HW 2 Solutions

4.2) $\Psi = A\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{a}\right)\sin\left(\frac{n_z \pi z}{a}\right)$

$E = \frac{\vec{n}^2 \pi^2 \hbar^2}{2ma^2}$ $\qquad \vec{n}^2 = n_x^2 + n_y^2 + n_z^2$

energy levels determined by n^2

E_1	$n^2 = 3$	$(1,1,1)$	example states
E_2	$n^2 = 6$	$(2,1,1)$	
E_3	$n^2 = 9$	$(2,2,1)$	
E_4	$n^2 = 11$	$(3,1,1)$	
⋮	$n^2 = 12$	$(2,2,2)$	
	$n^2 = 14$	$(3,2,1)$	
	$n^2 = 17$		
	$n^2 = 18$		
	$n^2 = 19$		
	$n^2 = 21$		
	n^2 22		
	$n^2 = 24$		
	$n^2 = 26$		
	$n^2 = 27$	$(3,3,3), (5,1,1), (1,5,1), (1,1,5)$	

degeneracy is $\boxed{d = 4}$

in all the cases above the degeneracy is either 3, 6, or 9
This case is unique because there are two different possible combinations of states. Permutations of $(3,3,3)$ and $(1,1,5)$

4.9) $V(r) = \begin{cases} -V_0 & \text{if } r \le a \\ 0 & r > a \end{cases}$

using the radial equation with $l=0$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + (V-E)u = 0 \quad \leftarrow \quad u = \psi/r$$

inside

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}(E+V_0)u = 0$$

outside

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}Eu = 0$$

for bound states: $E<0$, $V_0>0$

boundary conditions: ψ should vanish at ∞
ψ should not blow up at $r=0$

$u_{in} = C\sin(\beta r)$ $\quad$ $u_{out} = Ae^{-\kappa r}$ $\quad$ C, A are constants

$$\kappa = \sqrt{\frac{2m|E|}{\hbar}}$$

$$\beta = \sqrt{\frac{2m(E+V_0)}{\hbar}}$$

$u_{in}|_{r=a} = u_{out}|_{r=a}$

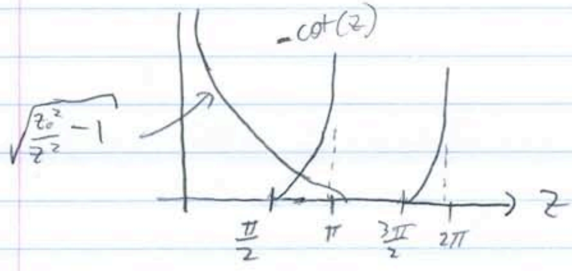
$u_{in}'|_{r=a} = u_{out}'|_{r=a}$

$\cot(\beta a) = -\frac{\kappa}{\beta}$ $\quad$ transcendental equation

as with the document I posted, and section 2.6 of Griffiths, define $\beta a = z$, and $z_0 = \sqrt{\frac{2mV_0}{\hbar^2}}\,a$

$\cot(z) = -\sqrt{(z_0/z)^2-1}$ $\quad$ plotting both sides of this equation reveals there is one solution for $\pi/2 < z < \pi$

4.9) rough sketch:



the first intersection occurs for $\pi/2 < z < \pi$

$$\frac{\pi}{2} < \sqrt{\frac{2m(E+V_0)}{\hbar^2}}\,a < \pi$$

$$\frac{\hbar^2\pi^2}{8ma^2} < E+V_0 < \frac{\hbar^2\pi^2}{2ma^2}$$ for the location of the first bound state (not required)

no bound state exists for $z_0 < \pi/2$

$$\frac{2mV_0a^2}{\hbar^2} < \frac{\pi^2}{4}$$

$$\boxed{V_0 a^2 < \frac{\hbar^2\pi^2}{8m}}$$

6.1) $H' = \alpha\,\delta(x-a/2)$

we need $\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

a) $E_n' = \langle \psi_n | H' | \psi_n \rangle = \frac{2}{a}\alpha \int \sin^2\left(\frac{n\pi x}{a}\right)\delta(x-a/2)$

$$\boxed{E_n' = \frac{2}{a}\alpha\sin^2\left(\frac{n\pi}{2}\right)}$$ for even n, $\sin^2\left(\frac{n\pi}{2}\right) = 0$

b) $\psi_1' = \sum_{n\neq 1} \frac{\langle \psi_n^0 | H' | \psi_1^0 \rangle}{E_1^0 - E_n^0}\psi_1^0$ $\quad E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$ $\quad$ first 3 corrections are $n = 3, 5, 7$

6.1) $\langle \psi_3^0 | H' | \psi_1^0 \rangle = \frac{2}{a}\alpha \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right) = -\frac{2\alpha}{a}$

$\langle \psi_5^0 | H' | \psi_1^0 \rangle = \frac{2}{a}\alpha \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{5\pi}{2}\right) = \frac{2\alpha}{a}$

$\langle \psi_7^0 | H' | \psi_1^0 \rangle = \frac{2}{a}\alpha \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{7\pi}{2}\right) = -\frac{2\alpha}{a}$

$$E_n^0 - E_1^0 = \frac{\pi^2\hbar^2}{2ma^2}(1-n^2)$$

$$\psi_1' = \frac{2\alpha}{a}\frac{2ma^2}{\pi^2\hbar^2}\left(\frac{1}{8}\psi_3^0 - \frac{\psi_5^0}{24} + \frac{\psi_7^0}{48}\right)$$

$$\psi_1' = \frac{m\alpha}{\pi^2\hbar^2}\sqrt{\frac{a}{2}}\left[\sin\left(\frac{3\pi x}{a}\right) - \frac{1}{3}\sin\left(\frac{5\pi x}{a}\right) + \frac{1}{6}\sin\left(\frac{7\pi x}{a}\right)\right]$$

6.2) $E_n = \hbar\omega(n+1/2)$

$k' = (1+\epsilon)k$

a) $E_n = (n+\frac{1}{2})\hbar\omega_0\sqrt{1+\epsilon}$ $\qquad \omega_0 = \sqrt{k/m}$

expanding $\sqrt{1+\epsilon}$:

$$\boxed{E_n = (n+\tfrac{1}{2})\hbar\omega_0\left(1+\tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2 + \cdots\right)}$$

b) $E_n' = \langle \psi_n | H' | \psi_n \rangle = \langle \psi_n | \frac{1}{2}\epsilon k x^2 | \psi_n \rangle$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \Rightarrow x^2 = \frac{\hbar}{2m\omega}(a_+^2 + a_+a_- + a_-a_+ + a_-^2)$$

$$E_n' = \frac{\hbar}{2m\omega}\frac{1}{2}\epsilon k\Big(\underbrace{\cancel{\langle \psi_n | a_+^2 | \psi_n \rangle}}_{0} + \underbrace{\cancel{\langle \psi_n | a_-^2 | \psi_n \rangle}}_{0} + \langle \psi_n | a_+a_- | \psi_n \rangle + \langle \psi_n | a_-a_+ | \psi_n \rangle\Big)$$

6.2) $H = \hbar\omega(a_+a_- + \frac{1}{2})$

$a_+a_- = \frac{H}{\hbar\omega} - 1/2$

we can use these to solve for the energy

$[a_-, a_+] = 1$

$a_-a_+ = 1 + a_+a_-$

$$E_n' = \frac{\hbar\epsilon k}{4m\omega}\left(\langle\psi_n| \frac{H}{\hbar\omega} \cancel{-\tfrac{1}{2}} |\psi_n\rangle + \langle\psi_n| \frac{H}{\hbar\omega} \cancel{+\tfrac{1}{2}} |\psi_n\rangle \right)$$

$$E_n' = \frac{\hbar\epsilon k}{2m\omega}\frac{1}{\hbar\omega}\langle\psi_n|H|\psi_n\rangle = \frac{\epsilon k}{2m\omega^2}\hbar\omega(n+1/2) \qquad \omega^2 = k/m$$

$$E_n' = \frac{\epsilon\hbar}{2\omega}\omega^2(n+1/2) = \boxed{\frac{\epsilon\hbar\omega(n+\frac{1}{2})}{2}}$$

6.3) $V(x_1, x_2) = -aV_0\,\delta(x_1 - x_2)$

for bosons: $\psi(x_1, x_2) = A[\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)]$

using $H = \sum \frac{p_i^2}{2m} + V(x_1, x_2, \ldots)$

$$H|\psi\rangle = (E_1 + E_2)|\psi\rangle \Rightarrow E_i = \frac{\pi^2\hbar^2 n_i^2}{2ma^2}$$

ground state is $n_1 = n_2 = 1$, which means $\psi_1 = \psi_2$

$$\boxed{\psi(x_1, x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)} \qquad E = \frac{\pi^2\hbar^2}{ma^2}$$

6.3) 1^{st} excited state $n_1 = 1,\ n_2 = 2$ or the other way around, doesn't matter

$$\psi_1 = \sqrt{2/a}\,\sin\left(\frac{\pi x}{a}\right)$$
$$\psi_2 = \sqrt{2/a}\,\sin\left(\frac{2\pi x}{a}\right)$$

$$\boxed{\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\frac{2}{a}\left(\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\right)}$$

$$E = \frac{5\pi^2\hbar^2}{2ma^2}$$

(b) $$E_1' = \langle \psi_1^0 | H' | \psi_1^0 \rangle = -aV_0\frac{4}{a^2}\int_0^a \sin^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right)\delta(x_1 - x_2)\,dx_1\,dx_2$$

$$E_1' = -4V_0\frac{1}{a}\int_0^a \sin^4\left(\frac{\pi x_1}{a}\right)dx_1$$

$$\boxed{E_1' = -3/2\,V_0}$$

ran out of room, but its the other half of ψ as shown above

$$E_2' = \langle \psi_2^0 | H' | \psi_2^0 \rangle = -aV_0\frac{2}{a^2}\int \delta(x_1 - x_2)\left(\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \cdots\right)^2 dx_1\,dx_2$$

$$E_2' = -2V_0\frac{1}{a}\int 2^2\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right)dx$$

$$E_2' = -8V_0\frac{1}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right)dx$$

$$\boxed{E_2' = -2V_0}$$