

Quantum Mechanics B (Physics 130B) Fall 2014 Worksheet 1 – Solutions

Announcements

- The 130B web site is:

<http://physics.ucsd.edu/students/courses/fall2014/physics130b/> .

Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to remind ourselves about 130A and the hydrogen atom.

This is going to be the only worksheet I ask you to turn in to me. Nothing is graded. Solutions will be posted on the course website later.

A quick note on notation. I use units where $\hbar = 1 = c$ and where the Pauli matrices are just X, Y, Z . You guys can plug in the factors back for fun.

Problems

1. Who are you? (And where did you go?)

Please share with me a bit about your interests in physics and what you are curious about. Possibilities include: experimental or theoretical particle physics, biophysics, string theory, experimental or theoretical condensed matter (soft or hard), astrophysics, atomic physics. The list goes on.

Please rate your comfort with the following claims about quantum mechanics. Use the following scale for reference:

- 1 "I'm just hearing this now."
- 2 "I ran into once on the street on a rainy weekend evening in 1997."
- 3 "I get a warm feeling of truthiness from it."
- 4 "I know it but maybe not why."
- 5 "This was my breakfast this morning"

Questions about Quantum:

- (a) Wavefunctions represent (potential) physical states.
- (b) Wavefunctions are vectors¹ in a vector space, often named after Hilbert.
- (c) Operators are used to describe physics. These are, sometimes infinite, matrices on the Hilbert space.
- (d) The coefficients of these vectors/matrices are in general complex numbers. This feature is responsible for 'quantum interference'.
- (e) Matrices in general don't commute, $AB \neq BA$. This feature is responsible for 'quantum uncertainty'.
- (f) 'Hermitian' matrices are their own complex conjugate transpose. This means they have real eigenvalues.
- (g) Hermitian represent (potentially) observable quantities like momentum or energy.
- (h) A wavefunction has a definite momentum, energy, spin, etc.. if it is an eigenvector of the associated Hermitian operator. The eigenvalue is this quantity.
- (i) There is an inner/dot product associated with this vector space. This inner product allows to define 'probability distributions' for physical quantities.
- (j) Quantum states need to be 'normalized'. The inner product with itself should be 1. This means the sum of probabilities is also 1.
- (k) Two quantum states are 'distinct' if the inner product between them is 0.
- (l) 'Degeneracy' means two distinct quantum states have the same eigenvalue of a given Hermitian operator.
- (m) The 'expectation value' of a Hermitian operator is the average value of it's eigenvalues you will measure for a given wavefunction.
- (n) The inverse of a 'Unitary' operator is it's complex conjugate transpose.
- (o) Unitary operators 'evolve' or change a quantum state. For example, time evolution is by a unitary operator.
- (p) Unitary operators are constructed by exponentiation of a Hermitian operator.

2. Particles on Spheres

Consider particle of mass M freely living on a ring of radius R . In position space it lives on the interval $x \in [0, 2\pi R)$ where $x \simeq x + 2\pi R$

- (a) Write down the Hamiltonian of the system in terms of L_z just from classical mechanics. Show how the Schroedinger equation is a position space realization of this. Use the angular coordinate $\phi \equiv \frac{x}{R}$

The Hamiltonian should be $\hat{H} = \frac{L_z^2}{2I}$ where $I \equiv MR^2$ is the moment of inertia.

The Schroedinger equation is $-\frac{1}{2M}\partial_x^2\psi(x) = E\psi(x) \rightarrow -\frac{1}{2MR^2}\partial_\phi^2\psi(\phi) = E\psi(\phi)$

This implies the position space representation of $L_z = -i\partial_\phi$ where the minus sign is conventional

¹More precisely they are *rays* which means multiplication by a complex phase doesn't change the state.

- (b) Recall from the theory of angular momentum we diagonalize $L_z|\psi\rangle = m|\psi\rangle$ For this example, what is the relation between m and our physical variables? What does this tell us about the spectra?

Recall $\hat{H}|\psi\rangle = E|\psi\rangle$. Substituting the Hamiltonian we get: $E|\psi\rangle = \frac{m^2}{2I}|\psi\rangle$. Therefore $m^2 = 2IE$.

Because $E \propto m^2$ there is a two-fold degeneracy of $\pm m$. This the fact that the energy is the same going widdershins or anti-widdershins.

- (c) Use the form of m you derive above to rewrite the Schroedinger equation and solve it. Demand that $\psi(\phi) = \psi(\phi + 2\pi)$. What does this tell you about m ?

The Schroedinger equation is simply: $\psi''(\phi) = -m^2\psi(\phi) \implies \psi(\phi) \propto e^{im\phi}$

Take $\phi \rightarrow \phi + 2\pi$. $\psi(\phi) \rightarrow e^{im\phi}e^{2\pi im}$ So we need to impose the demand that $e^{2\pi im} \equiv 1$, which implies $m \in \mathbb{Z}$. The angular momentum is quantized!

OK! Let's be more bold. Consider that particle constrained to the sphere of radius R .

Define spherical coordinates: $x = R \sin \theta \cos \phi$, $y = R \sin \theta \sin \phi$, $z = R \cos \theta$ where $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ Recall on the sphere $\nabla^2 = \frac{1}{R^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{R^2 \sin^2 \theta} \partial_\phi^2$

- (a) What is the Hamiltonian in terms of angular momentum operators?

In terms of angular momentum $\hat{H} = \frac{L^2}{2I} = \frac{1}{2I}(L_x^2 + L_y^2 + L_z^2)$ The position space Schroedinger equation is simply: $-\frac{1}{2I}(\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2)\psi(\theta, \phi) = E\psi(\theta, \phi)$

- (b) Recall again from angular momentum that we can diagonalize $L^2|\psi\rangle = \ell(\ell+1)|\psi\rangle$ in addition to L_z .² Use this to relate ℓ to the physical variables. Write the Schroedinger equation using this relation and find two ODE's by separation of variables $\psi(\theta, \phi) = \Theta(\theta)\Phi(\phi)$

$\ell(\ell+1) = 2IE$ similar to before

Let's do the separation out explicitly:

$$\begin{aligned} \left[\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right] \Theta(\theta)\Phi(\phi) &= -\ell(\ell+1)\Theta(\theta)\Phi(\phi) \\ \rightarrow \frac{\sin \theta}{\Theta} [\partial_\theta (\sin \theta \partial_\theta)] \Theta + \ell(\ell+1) \sin^2 \theta &= -\frac{\partial_\phi^2 \Phi}{\Phi} = C \end{aligned}$$

The equation $-\partial_\phi^2 \Phi = C\Phi$ is just $L_z^2 \Phi = C\Phi$ so $C = m^2$ and $\Phi(\phi) \propto e^{im\phi}$

Plugging this in we arrive at the differential equation for $\Theta(\theta)$:

$$\sin \theta [\partial_\theta (\sin \theta \partial_\theta)] \Theta + \ell(\ell+1) \sin^2 \theta \Theta - m^2 \Theta = 0$$

Solutions to the Θ ODE may be familiar as the (associated) Legendre polynomials. Similar to before, periodicity requires ℓ and m to be integral. Together we conclude:

$$\psi(\theta, \phi) \equiv \langle \theta, \phi | \psi \rangle = \frac{1}{\sqrt{2\pi}} P_\ell^m(\cos \theta) e^{im\phi} \equiv Y_\ell^m(\theta, \phi) \quad (1)$$

where Y_ℓ^m are known as the *spherical harmonics*. This is most of the solution to the hydrogen atom aside from the annoying radial piece. We'll construct these functions again in a slightly different manner when studying angular momentum.

²This is a consequence of $[L^2, L_z] = 0$