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Quantum Mechanics B (Physics 130B) Fall 2014
Worksheet 6 – Solutions

Announcements

- The 130B web site is:

  http://physics.ucsd.edu/students/courses/fall2014/physics130b/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to kick the harmonic oscillator and talk about spontaneous emission.

Problems

1. **Give it a Kick**

   Consider the $D = 1$ simple harmonic oscillator in it's groundstate. Suppose something kicks the system imparting an additional momentum p_0. What's the probability the system remains in the ground state?

   (a) What's the new Hamiltonian for the system? Express this in terms of the usual ladder operators $\hat{a}$ and $\hat{a}^\dagger$
   $H_{new} = \frac{(p+p_0)^2}{2m} + \frac{1}{2}m\omega^2 x^2 = H_{old} + \frac{p\ p_0}{m} + \frac{p_0^2}{2m} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \mathbf{i}\frac{p_0}{m}\sqrt{\frac{m\omega}{2}}(\hat{a}^\dagger - \hat{a}) + \frac{p_0^2}{2m}$

   (b) Define a new operator $\hat{A} \equiv \hat{a} - \beta$ where $\beta \equiv \frac{1}{\mathbf{i}\omega}\frac{p_0}{m}\sqrt{\frac{m\omega}{2}}$.
   Show that the $\hat{A}$ are ladder operators: $[\hat{A}, \hat{A}^\dagger] = 1$
   This follows immediately from $[\hat{a}, \hat{a}^\dagger] = 1$ and that β is a constant.

   (c) Rewrite the new Hamiltonian in terms of these operators, what do you find?
   $H_{new} = \omega(\hat{A}^\dagger\hat{A} + \frac{1}{2})$

   (d) Relate the original groundstate $|0\rangle$ to the new groundstate $|\beta\rangle$
   Since the new Hamiltonian is another harmonic oscillator it must be that:
   $\hat{A}|\beta\rangle = 0 = (\hat{a} - \beta)|\beta\rangle$ or in other words $\hat{a}|\beta\rangle = \beta|\beta\rangle$ this is a *coherent* state.

   (e) Using $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$ compute $P = |\langle 0|\beta\rangle|^2$
   Hint: Insert identity and use the relation above.
   $|\beta\rangle = \mathbb{1}|\beta\rangle = \sum_n |n\rangle\langle n|\beta\rangle = \sum_n |n\rangle\langle 0|\frac{(\hat{a})^n}{\sqrt{n!}}|\beta\rangle = (\sum_n \frac{\beta^n}{\sqrt{n!}}|n\rangle)\langle 0|\beta\rangle$

Knowing this consider $\langle\beta|\beta\rangle = 1 = (\sum_n \frac{(|\beta|^2)^n}{n!}\langle n|n\rangle)|\langle 0|\beta\rangle|^2 = e^{|\beta|^2}|\langle 0|\beta\rangle|^2$
Therefore: $|\langle 0|\beta\rangle|^2 = e^{-|\beta|^2}$

2. **Multipole transitions**

Consider an electric field of the form:

$$\vec{E}(r,t) = E_0(\cos\omega t + (k\cdot r)\sin\omega t)\hat{n} \tag{1}$$

which is coupling to a particle of charge q. Recall from lecture that the interaction Hamiltonian is: $H' = -qE(r,t)\hat{n}\cdot r$ and that the spatially independent term produces a spontaneous decay rate of:

$$R_{f\to i} = \frac{\omega^3 q^2|\langle f|(\hat{n}\cdot r)|i\rangle}{\pi\epsilon_0\hbar c^3} \tag{2}$$

(a) Write the expression analogous to 2 for the spatially varying piece

Everything is the same except the matrix element gives you $R_{f\to i} = \frac{\omega^3 q^2|\langle f|(\hat{n}\cdot r)(k\cdot r)|i\rangle|^2}{\pi\epsilon_0\hbar c}$

If you then pull out $|k| = \frac{\omega}{c}$ you then find $R \propto \frac{\omega^5}{c^5}$

(b) Consider this problem where the particle is in a $D = 1$ oscillator potential with frequency Ω. Calculate the transition rate from n to $n-2$; don't calculate the averaging over $\hat{n}$ or $\hat{k}$

Choose the oscillator in the $\hat{x}$-direction so that $\hat{n}\cdot r = x\ \hat{n}_x$ and $\hat{k}\cdot r = x\ \hat{k}_x$

One finds $R \propto |\langle n-2|x^2|n\rangle|^2$ which $x^2 \propto (a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2)$ and only the $\hat{a}^2$ term contributes