University of California at San Diego – Department of Physics – TA: Shauna Kravec

# Quantum Mechanics B (Physics 130B) Fall 2014
# Worksheet 5 – Solutions

## Announcements

- The 130B web site is:

  **http://physics.ucsd.edu/students/courses/fall2014/physics130b/** .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to add angular momentum.

## Problems

1. **Combine?**

   Consider a system of two particles, one of spin-1 and another of spin-2. Let $\{s_1, m_1; s_2, m_2\}$ denote their spins and $\mathcal{H}_1$ and $\mathcal{H}_2$ their Hilbert spaces respectively.

   Suppose they interact with a Hamiltonian of the form:

   $$H = -\epsilon \vec{S}_1 \cdot \vec{S}_2 \tag{1}$$

   Let's understand the space of states for these particles

   (a) How many different spin states are allowed for particle 1? Equivalently, what is the dimension of $\mathcal{H}_1$? Particle 2?

   Particle 1 has $s_1 = 1$ and thus $m_1 \in \{-1, 0, 1\}$ thus dim $\mathcal{H}_1 = 3$

   Particle 2 has $s_2 = 2$ and thus $m_2 \in \{-2, -1, 0, 1, 2\}$ thus dim $\mathcal{H}_1 = 5$

   What's the dimension of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ ?

   dim $\mathcal{H} = 3 \times 5 = 15$

   One possible basis for $\mathcal{H}$ is the tensor product of the bases for $\mathcal{H}_1$ and $\mathcal{H}_2$

   Denote this as:

   $$|m_1; m_2\rangle \equiv |s_1 = 1, m_1\rangle \otimes |s_2 = 2, m_2\rangle \tag{2}$$

   Another possible basis is that of a combined angular momentum operator:

   $$\vec{S} \equiv \vec{S}_1 + \vec{S}_2 \tag{3}$$

This operator allows us to analyze the Hamiltonian 1 in terms of better quantum numbers. It also makes physical sense as the spin of the composite system.
A basis $|S, M\rangle$ associated with $S^2$ and $S_z$ of the combined pair is:

$$S^2|S,M\rangle = S(S+1)|S,M\rangle \quad S_z|S,M\rangle = M|S,M\rangle \tag{4}$$

The values of $S$ are not independent of $s_1$ and $s_2$; they can be thought of as the lengths allowed by adding independent $S_i$ vectors.
The allowed range is thus[1]:

$$|s_1 - s_2| \leq S \leq s_1 + s_2 \tag{5}$$

The $M$ quantum number is also directly determinable from the $m_i$ of the tensor product states as we'll see.

(b) Determine the number of independent $|S, M\rangle$ states. Does this match the value for dim $\mathcal{H}$ obtained previously?
$S = 1 \implies M \in \{-1, 0, 1\}$
$S = 2 \implies M \in \{-2, -1, 0, 1, 2\}$
$S = 3 \implies M \in \{-3, -2, -1, 0, 1, 2, 3\}$
The number of $|S, M\rangle$ states is then $3 + 5 + 7 = 15$, this is consistent.

(c) Rewrite the Hamiltonian 1 in terms of $S^2$. What are the energies associated with the $|S, M\rangle$ states?
$H = -\epsilon \vec{S}_1 \cdot \vec{S}_2$ where we note that $S^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$
Therefore $H = -\frac{\epsilon}{2}(S^2 - S_1^2 - S_2^2)$ which we can replace the $S_i^2$ with their eigenvalues because we're acting on states with definite $s_i$
$H = -\frac{\epsilon}{2}(S^2 - s_1(s_1+1)\mathbb{1} - s_2(s_2+1)\mathbb{1}) = -\frac{\epsilon}{2}(S^2 - 8\mathbb{1})$
Note that the spectrum is degenerate in $M$
$S = 1 \implies E = -\frac{\epsilon}{2}(2 - 8) = 3\epsilon$ , $S = 2 \implies E = \epsilon$ and $S = 3 \implies E = -2\epsilon$
Now let's derive explicit relations between the two bases we've constructed.
Recall that we define $S_\pm \equiv S_x \pm \mathbf{i}S_y = S_{1,\pm} + S_{2,\pm}$ such that:

$$S_\pm|S,M\rangle = \sqrt{S(S+1) - M(M\pm1)}|S, M\pm1\rangle \tag{6}$$

The *highest weight* state is $|3,3\rangle \equiv |m_1 = 1; m_2 = 2\rangle$ such that $S_+|3,3\rangle = 0$

(d) Using the $S_-$ operator and normalization/orthogonality constraints determine the values $a, b$ for which:

$$|3,2\rangle = a|1;1\rangle + b|0;2\rangle \tag{7}$$

First we note $a^2 + b^2 = 1$ by normalization. Then $S_-|3,3\rangle = \sqrt{6}|3,2\rangle$
We can also decompose $S_- = S_{1,-} + S_{2,-}$ to infer $\sqrt{6}|3,2\rangle = (S_{1,-} + S_{2,-})|1;2\rangle$
$(S_{1,-} + S_{2,-})|1;2\rangle = \sqrt{2}|0;2\rangle + 2|1;1\rangle \implies |3,2\rangle = \sqrt{\frac{1}{3}}|0;2\rangle + \sqrt{\frac{2}{3}}|1;1\rangle$
These are known as *Clebsch-Gordan coefficients*

[1] This is the same fact as $1 \otimes 2 = 1 \otimes 2 \otimes 3$; we're multiplying different $SU(2)$ representations