HW 4

6.22) eq. 6.80: $E_{fs}' = \langle n\, l\, m_l\, m_s | (H_r' + H_{so}') | n\, l\, m_l\, m_s \rangle$

$$E_{fs}' = \underbrace{\langle n l m_l m_s | H_r' | n l m_l m_s \rangle}_{\text{eq. 6.57}} + \underbrace{\langle n l m_l m_s | H_{so}' | n l m_l m_s \rangle}_{\text{eq. 6.61}}$$

$$E_{fs}' = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right] + \langle n l m_l m_s | \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L} | n l m_l m_s \rangle$$

eq. 6.64 $\langle n l m_l m_s | \frac{1}{r^3} | n l m_l m_s \rangle = \frac{1}{l(l+1/2)(l+1)n^3a^3}$

$$\langle S\cdot L\rangle = \langle S_x L_x\rangle + \langle S_y L_y\rangle + \langle S_z L_z\rangle$$

but L and S commute, they act on completely different quantum numbers, we might as well write

eq 6.81 $\langle S\cdot L\rangle = \langle S_x\rangle\langle L_x\rangle + \langle S_y\rangle\langle L_y\rangle + \langle S_z\rangle\langle L_z\rangle = \hbar^2 m_l m_s$

we have:

$$\langle \tfrac{1}{r^3}\vec{S}\cdot\vec{L}\rangle$$

but if $[\frac{1}{r^3}, S\cdot L] = 0$ then we can write $\langle \frac{1}{r^3}\vec{S}\cdot\vec{L}\rangle = \langle\frac{1}{r^3}\rangle\langle\vec{S}\cdot\vec{L}\rangle$

neither $\vec{S}$ or $\vec{L}$ care about r

- $\vec{S}$ has nothing to do with position space
- $\vec{L} = \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$ only cares about θ, ϕ

$$E_{fs}' = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{l+1/2} - 3\right] + \frac{e^2}{8\pi\epsilon_0 m^2c^2}\,\frac{\hbar^2 m_l m_s}{l(l+1/2)(l+1)n^3a^3}$$

$$E_{fs}' = -\underbrace{\frac{2(E_n)^2}{mc^2}}_{=\frac{(13.6\,eV)\alpha^2}{n^4}}\left[\frac{n}{l+1/2} - \frac{3}{4}\right] + \underbrace{\frac{e^2\hbar^2}{8\pi\epsilon_0 m^2c^2a^3}}_{=(13.6\,eV)\alpha^2}\left[\frac{m_l m_s}{l(l+1/2)(l+1)n^3}\right]$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

$$E_{fs}' = \frac{(13.6\,eV)\alpha^2}{n^3}\left[\frac{3}{4n} - \frac{l(l+1) - m_l m_s}{l(l+1/2)(l+1)}\right]$$

6.24) eq 6.72: $E_Z' = \frac{e}{2m} \vec{B}_{ext} \cdot \langle \vec{L} + 2\vec{S} \rangle$

$$\langle \vec{L} + 2\vec{S} \rangle = \langle \vec{L} \rangle + 2 \langle \vec{S} \rangle$$

$$\langle \vec{L} \rangle = \langle L_x \rangle \hat{x} + \langle L_y \rangle \hat{y} + \langle L_z \rangle \hat{z}$$
$$= \tfrac{1}{2} \langle L_+ + L_- \rangle \hat{x} + \tfrac{1}{2i} \langle L_+ - L_- \rangle \hat{y} + \langle L_z \rangle \hat{z}$$

$\langle L_\pm \rangle = 0$ since $\langle l m | l \, m \pm 1 \rangle = 0$

$\langle \vec{L} \rangle = \langle L_z \rangle \hat{z} = 0$ for $l = 0$

$$\langle \vec{S} \rangle = \langle S_z \rangle \hat{z} = \hbar m_s$$

$$E_Z' = \frac{e}{2m} (B_{ext} \cdot \hat{z}) \, 2 m_s \hbar = 2 m_s \mu_B (B_{ext} \cdot \hat{z}) \qquad \mu_B = \frac{e\hbar}{2m}$$

eq 6.67

$$E_{nj} = \frac{-13.6\text{eV}}{n^2} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + 1/2} - \frac{3}{4} \right) \right]$$

$l = 0$, $s = 1/2$ (electron)
so $j = 1/2$ is the only option

$$E = \underbrace{\frac{-13.6\text{eV}}{n^2}}_{\text{unperturbed}} \Big[1 + \underbrace{\frac{\alpha^2}{n^2} \left(n - \frac{3}{4} \right)}_{\text{fine structure}} \Big] + \underbrace{2 m_s \mu_B (B_{ext} \cdot \hat{z})}_{\text{zeeman}}$$

back to 6.82, if we set the indeterminant to unity

$$E_{fs}' = \frac{13.6\text{eV}}{n^3} \alpha^2 \left(\frac{3}{4} n - 1 \right) = \frac{-13.6}{n^4} \alpha^2 \left(n - \frac{3}{4} \right)$$

3) $L_z|l,m\rangle = \hbar m|l,m\rangle$

$L^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$

$L_x = \frac{1}{2}(L_+ + L_-)$, $L_y = \frac{1}{2i}(L_+ - L_-)$

$$L_x|l,m\rangle = \tfrac{1}{2}L_+|l,m\rangle + \tfrac{1}{2}L_-|l,m\rangle$$

$$L_\pm|l,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\,|l,m\pm1\rangle$$

$$\boxed{L_x|l,m\rangle = \tfrac{1}{2}\hbar\left(l(l+1)-m(m+1)\right)^{1/2}|l,m+1\rangle + \tfrac{1}{2}\hbar\left(l(l+1)-m(m-1)\right)^{1/2}|l,m-1\rangle}$$

$$\boxed{L_y|l,m\rangle = \tfrac{1}{2i}\hbar\left(l(l+1)-m(m+1)\right)^{1/2}|l,m+1\rangle - \tfrac{1}{2i}\hbar\left(l(l+1)-m(m-1)\right)^{1/2}|l,m-1\rangle}$$

$$\begin{aligned}
L_x^2|l,m\rangle &= L_xL_x|l,m\rangle \\
&= \tfrac{1}{4}\hbar^2\left(l(l+1)-(m+1)(m+2)\right)^{1/2}\left(l(l+1)-m(m+1)\right)^{1/2}|l,m+2\rangle \\
&\quad + \tfrac{1}{4}\hbar^2\left(l(l+1)-(m+1)(m)\right)^{1/2}\left(l(l+1)-m(m+1)\right)^{1/2}|l,m\rangle \\
&\quad + \tfrac{1}{4}\hbar^2\left(l(l+1)-(m-1)(m)\right)^{1/2}\left(l(l+1)-m(m-1)\right)^{1/2}|l,m\rangle \\
&\quad + \tfrac{1}{4}\hbar^2\left(l(l+1)-(m-1)(m-2)\right)^{1/2}\left(l(l+1)-m(m-1)\right)^{1/2}|l,m-2\rangle
\end{aligned}$$

$$\boxed{\begin{aligned}
L_x^2|l,m\rangle &= \tfrac{1}{4}\hbar^2\left(l(l+1)-(m+1)(m+2)\right)^{1/2}\left(l(l+1)-m(m+1)\right)^{1/2}|l,m+2\rangle \\
&\quad + \tfrac{1}{4}\hbar^2\left(2l(l+1)-2m^2\right)|l,m\rangle \\
&\quad + \tfrac{1}{4}\hbar^2\left(l(l+1)-(m-1)(m-2)\right)^{1/2}\left(l(l+1)-m(m-1)\right)^{1/2}|l,m-2\rangle
\end{aligned}}$$

With the above information:

$$\boxed{\langle L_x\rangle = \langle L_y\rangle = 0 \qquad \langle L_x^2\rangle = \tfrac{1}{2}\hbar^2\left(l(l+1)-m^2\right)}$$

4) $H' = bx^2$

ⓐ $E_n^\circ = \hbar\omega(n+1/2)$

ⓑ $E_n' = \langle \psi_n^\circ | H' | \psi_n^\circ \rangle$

$E_n' = b\frac{\hbar}{2m\omega}\langle n|(a_+ + a_-)^2|n\rangle \qquad x = \sqrt{\frac{\hbar}{2m\omega}}(a_- + a_+)$

$E_n' = b\frac{\hbar}{2m\omega}\langle n|\cancel{a_+^2} + \cancel{a_-^2} + a_+a_- + a_-a_+|n\rangle$

$a_-|n\rangle = \sqrt{n}|n-1\rangle$
$a_+|n\rangle = \sqrt{n+1}|n+1\rangle$

$E_n' = b\frac{\hbar}{2m\omega}\left(\sqrt{n}\langle n|a_+|n-1\rangle + \sqrt{n+1}\langle n|a_-|n+1\rangle\right)$

$E_n' = b\frac{\hbar}{2m\omega}\left(n + n + 1\right)$

$$\boxed{E_n' = \frac{\hbar b}{m\omega}(n+1/2)}$$

ⓒ $E_n^2 = \sum_{m\neq n}\frac{b^2|\langle\psi_m|x^2|\psi_n\rangle|^2}{E_n^\circ - E_m^\circ}$

$E_n^2 = \frac{b^2}{\hbar\omega}\frac{\hbar^2}{4m^2\omega^2}\sum_{m\neq n}\frac{|\langle m|a_+^2 + a_-^2 + a_+a_- + a_-a_+|n\rangle|^2}{n-m}$

$= \frac{b^2\hbar}{4m^2\omega^3}\sum_{m\neq n}\frac{|\sqrt{n+1}\sqrt{n+2}\langle m|n+2\rangle + \sqrt{n}\sqrt{n-1}\langle m|n-2\rangle + \underbrace{\cancel{n\langle m|n\rangle}}_{m\neq n} + \underbrace{\cancel{(n+1)\langle m|n\rangle}}_{m\neq n}|^2}{n-m}$

$E_n^2 = \frac{b^2\hbar}{4m^2\omega^3}\left[\frac{(n+1)(n+2)}{-2} + \frac{n(n-1)}{2}\right]$

$E_n^2 = \frac{b^2\hbar}{8m^2\omega^3}(\cancel{n^2} - n - \cancel{n^2} - 3n - 2) = \boxed{\frac{-b^2\hbar}{2m^2\omega^3}(n+1/2)}$

5) neglecting spin, the $n=2$ states are

$$\psi_{200},\ \psi_{210},\ \psi_{211},\ \psi_{21-1}$$

using the notation $\psi = \psi_{n\ell m} = R_{n\ell} Y_\ell^m$

they all have the same energy, so we need degenerate perturbation theory

for an electric field $\mathcal{E}$ the perturbation is

$$H_s' = e\mathcal{E} z = e\mathcal{E} r\cos\theta$$ where I've chosen $\mathcal{E}$ to point in the z-direction

$$\langle n_1 \ell_1 m_1 | H_s' | n_2 \ell_2 m_2 \rangle = e\mathcal{E}\int dr\, r^2 R_{n_1\ell_1} R_{n_2\ell_2}\, r \int d\theta\, d\phi\, \sin\theta\, Y_{\ell_1}^{m_1 *} Y_{\ell_2}^{m_2} \cos\theta$$

first, I'll just look at the angular portion

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \qquad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$$

also note $\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\cos\theta = 0$ because it's an odd function in the interval $0<\theta<\pi$

and $\int_0^{2\pi} d\phi\, e^{\pm i\phi} = 0$

the only cases where both the θ and ϕ integrals are nonzero are

$$\langle 200 | H' | 210 \rangle, \quad \langle 210 | H' | 200 \rangle$$

$\langle 200 | H' | 210 \rangle = \langle 210 | H' | 200 \rangle$ both states are real

$$= e\mathcal{E}\, \frac{1}{\sqrt{2}} a^{-3/2} \frac{1}{\sqrt{24}} a^{-3/2} \frac{1}{\sqrt{4\pi}} \left(\frac{3}{4\pi}\right)^{1/2} \int_0^{\infty} dr\, r^3 \left(1 - \frac{1}{2}\frac{r}{a}\right)\frac{r}{a} e^{-r/a} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\cos^2\theta$$

$$= e\mathcal{E}\, 2\pi \frac{1}{\sqrt{2}} a^{-4} \frac{1}{\sqrt{24}} \frac{1}{\sqrt{4\pi}} \left(\frac{3}{4\pi}\right)^{1/2} \int_0^{\infty} dr\, r^3 \left(r - \frac{1}{2}\frac{r^2}{a}\right) e^{-r/a} \int_0^{\pi} d\theta\, \sin\theta\cos^2\theta$$

integrate by parts

u-substitution

5) $\langle 200|H'|210\rangle = \frac{e\mathcal{E}}{8a^4}(-36a^5)(2/3)$

$= -3e\mathcal{E}a$

$$W = \begin{array}{cc} \begin{pmatrix} 0 & -3e\mathcal{E}a & 0 & 0 \\ -3e\mathcal{E}a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \psi_{200} \\ \psi_{210} \\ \psi_{211} \\ \psi_{21-1} \end{array} \\ \begin{array}{cccc} \psi_{200} & \psi_{210} & \psi_{211} & \psi_{21-1} \end{array} & \end{array}$$

solving for the eigenvalues

$\lambda^2(\lambda^2 - 9e^2\mathcal{E}^2a^2) = 0$

$\lambda_1 = 0, \; \lambda_2 = 0, \; \lambda_3 = 3e\mathcal{E}a, \; \lambda_4 = -3e\mathcal{E}a$

degeneracy is partially broken, but we still have a two fold degeneracy left

eigenstates are

for $\lambda_1, \lambda_2 = 0$

$$\begin{pmatrix} 0 & -3e\mathcal{E}a & 0 & 0 \\ -3e\mathcal{E}a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \psi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; \psi_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

for $\lambda_3 = 3e\mathcal{E}a$

$$\psi_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

for $\lambda_4 = -3e\mathcal{E}a$

$$\psi_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$