varepsilon a$ is the focal distance. Now for any point on an ellipse, the sum of the distances to the left and right foci is a constant, and taking $\phi = 0$ we see that this constant is $2a$. Thus, $\rho = 2a - r$, and we have

$$\begin{aligned} (2a - r)^2 = 4a^2 - 4ar + r^2 &= r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow \quad r(1 - \varepsilon\cos\phi) &= a(1 - \varepsilon^2) \quad . \end{aligned} \tag{4.49}$$

Thus, we obtain

$$r(\phi) = \frac{a\,(1 - \varepsilon^2)}{1 - \varepsilon\cos\phi} \quad , \tag{4.50}$$

and we therefore conclude that

$$r_0 = \frac{\ell^2}{\mu k} = a\,(1 - \varepsilon^2) \quad . \tag{4.51}$$

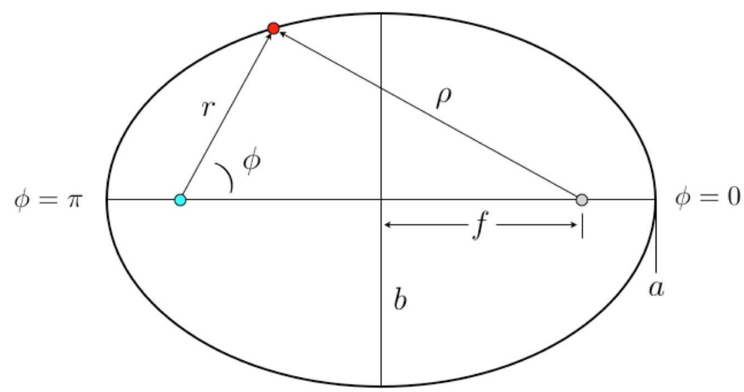

Figure 4.6: The Keplerian ellipse, with the force center at the left focus. The focal distance is $f = \varepsilon a$, where a is the semimajor axis length. The length of the semiminor axis is $b = \sqrt{1-\varepsilon^2}\,a$.

Next let us examine the energy,

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \tfrac{1}{2}\mu\left(\frac{\ell}{\mu r^2}\,\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \\ &= \frac{\ell^2}{2\mu}\left(\frac{ds}{d\phi}\right)^2 + \frac{\ell^2}{2\mu}\,s^2 - ks \quad , \end{aligned} \tag{4.52}$$

with

$$s = \frac{1}{r} = \frac{\mu k}{\ell^2}\Big(1 - \varepsilon\cos\phi\Big) \quad . \tag{4.53}$$

Thus,

$$\frac{ds}{d\phi} = \frac{\mu k}{\ell^2}\,\varepsilon\,\sin\phi \quad , \tag{4.54}$$

and

$$\begin{aligned} \left(\frac{ds}{d\phi}\right)^2 &= \frac{\mu^2 k^2}{\ell^4}\,\varepsilon^2\,\sin^2\!\phi \\ &= \frac{\mu^2 k^2 \varepsilon^2}{\ell^4} - \left(\frac{\mu k}{\ell^2} - s\right)^2 \\ &= -s^2 + \frac{2\mu k}{\ell^2}\,s + \frac{\mu^2 k^2}{\ell^4}\,\big(\varepsilon^2 - 1\big) \quad . \end{aligned} \tag{4.55}$$

Substituting this into eqn. 4.52, we obtain

$$E = \frac{\mu k^2}{2\ell^2}\,\big(\varepsilon^2 - 1\big) \quad . \tag{4.56}$$

For the hyperbolic orbit, depicted in fig. 4.7, we have $r - \rho = \mp 2a$, depending on whether we are on the attractive or repulsive branch, respectively. We then have

$$\begin{aligned} (r \pm 2a)^2 = 4a^2 \pm 4ar + r^2 &= r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow \quad r(\pm 1 + \varepsilon\cos\phi) &= a(\varepsilon^2 - 1) \quad . \end{aligned} \tag{4.57}$$

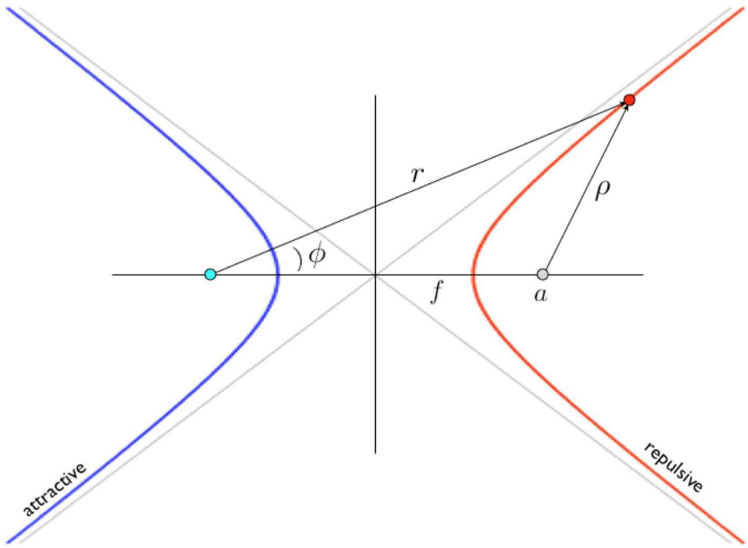

Figure 4.7: The Keplerian hyperbolae, with the force center at the left focus. The left (blue) branch corresponds to an attractive potential, while the right (red) branch corresponds to a repulsive potential. The equations of these branches are $r = \rho = \mp 2a$, where the top sign corresponds to the left branch and the bottom sign to the right branch.

This yields

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon \cos\phi} \quad . \tag{4.58}$$

4.4.4 Period of bound Kepler orbits

From $\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}}$, the period is $\tau = 2\mu \mathcal{A}/\ell$, where $\mathcal{A} = \pi a^2 \sqrt{1-\varepsilon^2}$ is the area enclosed by the orbit. This gives

$$\tau = 2\pi \left(\frac{\mu a^3}{k}\right)^{1/2} = 2\pi \left(\frac{a^3}{GM}\right)^{1/2} \tag{4.59}$$

as well as

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} \quad , \tag{4.60}$$

where $k = G m_1 m_2$ and $M = m_1 + m_2$ is the total mass. For planetary orbits, $m_1 = M_\odot$ is the solar mass and $m_2 = m_\mathrm{p}$ is the planetary mass. We then have

$$\frac{a^3}{\tau^2} = \left(1 + \frac{m_\mathrm{p}}{M_\odot}\right) \frac{GM_\odot}{4\pi^2} \approx \frac{GM_\odot}{4\pi^2} \quad , \tag{4.61}$$

which is to an excellent approximation independent of the planetary mass. (Note that $m_\mathrm{p}/M_\odot \approx 10^{-3}$ even for Jupiter.) This analysis also holds, *mutatis mutandis*, for the case of satellites orbiting the earth, and indeed in any case where the masses are grossly disproportionate in magnitude.

4.4.5 Escape velocity

The threshold for escape from a gravitational potential occurs at $E = 0$. Since $E = T + U$ is conserved, we determine the *escape velocity* for a body a distance r from the force center by setting

$$E = 0 = \tfrac{1}{2}\mu v_{\rm esc}^2(t) - \frac{Gm_1m_2}{r} \quad\Rightarrow\quad v_{\rm esc}(r) = \sqrt{\frac{2GM}{r}} \quad . \tag{4.62}$$

with $M = m_1 + m_2$. For an object on earth's surface, $v_{\rm esc} = \sqrt{2gR_{\rm E}} = 11.2\,{\rm km/s}$, assuming the object is much less massive than the earth itself.

4.4.6 Satellites and spacecraft

A satellite in a circular orbit a distance h above the earth's surface has an orbital period

$$\tau = \frac{2\pi}{\sqrt{GM_{\rm E}}}\,(R_{\rm E} + h)^{3/2} \quad , \tag{4.63}$$

where we take $m_{\rm satellite} \ll M_{\rm E}$. For low earth orbit (LEO), $h \ll R_{\rm E} = 6.37 \times 10^6\,{\rm m}$, in which case $\tau_{\rm LEO} = 2\pi\sqrt{R_{\rm E}/g} = 1.4\,{\rm hr}$.

Consider a weather satellite in an elliptical orbit whose closest approach to the earth (perigee) is 200 km above the earth's surface and whose farthest distance (apogee) is 7200 km above the earth's surface. What is the satellite's orbital period? From fig. 4.6, we see that

$$\begin{aligned} d_{\rm apogee} &= R_{\rm E} + 7200\,{\rm km} = 13571\,{\rm km} \\ d_{\rm perigee} &= R_{\rm E} + 200\,{\rm km} = 6971\,{\rm km} \\ a &= \tfrac{1}{2}(d_{\rm apogee} + d_{\rm perigee}) = 10071\,{\rm km} \quad . \end{aligned} \tag{4.64}$$

We then have

$$\tau = \left(\frac{a}{R_{\rm E}}\right)^{3/2} \cdot \tau_{\rm LEO} \approx 2.65\,{\rm hr} \quad . \tag{4.65}$$

What happens if a spacecraft in orbit about the earth fires its rockets? Clearly the energy and angular momentum of the orbit will change, and this means the shape will change. If the rockets are fired (in the direction of motion) at perigee, then perigee itself is unchanged, because $\boldsymbol{v} \cdot \boldsymbol{r} = 0$ is left unchanged at this point. However, E is increased, hence the eccentricity $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$ increases. This is the most efficient way of boosting a satellite into an orbit with higher eccentricity. Conversely, and somewhat paradoxically, when a satellite in LEO loses energy due to frictional drag of the atmosphere, the energy E decreases. Initially, because the drag is weak and the atmosphere is isotropic, the orbit remains circular. Since E decreases, $\langle T\rangle = -E$ must *increase*, which means that the frictional forces cause the satellite to speed up!

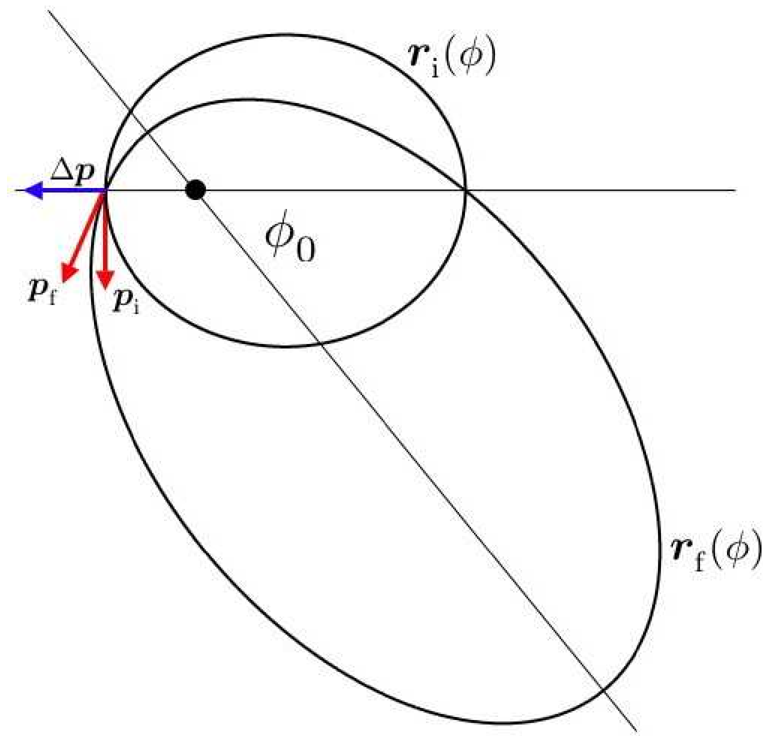

Figure 4.8: At perigee of an elliptical orbit $r_{\rm i}(\phi)$, a radial impulse $\Delta\boldsymbol{p}$ is applied. The shape of the resulting orbit $r_{\rm f}(\phi)$ is shown.

4.4.7 Two examples of orbital mechanics

- Problem #1: At perigee of an elliptical Keplerian orbit, a satellite receives an impulse $\Delta\boldsymbol{p} = p_0\hat{\boldsymbol{r}}$. Describe the resulting orbit.

- Solution #1: Since the impulse is radial, the angular momentum $\boldsymbol{\ell} = \boldsymbol{r}\times\boldsymbol{p}$ is unchanged. The energy, however, does change, with $\Delta E = p_0^2/2\mu$. Thus,

$$\varepsilon_{\rm f}^2 = 1 + \frac{2E_{\rm f}\,\ell^2}{\mu k^2} = \varepsilon_{\rm i}^2 + \left(\frac{\ell p_0}{\mu k}\right)^2 \quad . \tag{4.66}$$

The new semimajor axis length is

$$\begin{aligned} a_{\rm f} &= \frac{\ell^2/\mu k}{1-\varepsilon_{\rm f}^2} = a_{\rm i}\cdot\frac{1-\varepsilon_{\rm i}^2}{1-\varepsilon_{\rm f}^2} \\ &= \frac{a_{\rm i}}{1-(a_{\rm i}p_0^2/\mu k)} \quad . \end{aligned} \tag{4.67}$$

The shape of the final orbit must also be a Keplerian ellipse, described by

$$r_{\rm f}(\phi) = \frac{\ell^2}{\mu k}\cdot\frac{1}{1-\varepsilon_{\rm f}\cos(\phi+\delta)} \quad , \tag{4.68}$$

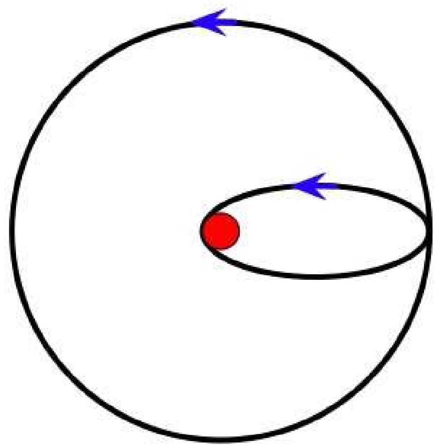

Figure 4.9: The larger circular orbit represents the orbit of the earth. The elliptical orbit represents that for an object orbiting the Sun with distance at perihelion equal to the Sun's radius.

where the phase shift δ is determined by setting

$$r_{\rm i}(\pi) = r_{\rm f}(\pi) = \frac{\ell^2}{\mu k} \cdot \frac{1}{1+\varepsilon_{\rm i}} \quad . \tag{4.69}$$

Solving for δ, we obtain

$$\delta = \cos^{-1}\big(\varepsilon_{\rm i}/\varepsilon_{\rm f}\big) \quad . \tag{4.70}$$

The situation is depicted in fig. 4.8.

- Problem #2: Which is more energy efficient – to send nuclear waste outside the solar system, or to send it into the Sun?

◦ Solution #2: Escape velocity for the solar system is $v_{{\rm esc},\odot}(r) = \sqrt{GM_\odot/r}$. At a distance $a_{\rm E}$, we then have $v_{{\rm esc},\odot}(a_{\rm E}) = \sqrt{2}\,v_{\rm E}$, where $v_{\rm E} = \sqrt{GM_\odot/a_{\rm E}} = 2\pi a_{\rm E}/\tau_{\rm E} = 29.9\,{\rm km/s}$ is the velocity of the earth in its orbit. The satellite is launched from earth, and clearly the most energy efficient launch will be one in the direction of the earth's motion, in which case the velocity after escape from earth must be $u = \big(\sqrt{2}-1\big)v_{\rm E} = 12.4\,{\rm km/s}$. The speed just above the earth's atmosphere must then be $\tilde{u}$, where

$$\tfrac{1}{2}m\tilde{u}^2 - \frac{GM_{\rm E}m}{R_{\rm E}} = \tfrac{1}{2}mu^2 \quad , \tag{4.71}$$

or, in other words,

$$\tilde{u}^2 = u^2 + v_{\rm esc,E}^2 \quad . \tag{4.72}$$

We compute $\tilde{u} = 16.7\,{\rm km/s}$.

The second method is to place the trash ship in an elliptical orbit whose perihelion is the Sun's radius, $R_\odot = 6.98\times10^8\,{\rm m}$, and whose aphelion is $a_{\rm E}$. Invoking the general equation for the shape of the Keplerian orbit $r(\phi) = (\ell^2/\mu k)/(1 - \varepsilon\cos\phi)$, we then solve the two equations

$$\begin{aligned} r(\phi=\pi) = R_\odot &= \frac{1}{1+\varepsilon}\cdot\frac{\ell^2}{\mu k} \\ r(\phi=0) = a_{\rm E} &= \frac{1}{1-\varepsilon}\cdot\frac{\ell^2}{\mu k} \quad . \end{aligned} \tag{4.73}$$

We thereby obtain

$$\varepsilon = \frac{a_{\rm E} - R_\odot}{a_{\rm E} + R_\odot} = 0.991 \quad , \tag{4.74}$$

which is a very eccentric ellipse, and

$$\begin{aligned} \frac{\ell^2}{\mu k} &= \frac{a_{\rm E}^2\, v^2}{G(M_\odot + m)} \approx a_{\rm E}\cdot\frac{v^2}{v_{\rm E}^2} \\ &= (1-\varepsilon)\, a_{\rm E} = \frac{2a_{\rm E}R_\odot}{a_{\rm E}+R_\odot} \quad . \end{aligned} \tag{4.75}$$

Hence,

$$v^2 = \frac{2R_\odot}{a_{\rm E}+R_\odot}\, v_{\rm E}^2 \quad , \tag{4.76}$$

and the necessary velocity relative to earth is

$$u = \left(\sqrt{\frac{2R_\odot}{a_{\rm E}+R_\odot}} - 1\right) v_{\rm E} \approx -0.904\, v_{\rm E} \quad , \tag{4.77}$$

i.e. $u = -27.0\,{\rm km/s}$. Launch is in the opposite direction from the earth's orbital motion, and from $\tilde u^2 = u^2 + v_{\rm esc,E}^2$ we find $\tilde u = -29.2\,{\rm km/s}$, which is larger (in magnitude) than in the first scenario. Thus, it is cheaper to ship the trash out of the solar system than to send it crashing into the Sun, by a factor $\tilde u_{\rm I}^2/\tilde u_{\rm II}^2 = 0.327$.

4.5 Mission to Neptune

Four earth-launched spacecraft have escaped the solar system: *Pioneer 10* (launch 3/3/72), *Pioneer 11* (launch 4/6/73), *Voyager 1* (launch 9/5/77), and *Voyager 2* (launch 8/20/77).[1] The latter two are still functioning, and each are moving away from the Sun at a velocity of roughly 3.5 AU/yr.

As the first objects of earthly origin to leave our solar system, both *Pioneer* spacecraft featured a graphic message in the form of a 6″ x 9″ gold anodized plaque affixed to the spacecrafts' frame. This plaque was designed in part by the late astronomer and popular science writer Carl Sagan. The humorist Dave Barry, in an essay entitled *Bring Back Carl's Plaque*, remarks,

[1]There is a very nice discussion in the Barger and Olsson book on 'Grand Tours of the Outer Planets'. Here I reconstruct and extend their discussion.

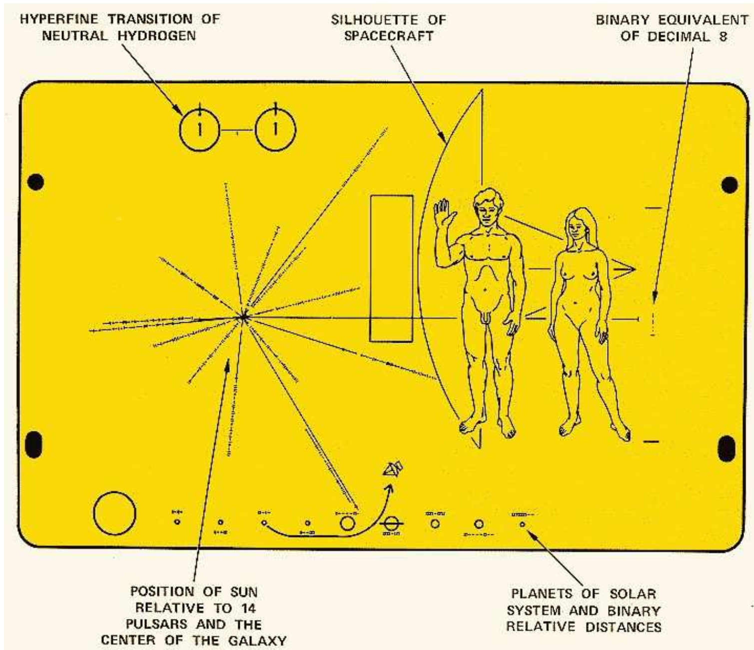

Figure 4.10: The unforgivably dorky *Pioneer 10* and *Pioneer 11* plaque.

> But the really bad part is what they put on the plaque. I mean, if we're going to have a plaque, it ought to at least show the aliens what we're really like, right? Maybe a picture of people eating cheeseburgers and watching "The Dukes of Hazzard." Then if aliens found it, they'd say, "Ah. Just plain folks."
>
> But no. Carl came up with this incredible science-fair-wimp plaque that features drawings of – you are not going to believe this – a hydrogen atom and naked people. To represent the entire Earth! This is crazy! Walk the streets of any town on this planet, and the two things you will almost never see are hydrogen atoms and naked people.

During August, 1989, *Voyager 2* investigated the planet Neptune. A direct trip to Neptune along a Keplerian ellipse with $r_{\rm p} = a_{\rm E} = 1\,{\rm AU}$ and $r_{\rm a} = a_{\rm N} = 30.06\,{\rm AU}$ would take 30.6 years. To see this, note that $r_{\rm p} = a\,(1-\varepsilon)$ and $r_{\rm a} = a\,(1+\varepsilon)$ yield

$$a = \tfrac{1}{2}\big(a_{\rm E} + a_{\rm N}\big) = 15.53\,{\rm AU} \quad , \quad \varepsilon = \frac{a_{\rm N} - a_{\rm E}}{a_{\rm N} + a_{\rm E}} = 0.9356 \quad . \tag{4.78}$$

Thus,

$$\tau = \tfrac{1}{2}\,\tau_{\rm E} \cdot \left(\frac{a}{a_{\rm E}}\right)^{3/2} = 30.6\,{\rm yr} \quad . \tag{4.79}$$

The energy cost per kilogram of such a mission is computed as follows. Let the speed of the probe after its escape from earth be $v_{\rm p} = \lambda v_{\rm E}$, and the speed just above the atmosphere (*i.e.* neglecting atmospheric friction) is v_0. For the most efficient launch possible, the probe is shot in the direction of earth's instantaneous motion about the Sun. Then we must have

$$\tfrac{1}{2} m\, v_0^2 - \frac{G M_{\rm E} m}{R_{\rm E}} = \tfrac{1}{2} m\,(\lambda - 1)^2\, v_{\rm E}^2 \quad , \tag{4.80}$$

since the speed of the probe in the frame of the earth is $v_{\rm p} - v_{\rm E} = (\lambda - 1)\, v_{\rm E}$. Thus,

$$\begin{aligned} \frac{E}{m} &= \tfrac{1}{2} v_0^2 = \Big[\tfrac{1}{2}(\lambda-1)^2 + h\Big] v_{\rm E}^2 \\ v_{\rm E}^2 &= \frac{GM_\odot}{a_{\rm E}} = 6.24 \times 10^7 \, R_{\rm J}/{\rm kg} \quad , \end{aligned} \tag{4.81}$$

where

$$h \equiv \frac{M_{\rm E}}{M_\odot} \cdot \frac{a_{\rm E}}{R_{\rm E}} = 7.050 \times 10^{-2} \quad . \tag{4.82}$$

Therefore, a convenient dimensionless measure of the energy is

$$\eta \equiv \frac{2E}{m v_{\rm E}^2} = \frac{v_0^2}{v_{\rm E}^2} = (\lambda - 1)^2 + 2h \quad . \tag{4.83}$$

As we shall derive below, a direct mission to Neptune requires

$$\lambda \geq \sqrt{\frac{2a_{\rm N}}{a_{\rm N} + a_{\rm E}}} = 1.3913 \quad , \tag{4.84}$$

which is close to the criterion for escape from the solar system, $\lambda_{\rm esc} = \sqrt{2}$. Note that about 52% of the energy is expended after the probe escapes the Earth's pull, and 48% is expended in liberating the probe from Earth itself.

This mission can be done much more economically by taking advantage of a Jupiter flyby, as shown in fig. 4.11. The idea of a flyby is to steal some of Jupiter's momentum and then fly away very fast before Jupiter realizes and gets angry. The CM frame of the probe-Jupiter system is of course the rest frame of Jupiter, and in this frame conservation of energy means that the final velocity $\boldsymbol{u}_{\rm f}$ is of the same magnitude as the initial velocity $\boldsymbol{u}_{\rm i}$. However, in the frame of the Sun, the initial and final velocities are $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm i}$ and $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm f}$, respectively, where $\boldsymbol{v}_{\rm J}$ is the velocity of Jupiter in the rest frame of the Sun. If, as shown in the inset to fig. 4.11, $\boldsymbol{u}_{\rm f}$ is roughly parallel to $\boldsymbol{v}_{\rm J}$, the probe's velocity in the Sun's frame will be enhanced. Thus, the motion of the probe is broken up into three segments:

I : Earth to Jupiter
II : Scatter off Jupiter's gravitational pull
III : Jupiter to Neptune

We now analyze each of these segments in detail. In so doing, it is useful to recall that the general form of a Keplerian orbit is

$$r(\phi) = \frac{d}{1 - \varepsilon \cos\phi} \quad , \quad d = \frac{\ell^2}{\mu k} = \big|\varepsilon^2 - 1\big|\, a \quad . \tag{4.85}$$

The energy is

$$E = (\varepsilon^2 - 1)\, \frac{\mu k^2}{2\ell^2} \quad , \tag{4.86}$$

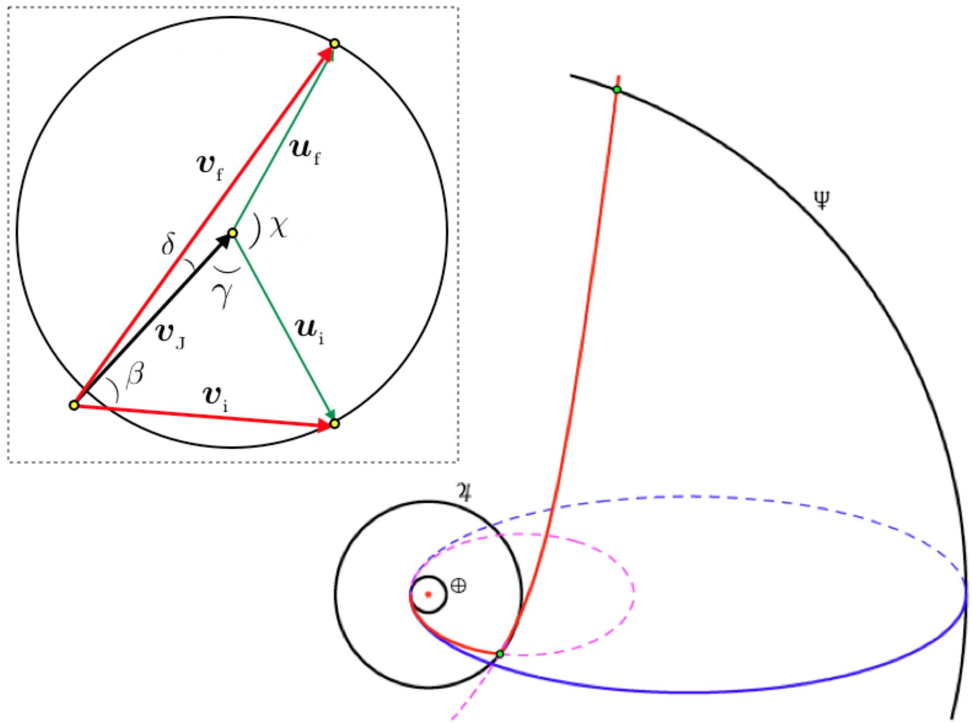

Figure 4.11: Mission to Neptune. The figure at the lower right shows the orbits of Earth, Jupiter, and Neptune in black. The cheapest (in terms of energy) direct flight to Neptune, shown in blue, would take 30.6 years. By swinging past the planet Jupiter, the satellite can pick up great speed and with even less energy the mission time can be cut to 8.5 years (red curve). The inset in the upper left shows the scattering event with Jupiter.

with $k = GMm$, where M is the mass of either the Sun or a planet. In either case, M dominates, and $\mu = Mm/(M+m) \simeq m$ to extremely high accuracy. The time for the trajectory to pass from $\phi = \phi_1$ to $\phi = \phi_2$ is

$$T = \int dt = \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\dot\phi} = \frac{\mu}{\ell} \int\limits_{\phi_1}^{\phi_2} d\phi \, r^2(\phi) = \frac{\ell^3}{\mu k^2} \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\big[1 - \varepsilon \cos\phi\big]^2} \quad . \tag{4.87}$$

For reference,

$$\begin{array}{lll} a_{\rm E} = 1\,{\rm AU} & a_{\rm J} = 5.20\,{\rm AU} & a_{\rm N} = 30.06\,{\rm AU} \\ M_{\rm E} = 5.972 \times 10^{24}\,{\rm kg} & M_{\rm J} = 1.900 \times 10^{27}\,{\rm kg} & M_\odot = 1.989 \times 10^{30}\,{\rm kg} \end{array}$$

with $1\,{\rm AU} = 1.496 \times 10^8\,{\rm km}$. Here $a_{\rm E,J,N}$ and $M_{{\rm E,J},\odot}$ are the orbital radii and masses of Earth, Jupiter, and Neptune, and the Sun. The last thing we need to know is the radius of Jupiter,

$$R_{\rm J} = 9.558 \times 10^{-4}\,{\rm AU} \quad .$$

We need $R_{\rm J}$ because the distance of closest approach to Jupiter, or *perijove*, must be $R_{\rm J}$ or greater, or else the probe crashes into Jupiter!

4.5.1 Earth to Jupiter (Phase I)

The probe's velocity at perihelion is $v_{\rm p} = \lambda v_{\rm E}$. The angular momentum is $\ell = \mu a_{\rm E} \cdot \lambda v_{\rm E}$, whence

$$d = \frac{(a_{\rm E} \lambda v_{\rm E})^2}{GM_\odot} = \lambda^2 \, a_{\rm E} \quad . \tag{4.88}$$

From $r(\pi) = a_{\rm E}$, we obtain

$$\varepsilon_{\rm I} = \lambda^2 - 1 \quad . \tag{4.89}$$

This orbit will intersect the orbit of Jupiter if $r_{\rm a} \geq a_{\rm J}$, which means

$$\frac{d}{1 - \varepsilon_{\rm I}} \geq a_{\rm J} \quad \Rightarrow \quad \lambda \geq \sqrt{\frac{2a_{\rm J}}{a_{\rm J} + a_{\rm E}}} = 1.2952 \quad . \tag{4.90}$$

If this inequality holds, then intersection of Jupiter's orbit will occur for

$$\phi_{\rm J} = 2\pi - \cos^{-1}\!\left(\frac{a_{\rm J} - \lambda^2 a_{\rm E}}{(\lambda^2 - 1)\, a_{\rm J}}\right) \quad . \tag{4.91}$$

Finally, the time for this portion of the trajectory is

$$\tau_{\rm EJ} = \tau_{\rm E} \cdot \lambda^3 \int\limits_{\pi}^{\phi_{\rm J}} \frac{d\phi}{2\pi} \, \frac{1}{\big[1 - (\lambda^2 - 1)\cos\phi\big]^2} \quad . \tag{4.92}$$

4.5.2 Encounter with Jupiter (Phase II)

We are interested in the final speed $\boldsymbol{v}_{\rm f}$ of the probe after its encounter with Jupiter. We will determine the speed $v_{\rm f}$ and the angle δ which the probe makes with respect to Jupiter after its encounter. According to the geometry of fig. 4.11,

$$\begin{aligned} v_{\rm f}^2 &= v_{\rm J}^2 + u^2 - 2uv_{\rm J}\cos(\chi + \gamma) \\ \cos\delta &= \frac{v_{\rm J}^2 + v_{\rm f}^2 - u^2}{2v_{\rm f} v_{\rm J}} \end{aligned} \tag{4.93}$$

Note that

$$v_{\rm J}^2 = \frac{GM_\odot}{a_{\rm J}} = \frac{a_{\rm E}}{a_{\rm J}} \cdot v_{\rm E}^2 \quad . \tag{4.94}$$

But what are u, χ, and γ?

To determine u, we invoke

$$u^2 = v_{\rm J}^2 + v_{\rm i}^2 - 2v_{\rm J} v_{\rm i} \cos\beta \quad . \tag{4.95}$$

The initial velocity (in the frame of the Sun) when the probe crosses Jupiter's orbit is given by energy conservation:

$$\tfrac{1}{2} m (\lambda v_{\rm E})^2 - \frac{GM_\odot m}{a_{\rm E}} = \tfrac{1}{2} m v_{\rm i}^2 - \frac{GM_\odot m}{a_{\rm J}} \quad , \tag{4.96}$$

which yields

$$v_i^2 = \left(\lambda^2 - 2 + \frac{2a_{\rm E}}{a_{\rm J}}\right) v_{\rm E}^2 \quad . \tag{4.97}$$

As for β, we invoke conservation of angular momentum:

$$\mu(v_{\rm i}\cos\beta)a_{\rm J} = \mu(\lambda v_{\rm E})a_{\rm E} \quad \Rightarrow \quad v_{\rm i}\cos\beta = \lambda\,\frac{a_{\rm E}}{a_{\rm J}}\,v_{\rm E} \quad . \tag{4.98}$$

The angle γ is determined from

$$v_{\rm J} = v_{\rm i}\cos\beta + u\cos\gamma \quad . \tag{4.99}$$

Putting all this together, we obtain

$$\begin{aligned} v_{\rm i} &= v_{\rm E}\sqrt{\lambda^2 - 2 + 2x} \\ u &= v_{\rm E}\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}} \\ \cos\gamma &= \frac{\sqrt{x} - \lambda x}{\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}}} \quad , \end{aligned} \tag{4.100}$$

where

$$x \equiv \frac{a_{\rm E}}{a_{\rm J}} = 0.1923 \quad . \tag{4.101}$$

We next consider the scattering of the probe by the planet Jupiter. In the Jovian frame, we may write

$$r(\phi) = \frac{\kappa R_{\rm J}\,(1 + \varepsilon_{\rm J})}{1 + \varepsilon_{\rm J}\cos\phi} \quad , \tag{4.102}$$

where perijove occurs at

$$r(0) = \kappa R_{\rm J} \quad . \tag{4.103}$$

Here, κ is a dimensionless quantity, which is simply perijove in units of the Jovian radius. Clearly we require $\kappa > 1$ or else the probe crashes into Jupiter! The probe's energy in this frame is simply $E = \frac{1}{2}mu^2$, which means the probe enters into a hyperbolic orbit about Jupiter. Next, from

$$\begin{aligned} E &= \frac{k}{2}\,\frac{\varepsilon^2 - 1}{\ell^2/\mu k} \\ \frac{\ell^2}{\mu k} &= (1 + \varepsilon)\,\kappa R_{\rm J} \end{aligned} \tag{4.104}$$

we find

$$\varepsilon_{\rm J} = 1 + \kappa\left(\frac{R_{\rm J}}{a_{\rm E}}\right)\left(\frac{M_\odot}{M_{\rm J}}\right)\left(\frac{u}{v_{\rm E}}\right)^2 \quad . \tag{4.105}$$

The opening angle of the Keplerian hyperbola is then $\phi_{\rm c} = \cos^{-1}\left(\varepsilon_{\rm J}^{-1}\right)$, and the angle χ is related to $\phi_{\rm c}$ through

$$\chi = \pi - 2\phi_{\rm c} = \pi - 2\cos^{-1}\left(\frac{1}{\varepsilon_{\rm J}}\right) \quad . \tag{4.106}$$

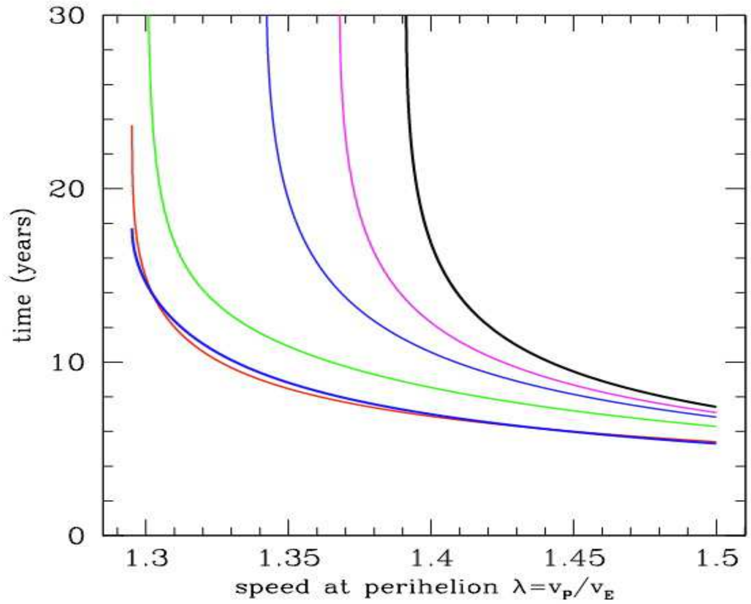

Figure 4.12: Total time for Earth-Neptune mission as a function of dimensionless velocity at perihelion, $\lambda = v_{\rm p}/v_{\rm E}$. Six different values of κ, the value of perijove in units of the Jovian radius, are shown: $\kappa = 1.0$ (thick blue), $\kappa = 5.0$ (red), $\kappa = 20$ (green), $\kappa = 50$ (blue), $\kappa = 100$ (magenta), and $\kappa = \infty$ (thick black).

Therefore, we may finally write

$$v_{\rm f} = \sqrt{x\, v_{\rm E}^2 + u^2 + 2\, u\, v_{\rm E} \sqrt{x} \cos(2\phi_{\rm c} - \gamma)} \tag{4.107}$$

$$\cos\delta = \frac{x\, v_{\rm E}^2 + v_{\rm f}^2 - u^2}{2\, v_{\rm f}\, v_{\rm E} \sqrt{x}} \quad . \tag{4.108}$$

4.5.3 Jupiter to Neptune (Phase III)

Immediately after undergoing gravitational scattering off Jupiter, the energy and angular momentum of the probe are

$$E = \tfrac{1}{2} m v_{\rm f}^2 - \frac{GM_\odot m}{a_{\rm J}} \tag{4.109}$$

and

$$\ell = \mu\, v_{\rm f}\, a_{\rm J} \cos\delta \quad . \tag{4.110}$$

We write the geometric equation for the probe's orbit as

$$r(\phi) = \frac{d}{1 + \varepsilon \cos(\phi - \phi_{\rm J} - \alpha)} \quad , \tag{4.111}$$

where

$$d = \frac{\ell^2}{\mu k} = \left(\frac{v_{\rm f}\, a_{\rm J} \cos\delta}{v_{\rm E}\, a_{\rm E}} \right)^2 a_{\rm E} \quad . \tag{4.112}$$

Setting $E = (\mu k^2/2\ell^2)(\varepsilon^2 - 1)$, we obtain the eccentricity

$$\varepsilon = \sqrt{1 + \left(\frac{v_{\rm f}^2}{v_{\rm E}^2} - \frac{2a_{\rm E}}{a_{\rm J}}\right)\frac{d}{a_{\rm E}}} \quad . \tag{4.113}$$

Note that the orbit is hyperbolic – the probe will escape the Sun – if $v_{\rm f} > v_{\rm E} \cdot \sqrt{2x}$. The condition that this orbit intersect Jupiter at $\phi = \phi_{\rm J}$ yields

$$\cos\alpha = \frac{1}{\varepsilon}\left(\frac{d}{a_{\rm J}} - 1\right) \quad , \tag{4.114}$$

which determines the angle α. Interception of Neptune occurs at

$$\frac{d}{1 + \varepsilon\cos(\phi_{\rm N} - \phi_{\rm J} - \alpha)} = a_{\rm N} \qquad \Rightarrow \qquad \phi_{\rm N} = \phi_{\rm J} + \alpha + \cos^{-1}\frac{1}{\varepsilon}\left(\frac{d}{a_{\rm N}} - 1\right) \quad . \tag{4.115}$$

We then have

$$\tau_{\rm JN} = \tau_{\rm E} \cdot \left(\frac{d}{a_{\rm E}}\right)^3 \int\limits_{\phi_{\rm J}}^{\phi_{\rm N}} \frac{d\phi}{2\pi} \, \frac{1}{\big[1 + \varepsilon\cos(\phi - \phi_{\rm J} - \alpha)\big]^2} \quad . \tag{4.116}$$

The total time to Neptune is then the sum,

$$\tau_{\rm EN} = \tau_{\rm EJ} + \tau_{\rm JN} \quad . \tag{4.117}$$

In fig. 4.12, we plot the mission time $\tau_{\rm EN}$ *versus* the velocity at perihelion, $v_{\rm p} = \lambda v_{\rm E}$, for various values of κ. The value $\kappa = \infty$ corresponds to the case of no Jovian encounter at all.

4.6 Restricted Three-Body Problem

Problem : Consider the 'restricted three body problem' in which a light object of mass m (*e.g.* a satellite) moves in the presence of two celestial bodies of masses m_1 and m_2 (*e.g.* the sun and the earth, or the earth and the moon). Suppose m_1 and m_2 execute stable circular motion about their common center of mass. You may assume $m \ll m_2 \le m_1$.

(a) Show that the angular frequency for the motion of masses 1 and 2 is related to their (constant) relative separation, by

$$\omega_0^2 = \frac{GM}{r_0^3} \quad , \tag{4.118}$$

where $M = m_1 + m_2$ is the total mass.

Solution : For a Kepler potential $U = -k/r$, the circular orbit lies at $r_0 = \ell^2/\mu k$, where $\ell = \mu r^2\dot\phi$ is the angular momentum and $k = G m_1 m_2$. This gives

$$\omega_0^2 = \frac{\ell^2}{\mu^2 \, r_0^4} = \frac{k}{\mu r_0^3} = \frac{GM}{r_0^3} \quad , \tag{4.119}$$

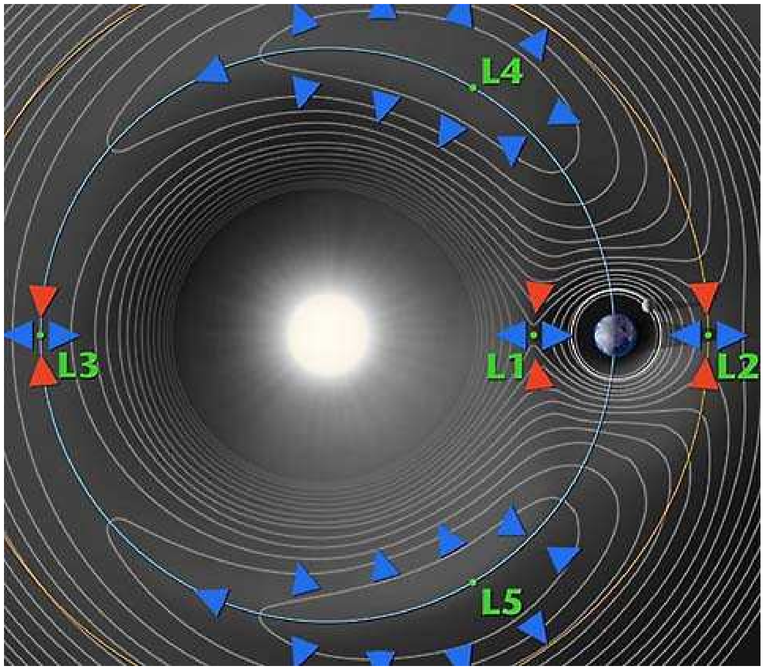

Figure 4.13: The Lagrange points for the earth-sun system. *Credit: WMAP project*.

with $\omega_0 = \dot{\phi}$.

(b) The satellite moves in the combined gravitational field of the two large bodies; the satellite itself is of course much too small to affect their motion. In deriving the motion for the satellilte, it is convenient to choose a reference frame whose origin is the CM and which rotates with angular velocity ω_0. In the rotating frame the masses m_1 and m_2 lie, respectively, at $x_1 = -\alpha r_0$ and $x_2 = \beta r_0$, with

$$\alpha = \frac{m_2}{M} \qquad , \qquad \beta = \frac{m_1}{M} \tag{4.120}$$

and with $y_1 = y_2 = 0$. Note $\alpha + \beta = 1$.

Show that the Lagrangian for the satellite in this rotating frame may be written

$$L = \tfrac{1}{2} m \big(\dot{x} - \omega_0 \, y\big)^2 + \tfrac{1}{2} m \big(\dot{y} + \omega_0 \, x\big)^2 + \frac{G \, m_1 \, m}{\sqrt{(x + \alpha r_0)^2 + y^2}} + \frac{G \, m_2 \, m}{\sqrt{(x - \beta r_0)^2 + y^2}} \quad . \tag{4.121}$$

Solution : Let the original (inertial) coordinates be (x_0, y_0). Then let us define the rotated coordinates (x, y) as

$$\begin{aligned} x &= \cos(\omega_0 t) \, x_0 + \sin(\omega_0 t) \, y_0 \\ y &= -\sin(\omega_0 t) \, x_0 + \cos(\omega_0 t) \, y_0 \quad . \end{aligned} \tag{4.122}$$

Therefore,

$$\begin{aligned} \dot{x} &= \cos(\omega_0 t) \, \dot{x}_0 + \sin(\omega_0 t) \, \dot{y}_0 + \omega_0 \, y \\ \dot{y} &= -\sin(\omega_0 t) \, x_0 + \cos(\omega_0 t) \, y_0 - \omega_0 \, x \quad . \end{aligned} \tag{4.123}$$

Therefore

$$(\dot{x} - \omega_0\, y)^2 + (\dot{y} + \omega_0\, x)^2 = \dot{x}_0^2 + \dot{y}_0^2 \quad , \tag{4.124}$$

The Lagrangian is then

$$L = \tfrac{1}{2}m\big(\dot{x} - \omega_0\, y\big)^2 + \tfrac{1}{2}m\big(\dot{y} + \omega_0\, x\big)^2 + \frac{G\, m_1\, m}{\sqrt{(x - x_1)^2 + y^2}} + \frac{G\, m_2\, m}{\sqrt{(x - x_2)^2 + y^2}} \quad , \tag{4.125}$$

which, with $x_1 \equiv -\alpha r_0$ and $x_2 \equiv \beta r_0$, agrees with eqn. 4.121

(c) Lagrange discovered that there are five special points where the satellite remains fixed in the rotating frame. These are called the *Lagrange points* {L1, L2, L3, L4, L5}. A sketch of the Lagrange points for the earth-sun system is provided in fig. 4.13. *Observation: In working out the rest of this problem, I found it convenient to measure all distances in units of r_0 and times in units of ω_0^{-1}, and to eliminate G by writing $Gm_1 = \beta\, \omega_0^2\, r_0^3$ and $Gm_2 = \alpha\, \omega_0^2\, r_0^3$.*

Assuming the satellite is stationary in the rotating frame, derive the equations for the positions of the Lagrange points.

Solution : At this stage it is convenient to measure all distances in units of r_0 and times in units of ω_0^{-1} to factor out a term $m\, r_0^2\, \omega_0^2$ from L, writing the dimensionless Lagrangian $\widetilde{L} \equiv L/(m\, r_0^2\, \omega_0^2)$. Using as well the definition of ω_0^2 to eliminate G, we have

$$\widetilde{L} = \tfrac{1}{2}\,(\dot{\xi} - \eta)^2 + \tfrac{1}{2}(\dot{\eta} + \xi)^2 + \frac{\beta}{\sqrt{(\xi + \alpha)^2 + \eta^2}} + \frac{\alpha}{\sqrt{(\xi - \beta)^2 + \eta^2}} \quad , \tag{4.126}$$

with

$$\xi \equiv \frac{x}{r_0} \quad , \quad \eta \equiv \frac{y}{r_0} \quad , \quad \dot{\xi} \equiv \frac{1}{\omega_0\, r_0}\frac{dx}{dt} \quad , \quad \dot{\eta} \equiv \frac{1}{\omega_0\, r_0}\frac{dy}{dt} \quad . \tag{4.127}$$

The equations of motion are then

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi - \frac{\beta(\xi + \alpha)}{d_1^3} - \frac{\alpha(\xi - \beta)}{d_2^3} \\ \ddot{\eta} + 2\dot{\xi} &= \eta - \frac{\beta\eta}{d_1^3} - \frac{\alpha\eta}{d_2^3} \quad , \end{aligned} \tag{4.128}$$

where

$$d_1 = \sqrt{(\xi + \alpha)^2 + \eta^2} \quad , \quad d_2 = \sqrt{(\xi - \beta)^2 + \eta^2} \quad . \tag{4.129}$$

Here, $\xi \equiv x/r_0$, $\xi = y/r_0$, *etc.* Recall that $\alpha + \beta = 1$. Setting the time derivatives to zero yields the static equations for the Lagrange points:

$$\begin{aligned} \xi &= \frac{\beta(\xi + \alpha)}{d_1^3} + \frac{\alpha(\xi - \beta)}{d_2^3} \\ \eta &= \frac{\beta\eta}{d_1^3} + \frac{\alpha\eta}{d_2^3} \quad , \end{aligned} \tag{4.130}$$

(d) Show that the Lagrange points with $y = 0$ are determined by a single nonlinear equation. Show graphically that this equation always has three solutions, one with $x < x_1$, a second with $x_1 < x < x_2$, and a third with $x > x_2$. These solutions correspond to the points L3, L1, and L2, respectively.

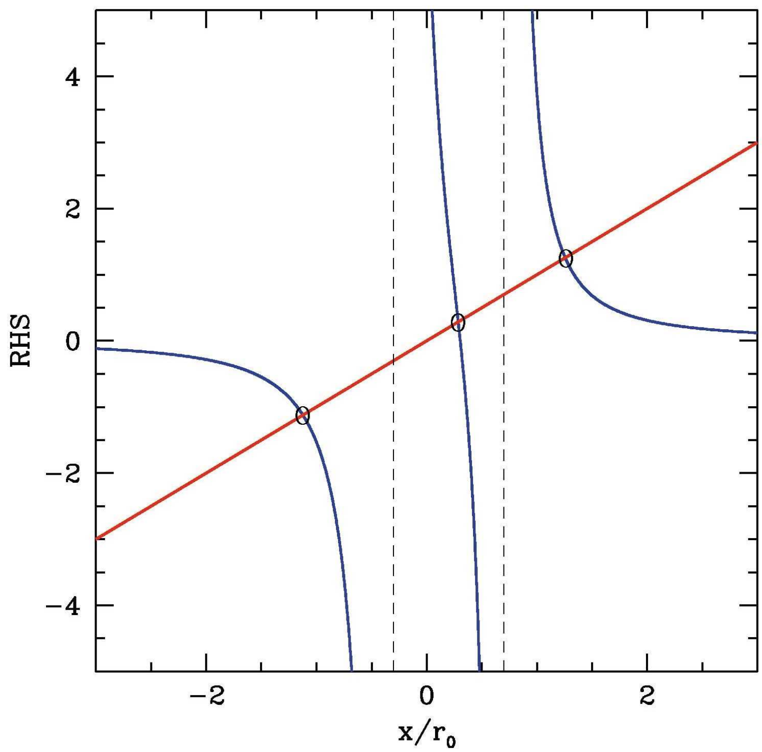

Figure 4.14: Graphical solution for the Lagrange points L1, L2, and L3.

Solution : If $\eta = 0$ the second equation is automatically satisfied. The first equation then gives

$$\xi = \beta \cdot \frac{\xi + \alpha}{\left|\xi + \alpha\right|^3} + \alpha \cdot \frac{\xi - \beta}{\left|\xi - \beta\right|^3} \quad . \tag{4.131}$$

The RHS of the above equation diverges to $+\infty$ for $\xi = -\alpha + 0^+$ and $\xi = \beta + 0^+$, and diverges to $-\infty$ for $\xi = -\alpha - 0^+$ and $\xi = \beta - 0^+$, where 0^+ is a positive infinitesimal. The situation is depicted in fig. 4.14. Clearly there are three solutions, one with $\xi < -\alpha$, one with $-\alpha < \xi < \beta$, and one with $\xi > \beta$.

(e) Show that the remaining two Lagrange points, L4 and L5, lie along equilateral triangles with the two masses at the other vertices.

Solution : If $\eta \neq 0$, then dividing the second equation by η yields

$$1 = \frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3} \quad . \tag{4.132}$$

Substituting this into the first equation,

$$\xi = \left(\frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3}\right)\xi + \left(\frac{1}{d_1^3} - \frac{1}{d_2^3}\right)\alpha\beta \quad , \tag{4.133}$$

gives

$$d_1 = d_2 \quad . \tag{4.134}$$

Reinserting this into the previous equation then gives the remarkable result,

$$d_1 = d_2 = 1 \quad , \tag{4.135}$$

which says that each of L4 and L5 lies on an equilateral triangle whose two other vertices are the masses m_1 and m_2. The side length of this equilateral triangle is r_0. Thus, the dimensionless coordinates of L4 and L5 are

$$\left(\xi_{\rm L4},\,\eta_{\rm L4}\right) = \left(\tfrac{1}{2} - \alpha,\, \tfrac{\sqrt{3}}{2}\right) \qquad , \qquad \left(\xi_{\rm L5},\,\eta_{\rm L5}\right) = \left(\tfrac{1}{2} - \alpha,\, -\tfrac{\sqrt{3}}{2}\right) \quad . \tag{4.136}$$

It turns out that L1, L2, and L3 are always unstable. Satellites placed in these positions must undergo periodic course corrections in order to remain approximately fixed. The SOlar and Heliopheric Observation satellite, *SOHO*, is located at L1, which affords a continuous unobstructed view of the Sun.

(f) Show that the Lagrange points L4 and L5 are stable (obviously you need only consider one of them) provided that the mass ratio m_1/m_2 is sufficiently large. Determine this critical ratio. Also find the frequency of small oscillations for motion in the vicinity of L4 and L5.

Solution : Now we write

$$\xi = \xi_{\rm L4} + \delta\xi \qquad , \qquad \eta = \eta_{\rm L4} + \delta\eta \quad , \tag{4.137}$$

and derive the linearized dynamics. Expanding the equations of motion to lowest order in $\delta\xi$ and $\delta\eta$, we have

$$\begin{aligned} \delta\ddot\xi - 2\delta\dot\eta &= \left(1 - \beta + \tfrac{3}{2}\beta\,\frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} - \alpha - \tfrac{3}{2}\alpha\,\frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\right)\delta\xi + \left(\tfrac{3}{2}\beta\,\frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} - \tfrac{3}{2}\alpha\,\frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\right)\delta\eta \\ &= \tfrac{3}{4}\,\delta\xi + \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\eta \end{aligned} \tag{4.138}$$

and

$$\begin{aligned} \delta\ddot\eta + 2\delta\dot\xi &= \left(\tfrac{3\sqrt{3}}{2}\beta\,\frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\,\frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\right)\delta\xi + \left(\tfrac{3\sqrt{3}}{2}\beta\,\frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\,\frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\right)\delta\eta \\ &= \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\xi + \tfrac{9}{4}\,\delta\eta \quad , \end{aligned} \tag{4.139}$$

where we have defined

$$\varepsilon \equiv \beta - \alpha = \frac{m_1 - m_2}{m_1 + m_2} \quad . \tag{4.140}$$

As defined, $\varepsilon \in [0,1]$.

Fourier transforming the differential equation, we replace each time derivative by $(-i\nu)$, and thereby obtain

$$\begin{pmatrix} \nu^2 + \frac{3}{4} & -2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon \\ 2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon & \nu^2 + \frac{9}{4} \end{pmatrix} \begin{pmatrix} \delta\hat\xi \\ \delta\hat\eta \end{pmatrix} = 0 \quad . \tag{4.141}$$

Nontrivial solutions exist only when the determinant D vanishes. One easily finds

$$D(\nu^2) = \nu^4 - \nu^2 + \tfrac{27}{16}\left(1 - \varepsilon^2\right) \quad , \tag{4.142}$$

which yields a quadratic equation in ν^2, with roots

$$\nu^2 = \tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{27\,\varepsilon^2 - 23} \quad . \tag{4.143}$$

These frequencies are dimensionless. To convert to dimensionful units, we simply multiply the solutions for ν by ω_0, since we have rescaled time by ω_0^{-1}.

Note that the L4 and L5 points are stable only if $\varepsilon^2 > \frac{23}{27}$. If we define the mass ratio $\gamma \equiv m_1/m_2$, the stability condition is equivalent to

$$\gamma = \frac{m_1}{m_2} > \frac{\sqrt{27}+\sqrt{23}}{\sqrt{27}-\sqrt{23}} = 24.960 \quad , \tag{4.144}$$

which is satisfied for both the Sun-Jupiter system ($\gamma = 1047$) – and hence for the Sun and any planet – and also for the Earth-Moon system ($\gamma = 81.2$).

Objects found at the L4 and L5 points are called *Trojans*, after the three large asteroids Agamemnon, Achilles, and Hector found orbiting in the L4 and L5 points of the Sun-Jupiter system. No large asteroids have been found in the L4 and L5 points of the Sun-Earth system.

Personal aside : David T. Wilkinson

The image in fig. 4.13 comes from the education and outreach program of the Wilkinson Microwave Anisotropy Probe (WMAP) project, a NASA mission, launched in 2001, which has produced some of the

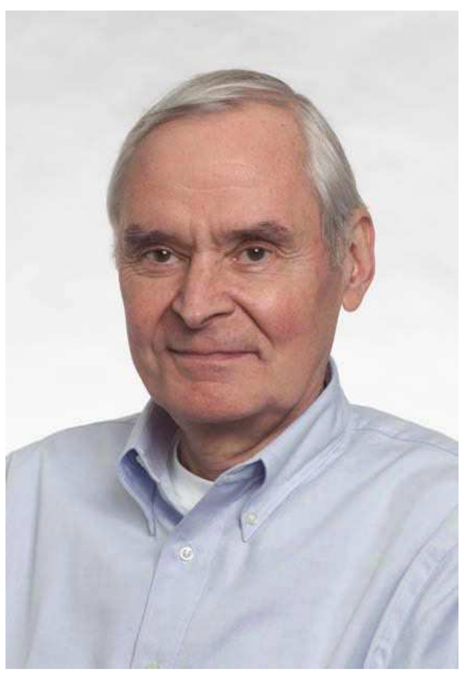

Figure 4.15: David T. Wilkinson (1935 – 2002).

most important recent data in cosmology. The project is named in honor of David T. Wilkinson, who was a leading cosmologist at Princeton, and a founder of the Cosmic Background Explorer (COBE) satellite (launched in 1989). WMAP was sent to the L2 Lagrange point, on the night side of the earth, where it can constantly scan the cosmos with an ultra-sensitive microwave detector, shielded by the earth from interfering solar electromagnetic radiation. The L2 point is of course unstable, with a time scale of about 23 days. Satellites located at such points must undergo regular course and attitude corrections to remain situated.

During the summer of 1981, as an undergraduate at Princeton, I was a member of Wilkinson's "gravity group," working under Jeff Kuhn and Ken Libbrecht. It was a pretty big group and Dave – everyone would call him Dave – used to throw wonderful parties at his home, where we'd always play volleyball. I was very fortunate to get to know David Wilkinson a bit – after working in his group that summer I took a class from him the following year. He was a wonderful person, a superb teacher, and a world class physicist.