University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics B (Physics 130B) Fall 2014
Worksheet 3

Announcements

- The 130B web site is:

  http://physics.ucsd.edu/students/courses/fall2014/physics130b/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to learn about spin, rotations, representations, and all that jazz.

Problems

Suppose we are studying a system with a rotational symmetry. So we need understand how to *represent* this symmetry on our Hilbert space of states. This involves creating matrices which do all the things we expect.

1. **Do a Barrel Roll**

   Recall that in 3-dimensional space[1] we can derive the following rotation matrices from geometry:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

   where R_i is a rotation about the i-th axis by an angle θ. Consider a rotation with an infinitesimal $\theta = \delta\theta$.

   (a) Express each rotation in 1 as $R_i(\theta = \delta\theta) = \mathbb{1} - \mathbf{i}(\delta\theta)X_i$ for some matrices X_i. These are the *generators* of rotations as we'll see in a moment.[2]

[1] Euclidean. Over $\mathbb{R}$. Don't get cheeky.

[2] Note that the factor of $\mathbf{i}$ is conventional.

Spoilers. The form of X_i is simply:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathbf{i} \\ 0 & \mathbf{i} & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & \mathbf{i} \\ 0 & 0 & 0 \\ -\mathbf{i} & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & -\mathbf{i} & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

(b) Show explicitly that each X_i is Hermitian: $X^\dagger = X$

(c) I claim that the X_i of 2 satisfy the following algebra[3]

$$[X_i, X_j] = \mathbf{i}\epsilon^{ijk} X_k \tag{3}$$

Convince yourself of this by checking a few examples.

Given a hermitian matrix X one can construct a unitary matrix $U = e^{-\mathbf{i}Xa}$ which 'evolves' a state by an amount a. For example the Hamiltonian $\hat{H}$ is hermitian and leads to the 'time-evolution' operator $U = e^{-\mathbf{i}\hat{H}t}$.

In this way $\hat{H}$ *generates* time evolution. Can you guess where this is going?

(d) Consider the unitary matrices given by $U_i = e^{-\mathbf{i}X_i\theta}$ for each X_i in 2. Show, using Taylor's theorem, that $U_i = R_i$; they are the rotation matrices of 1.

2. **What *is* Spin?**

The fact there are spin-$\frac{1}{2}$ particles is one of the most deeply quantum features of nature.

We can think of the spin of an electron as an additional degree of freedom. This is represented quantum mechanically is a two dimensional Hilbert space $\mathcal{H}_2$ spanned by two vectors $\{|\uparrow\rangle, |\downarrow\rangle\}$

Now, how can we represent rotations on this space?

Consider the following matrices:

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

These, up to that factor of $\frac{1}{2}$, are known as the Pauli matrices.

(a) Show that S_i are hermitian. Show explicitly that the following algebra is satisfied:

$$[S_i, S_j] = \mathbf{i}\epsilon^{ijk} S_k \tag{5}$$

This is the same algebra as 3, between the generators of rotations![4] Together these imply we are constructing something like angular momentum.

Now let's construct the analog of rotation matrices for these objects.

(b) Define $U_i = e^{-\mathbf{i}\theta S_i}$ and write a simple matrix expression for it.

Hint: Use the fact $\sigma_i^2 = \mathbb{1}$ where σ_i is a Pauli matrix.

(c) Now consider $U_i(\theta = 2\pi)$, what has happened?

[3] This is known as a Lie algebra

[4] Fancy math point, this is the statement $SO(3)$ and $SU(2)$ have the same Lie algebra.