Problem Set II: Due Wednesday, February 13, 2013

1.)   Consider a magnetic flux tube frozen into a moving fluid.

a.)   If c_1 and c_2 are any two curves encircling the flux tube, show

$$\oint_{c_1} \underline{A} \cdot d\ell = \oint_{c_2} A \cdot d\ell.$$

b.)   Show that the strength of the flux tube is constant in time. 'Strength' is defined by the integral in part a.).

2a.)   Show that for incompressible MHD in two dimensions, the basic equations can be written as:

$$\left(\partial_t + \underline{v} \cdot \nabla\right)\nabla^2 \phi = \left(B \cdot \nabla\right)\nabla^2 A + v\nabla^2\nabla^2\phi + \tilde{f}$$

$$\left(\partial_t + \underline{v} \cdot \nabla\right)A = \eta\nabla^2 A.$$

Here v is viscosity, η is resistivity, $\underline{v} = \underline{\nabla}\phi \times \hat{z}$ and $\underline{B} = \underline{\nabla}A \times \hat{z}$. $\tilde{f}$ is a random force. Take $P = P(\rho)$.

b.)   Take $\underline{B} = B_0 \hat{x}$ to be a weak in-plane magnetic field. Calculate the real frequency and damping for Alfven waves.

c.)   Using quasilinear theory, calculate the turbulent resistivity induced by a spectrum of Alfven waves in 2D MHD. For $v \to 0$, interpret your result in terms of the freezing-in-law. Why does viscosity enter your result for part i.)? Why does η enter? Contrast these.

d.)   Taking $\underline{B} = B_0 \hat{x}$ and $\langle \tilde{V}_y \tilde{A}\rangle = -\eta_T \partial A_0/\partial y$ as a definition of turbulent resistivity η_T. Show that at stationarity

$$\eta_T = \eta\langle \tilde{B}^2\rangle\big/B_0^2,$$

assuming the system has periodic boundary conditions. Discuss your result and its implications. This is a famous result, referred to as the Zeldovich Theorem, after Ya.B. Zeldovich.

e.)   What happens if one pair of boundaries are open? (Hint: Consider flux thru surface.)

3a.)   Derive the tensor virial theorem for a warm, self-gravitating fluid in an external gravitational potential $\phi_{ext}(\underline{x})$. In particular, how does $\phi_{ext}(\underline{x})$ change the virial balance?

(a.)   Describe the structure of $\phi_{ext}(\underline{x})$, relative to the blob, which is required to confine the fluid.

4a.)   Consider a self-gravitating fluid with speed of sound c_s. Derive the Jeans length-i.e. the scale which is marginal to collapse instability.

b.)   Now consider a blob of collisionless self-gravitating matter. For simplicity, take $f_0(v) = f_0$, a constant, for $u_0 - \Delta V < v < u_0 + \Delta V$. Calculate the Jeans length for this system.

c.)   Now consider a similar distribution in a plasma. Characterize the marginal states for that system.

d.)   For a realistic gas, how does the Jeans length evolve as collapse proceeds?

5a.)   Kulsrud 5.1

b.)   Derive an energy balance theorem for acoustic waves. Discuss the structure.

6.)   Kulsrud 5.2 – Ignore the last sentence of the problem.

7.)   Kulsrud 5.3

8.)   Kulsrud 5.4

9.)   Kulsrud 11.1 – Omit the second paragraph.

10.)   Kulsrud 7.1

11.)   Kulsrud 7.2

12a.)   Use quasi-linear theory to write a mean field electron momentum equation for current-driven ion acoustic waves. Include weak collisions.

b.)   What does your result imply about Sweet-Parker type reconnection at low collisionality? In your answer, treat the electron distribution self-consistently. Give the anomalous resistivity explicitly, in terms of the wave spectrum.

c.)   Use the approach in Section 14.7 of Kulsrud to quantify how far above marginality the CDIA can be expected to get. Express your answer in terms of the parameters of the problem. Clarify the momentum transfer process and the ultimate fate of the momentum.

13a.)   Revisit the Sweet-Parker reconnection problem, discussed in class. Now, consider a very weakly collisional plasma, so that *electron viscosity* is the dominant dissipation coefficient, i.e. now

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times \underline{v} \times \underline{B} + \eta \nabla^2 \underline{B} - \mu \nabla^2 \nabla^2 \underline{B} \, ,$$

where μ has dimensions of $\ell^2 *$ diffusion. Hereafter, neglect η. Following the analysis in class, calculate the thickness of the reconnection layer and the reconnection velocity. Express your answer in terms of an effective "So-and-So Number". Discuss your result as compared to the classic Sweet-Parker result.

b.)   What would be the effect of finite fluid viscosity on the Sweet-Parker process?

c.)   Derive an equation for the rate of increase in temperature due reconnection-induced ohmic heating. What is the temperature in the layer?