HW4 Solution

2 (FW 7.16)

$$L = \frac{1}{2} m \sum_{i=1}^{N} \dot{\eta}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} (\eta_{i+1} - \eta_i)^2 \qquad \eta_0 = \eta_{N+1} = 0$$

a) $$p_i = \frac{\partial L}{\partial \dot{\eta}_i} = m \dot{\eta}_i$$

$$H = \sum_i p_i \dot{\eta}_i - L = \frac{1}{2} m \sum_i \dot{\eta}_i^2 + \frac{1}{2} k \sum_i (\eta_{i+1} - \eta_i)^2$$

$$= \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} k \sum_i (\eta_{i+1} - \eta_i)^2$$

b) Rewrite $$H = \frac{1}{2}\left(\frac{m}{a}\right) \sum_{i=1} a \dot{\eta}^2 + \frac{1}{2} ka \sum_{i=0}^{N} a \left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2$$

Take $a \to 0$, $\frac{m}{a} \to \sigma$, $\dot{\eta} \to u_t$, $\frac{\eta_{i+1} - \eta_i}{a} \to u_x$, $ka \to \tau$

$$\sum_i a \to \sum_i \Delta x_i \to \int dx$$

$$\Rightarrow H = \frac{1}{2} \sigma \int u_t^2 \, dx + \frac{1}{2} \tau \int u_x^2 \, dx \Rightarrow \mathcal{H} = \frac{1}{2} \sigma u_t^2 + \frac{1}{2} \tau u_x^2$$ //

For p, $$p_{total} = \sum_i p_i = \sum_i \left(\frac{m}{a}\right) a \dot{\eta} \Rightarrow \int \sigma u_t \, dx$$

$\Rightarrow \wp = \sigma \dot{u}$ (momentum density) //

4 $$\mathcal{L} = \frac{h^2}{8\pi^2 m}\nabla\psi\cdot\nabla\psi^* + V\psi^*\psi + \frac{h}{2\pi i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right)$$

ψ: $$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial\underline{x})} = \frac{h^2}{8\pi^2 m}\nabla\psi^*$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = V\psi^* - \frac{h}{2\pi i}\dot{\psi}^* = \frac{d}{d\underline{x}}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial\underline{x})} = \frac{h^2}{8\pi^2 m}\nabla^2\psi^*$$

$$\Rightarrow -\frac{h^2}{8\pi^2 m}\nabla^2\psi^* + V\psi^* = -\frac{ih}{2\pi}\frac{\partial\psi}{\partial t}$$

c.c.

$$\Rightarrow -\frac{h^2}{8\pi^2 m}\nabla^2\psi + V\psi = \frac{ih}{2\pi}\frac{\partial\psi}{\partial t} \quad //$$

Canonical momenta:

$$\cancel{\underline{p} = \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial\underline{x})} = \frac{h^2}{8\pi^2 m}\nabla\psi^* \quad \text{and} \quad \underline{p}^* = \frac{h^2}{8\pi^2 m}\nabla\psi}$$

$$\cancel{\mathcal{H} = \underline{p}(\partial\psi/\partial\underline{x}) + \underline{p}^*(\partial\psi^*/\partial\underline{x}) - \mathcal{L}}$$

$$\cancel{= \frac{h^2}{8\pi^2 m}\nabla\psi\cdot\nabla\psi^* - V\psi^*\psi - \frac{h}{2\pi i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right)}$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{h}{2\pi i}\psi^* \qquad \pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = -\frac{h}{2\pi i}\psi$$

$$\Rightarrow \mathcal{H} = \pi\dot{\psi} + \pi^*\dot{\psi}^* - \mathcal{L}$$

$$= \frac{h}{2\pi i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right) - \mathcal{L}$$

$$= -\frac{h^2}{8\pi^2 m}\nabla\psi\cdot\nabla\psi^* - V\psi^*\psi \quad //$$

5. $\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3$ $\qquad \alpha, \beta$ small $\sim \epsilon$

Let $x = x_0 + x_1 + \cdots \mathcal{O}(\epsilon^2)$ $\qquad$ Also, $x_0 = a\cos\omega t$
$\omega = \omega_0 + \omega_1 + \cdots \mathcal{O}(\epsilon^2)$

Rewrite: $\quad \frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}$

Plug in $x = x_0 + x_1$

$$\Rightarrow \frac{\omega_0^2}{\omega^2}\left(\cancel{-\omega^2 x_0} + \ddot{x}_1\right) + \omega_0^2(\cancel{x_0} + x_1) = -\alpha(x_0 + x_1)^2 - \beta(x_0 + x_1)^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)(-\omega^2 x_0 - \ddot{x}_1)$$

$\mathcal{O}(\epsilon)$

$$\Rightarrow \frac{\omega_0^2}{\omega^2}\ddot{x}_1 + \omega_0^2 x_1 \simeq -\alpha x_0^2 - \beta x_0^3 + (\omega^2 - \omega_0^2)x_0 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}_1$$

$$\Rightarrow \ddot{x}_1 + \omega_0^2 x_1 \simeq -\alpha x_0^2 - \beta x_0^3 + (\omega^2 - \omega_0^2)\,\cancel{\cos}\, x_0 \qquad \circledast$$

Since $\omega \simeq \omega_0 + \omega_1$

$$\omega^2 = \omega_0^2 + 2\omega_0\omega_1 + \mathcal{O}(\epsilon^2) \Rightarrow \omega^2 - \omega_0^2 = 2\omega_0\omega_1$$

Also $\quad x_0^2 = a^2\cos^2\omega t = \frac{a^2}{2}(1 + \cos 2\omega t)$
$\qquad x_0^3 = a^3\cos^3\omega t = \frac{a^3}{4}(3\cos\omega t + \cos 3\omega t)$

$$\circledast \Rightarrow \ddot{x}_1 + \omega_0^2 x_1 = -\frac{\alpha a^2}{2} - \frac{3a^3\beta}{4}\cos\omega t - \frac{\alpha a^2}{2}\cos 2\omega t - \frac{a^3\beta}{4}\cos 3\omega t + 2\omega_0\omega_1 a\cos\omega t$$

eliminat $\cos\omega t$ term
$\Rightarrow \qquad -\frac{3a^3\beta}{4} + 2\omega_0\omega_1 a = 0 \Rightarrow \omega_1 = \frac{3}{8}\frac{a^2\beta}{\omega_0}$

As $\circledast$ is linear in x_1, we find $x_1 = f(\omega)\cos 3\omega t + g(\omega)\cos 2\omega t + h(\omega)$

5 Cont'd, ① $\ddot{x}_1 + \omega_0^2 x_1 = -\frac{a^3\beta}{4}\cos 3\omega t$

$x_1 = f(\omega)\cos 3\omega t \qquad f \sim O(\epsilon)$

$\Rightarrow (-9\omega^2 + \omega_0^2) f(\omega) = -\frac{a^3\beta}{4}$

$-8\omega_0^2 f = -\frac{a^3\beta}{4} \Rightarrow f = \frac{a^3\beta}{32\omega_0^2} = \frac{1}{2}\left(\frac{\beta a^3}{16\omega_0^2}\right)$ //

② $\ddot{x}_1 + \omega_0^2 x_1 = -\frac{\alpha a^2}{2}\cos 2\omega t$

Plug in $g(\omega)\cos 2\omega t \Rightarrow (-4\omega^2 + \omega_0^2) g(\omega) = -\frac{\alpha a^2}{2}$

$\Rightarrow g(\omega) = \frac{1}{2}\left(\frac{\alpha a^2}{3\omega_0^2}\right)$

$$\Rightarrow \begin{cases} x_1 = -\frac{1}{2}\left(\frac{\alpha a^2}{\omega_0^2}\right) + \frac{1}{2}\left(\frac{\alpha a^2}{3\omega_0^2}\right)\cos 2\omega t + \frac{1}{2}\left(\frac{\beta a^3}{16\omega_0^2}\right)\cos 3\omega t \\ \omega_1 = \frac{3}{8}\frac{a^2\beta}{\omega_0} \end{cases}$$ //

//

7. $\dot{x} = -cxy \quad , \quad \dot{y} = cxy - by$

Since $x, y \geqslant 0$ $\quad \dot{x} < 0$ for all time. $\Rightarrow x(0)$ is maximum

$$\dot{y} = c(x - \tfrac{b}{c})y$$

$\Rightarrow$ if $x(t=0) < b/c \Rightarrow x(t) < b/c \Rightarrow \dot{y}(t) < 0$

If $x(0) > b/c \Rightarrow \dot{y}(t) > 0$ when $x(t) > b/c$

$\overset{\text{after some time}}{\Longrightarrow}$ $\dot{y}(t) < 0$ w/ $x(t) < b/c$ //

6. $\ddot{x} + f(x)\dot{x} + g(x) = 0$

i) $\dot{x} = y - F(x) \qquad \dot{y} = -g(x)$, $\quad F(x) = \int^x f(u)\,du$

$$\Rightarrow \ddot{x} = \dot{y} - \frac{d}{dt}F(x)$$
$$= -g(x) - \frac{dx}{dt}F' = -g(x) - \dot{x}f(u)$$

$$\Rightarrow \ddot{x} + f(x)\dot{x} + g(x) = 0 \;//$$

ii) $g(x) = x$, Point P on Lienard plane $\Rightarrow$ direction of orbit $= dy/dx$

Construction: $P = (x_0, y_0)$

$\rightarrow Q = (x_0, F(x_0))$ $\quad \because Q$ on $y = F(x)$ and same x as P

$\rightarrow R = (0, F(x_0))$ $\quad \because R$ on y-axis, and same y as R

$$\Rightarrow \text{Slope of } \overrightarrow{RP} = \frac{y_0 - F(x_0)}{x_0}$$

$$\Rightarrow \text{direction} = -1/\text{slope} = \frac{-x_0}{y_0 - F(x_0)} = \frac{-g(x_0)}{y_0 - F(x_0)} = \frac{\dot{y}(x_0)}{\dot{x}(x_0)} = \left.\frac{dy}{dx}\right|_{x_0, y_0} //$$

8 i) $\dot{x} = x(a - cy) \qquad \dot{y} = -y(b - cx)$

$$\ddot{x} = \dot{x}(a - cy) - cx\dot{y}$$

$$= x(a-cy)^2 + cxy(b-cx) \qquad \ddot{x}(x=0, y=0) = 0$$

$$\ddot{y} = -\dot{y}(b-cx) + cy\dot{x}$$

$$= +y(b-cx)^2 + cxy(a-cy) \qquad \ddot{y}(x=0, y=0) = 0$$

$\Rightarrow \mathbf{X} = (0,0)$ is a saddle point.

ii) $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Rightarrow \begin{matrix} a - cy = 0 \\ b - cx = 0 \end{matrix} \Rightarrow \begin{matrix} y = a/c \\ x = b/c \end{matrix}$

iii) $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = -\dfrac{y(b-cx)}{x(a-cy)}$

$$\Rightarrow \frac{a-cy}{y}\,dy + \frac{b-cx}{x}\,dx = 0$$

$$\Rightarrow a\ln y - cy + b\ln x - cx = d$$

d is integration const.

Small oscillation: $y = a/c + \xi, \quad x = \frac{b}{c} + \eta$

$$\dot{x} = \dot{\eta} = \left(\frac{b}{c} + \eta\right)(a - a - c\xi) = b\xi + \mathcal{O}(\eta\xi)$$

$$\dot{y} = \dot{\xi} = -\left(\frac{a}{c} + \xi\right)(b - b - c\eta) = -a\eta$$

$$\Rightarrow \ddot{\eta} = -ab\eta \Rightarrow \omega = \frac{2\pi}{T} = \sqrt{ab} \Rightarrow T = \frac{2\pi}{\sqrt{ab}} \;//$$

(10)

9.4 FW

Let $\underline{v} = \underline{u} + \tilde{u}$ $\quad\quad \underline{u} = \text{const.}\ ,\ \tilde{u} = -\nabla\Phi$

$\rho = \rho_0 + \tilde{\rho}$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0 \Rightarrow \frac{\partial}{\partial t}\tilde{\rho} + \rho_0\nabla\cdot\tilde{u} + \cancel{\tilde{\rho}\nabla\cdot\underline{u}} + \cancel{\tilde{u}\cdot\nabla\rho_0} + \underline{u}\cdot\nabla\tilde{\rho} = 0$$

$$\frac{\partial}{\partial t}\underline{v} + \underline{v}\cdot\nabla\underline{v} = -\frac{1}{\rho}\nabla P \quad\quad \text{Let } P = P(\rho) \Rightarrow \frac{1}{\rho}\nabla P = \frac{c^2}{\rho}\nabla\rho$$

$$\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\tilde{\rho} = -\rho_0\nabla\cdot\tilde{u}$$

$$\Rightarrow \frac{\partial}{\partial t}\tilde{u} + \underline{u}\cdot\nabla\tilde{u} = -\frac{c^2}{\rho_0}\nabla\tilde{\rho}$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\Phi = \frac{c^2}{\rho_0}\tilde{\rho} \quad\quad \text{by } \nabla\cdot\underline{u} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)^2\Phi = \frac{c^2}{\rho_0}\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\tilde{\rho}$$
$$= -c^2\,\nabla\cdot\tilde{u}$$
$$= c^2\,\nabla^2\Phi$$

$$\Rightarrow \left[\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)^2\right]\Phi = 0 \quad //$$

(11)

9.5 $\quad \underline{\zeta} \equiv \nabla\times\underline{v}$

a) Since $\nabla\cdot\underline{\zeta} = 0 \Rightarrow$ no source/sink

$\Rightarrow$ field line does not start/~~the~~ end unless at boundary

b) Isentropic Flow: $P = k\rho^{\gamma} \Rightarrow \rho^{-1}\nabla P = \nabla h \quad\quad h = \int\frac{dP}{\rho}$

$$\Rightarrow \frac{\partial}{\partial t}\underline{v} + \underline{v}\cdot\nabla\underline{v} = -\nabla h$$

Curl $\Rightarrow \frac{\partial}{\partial t}\underline{\zeta} - \nabla\times(\underline{v}\times\underline{\zeta}) = 0 \Rightarrow \frac{\partial\zeta}{\partial t} = \nabla\times(\underline{v}\times\underline{\zeta}) \quad //$

c) Incompressible: $\frac{D\rho}{Dt} = \rho\nabla\cdot\underline{v} = 0$

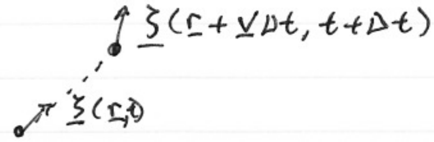
$$\frac{\partial}{\partial t}\underline{\zeta} = \underline{\zeta}\cdot\nabla\underline{v} - \underline{v}\cdot\nabla\underline{\zeta} + \underline{v}\cancel{(\nabla\cdot\zeta)} - \underline{\zeta}\cancel{\nabla\cdot\underline{v}} \Rightarrow \frac{d\zeta}{dt} \equiv \frac{\partial\zeta}{\partial t} + \underline{v}\cdot\nabla\zeta = (\zeta\cdot\nabla)\underline{v} \quad //$$

$\nabla\cdot\nabla\times = 0$

(11)

9.5

c) Cont'd ; Recall



Change of ζ along same fluid element:

$$\delta\underline{\zeta} = \underline{\zeta}(\underline{r}+\underline{V}\Delta t, t+\Delta t) - \zeta(\underline{r},t) = \left(\frac{\partial}{\partial t} + \underline{V}\cdot\nabla\right)\underline{\zeta}\,\Delta t$$

While for RHS : $\zeta\cdot\nabla\underline{V}$ is the rate of change of $\underline{V}$ along $\underline{\zeta}$

$\Rightarrow$ ζ move as if rigidly attached with the fluid (moving w/ $\underline{V}$)

d) For compressible flow : $\frac{D\rho}{Dt} = -\rho\,\nabla\cdot\underline{V}$

$$\frac{D}{Dt}\rho^{-1} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = +\frac{1}{\rho}\nabla\cdot\underline{V}$$

Since, $\frac{\partial\zeta}{\partial t} = \nabla\times(\underline{V}\times\underline{\zeta}) = -\underline{V}\cdot\nabla\zeta + \zeta\cdot\nabla\underline{V} + \underline{V}\overbrace{\cancel{(\nabla\cdot\zeta)}}^{0} - \zeta(\nabla\cdot\underline{V})$

$$\Rightarrow \frac{D\underline{\zeta}}{Dt} = \underline{\zeta}\cdot\nabla\underline{V} - \underline{\zeta}(\nabla\cdot\underline{V})$$

$$= \underline{\zeta}\cdot\nabla\underline{V} - \rho\frac{D}{Dt}\rho^{-1}$$

$$\Rightarrow \frac{D}{Dt}\left(\zeta/\rho\right) = \left(\zeta/\rho\cdot\nabla\right)\underline{V} \quad //$$

Thus, ζ/ρ is the quantity that moves with the fluid.

9.27 Euler Equation in rotating frame

Let $\underline{\Omega} = \Omega \hat{z}$ and we work in cylindrical coordinate sys.

$(\varpi, \phi, z) \to (\varpi, \varphi + \Omega t, z)$ ϖ : radial $= r$

Non-rotating : $\frac{\partial}{\partial t} \underline{v} + \underline{v} \cdot \nabla \underline{v} = - \frac{\nabla P}{\rho} + \underline{f}$

can do completely in vector-form or other coordinate..

Transform velocity and derivatives

Velocity :
$$v_\varpi = v_\varpi$$
$$v_\phi = v_\varphi + \Omega \varpi$$
$$v_z = v_z$$

Derivatives : Since $\phi = \varphi + \Omega t$, $d\phi = d\varphi + \Omega dt$

Total differential of arbitrary f :

$$df = \left(\frac{\partial f}{\partial t}\right)_{\varpi, \phi} dt + \left(\frac{\partial f}{\partial \varpi}\right)_{\phi, t} d\varpi + \left(\frac{\partial f}{\partial \phi}\right)_{\varpi, t} d\phi + \cdots$$

z-term is simple

$$= \left[\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \phi} \right] dt + \left(\frac{\partial f}{\partial \varpi}\right)_{\phi, t} d\varpi + \left(\frac{\partial f}{\partial \phi}\right)_{\varpi, t} d\varphi + \cdots$$

$$\Rightarrow \underbrace{\left(\frac{\partial f}{\partial t}\right)_{\varphi}}_{\text{"rotating"}} = \left(\frac{\partial f}{\partial t}\right)_{\phi} + \Omega \left(\frac{\partial f}{\partial \phi}\right)_{t}, \quad \left(\frac{\partial f}{\partial \varpi}\right)_{\varphi} = \left(\frac{\partial f}{\partial \varpi}\right)_{\phi}, \quad \cdots$$

Normal vectors : $\hat{e}_\varpi, \hat{e}_\phi, \hat{e}_z$

We need $\frac{d}{d\phi} e_\phi = - \hat{e}_\varpi$, $\frac{d}{d\phi} \hat{e}_\varpi = \hat{e}_\phi$, $\cancel{\frac{d\hat{e}_\phi}{dt} = - \Omega \hat{e}_\varpi}$

$\cancel{\frac{d}{dt} \hat{e}_\varpi = \Omega \hat{e}_\phi}$

Writing the eq. in component form:

$$\varpi: \quad LHS = \frac{\partial}{\partial t} v_\varpi + v_\varpi \frac{\partial}{\partial \varpi} v_\varpi + \frac{v_\phi}{\varpi}\frac{\partial}{\partial \phi} v_\varpi - \frac{v_\phi^2}{\varpi} + v_z \frac{\partial}{\partial z} v_\varpi$$

$$= \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right) V_\varpi + V_\varpi \frac{\partial}{\partial \varpi} V_\varpi + \frac{V_\varphi}{\varpi}\frac{\partial}{\partial \phi} V_\varpi - \frac{V_\varphi^2}{\varpi} - 2\Omega V_\varphi$$

$$- \varpi\Omega^2 + V_z \frac{\partial}{\partial z} V_\varpi \qquad \text{by} \quad v_\phi = V_\varphi + \varpi\Omega$$

$$= \left(\frac{\partial}{\partial t}\right)_\varphi V_\varpi + V_\varpi \frac{\partial}{\partial \varpi} V_\varpi + \frac{V_\varphi}{\varpi}\frac{\partial}{\partial \varphi} V_\varpi + V_z \frac{\partial}{\partial z} V_\varpi - \frac{V_\varphi^2}{\varpi} - 2\Omega V_\varphi - \varpi\Omega^2$$

$$= \underbrace{\left[\frac{\partial}{\partial t}\underline{v}' + \underline{v}' \cdot \nabla \underline{v}'\right]_\varpi}_{\text{rotating frame}} \underbrace{- 2\Omega V_\varphi}_{\text{Coriolis force}} \underbrace{- \varpi\Omega^2}_{\text{centrifugal}}$$

$$\phi: \quad LHS = \frac{\partial}{\partial t} v_\phi + v_\varpi \frac{\partial}{\partial \varpi} v_\phi + \frac{v_\phi}{\varpi}\frac{\partial}{\partial \phi} v_\phi + v_z \frac{\partial}{\partial z} v_\phi + \frac{v_\varpi v_\phi}{\varpi}$$

$$= \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right)(V_\varphi + \varpi\Omega) + V_\varpi \frac{\partial}{\partial \varpi}(V_\varphi + \varpi\Omega) + \frac{V_\varphi}{\varpi}\frac{\partial}{\partial \phi} V_\varphi + V_z \frac{\partial}{\partial z} V_\varphi$$

$$+ \frac{V_\varpi V_\varphi}{\varpi} + \Omega V_\varpi$$

$$= \left(\frac{\partial}{\partial t} V_\varphi\right)_\varphi + V_\varpi \frac{\partial}{\partial \varpi} V_\varphi + \frac{V_\varphi}{\varpi}\frac{\partial}{\partial \varphi} V_\varphi + V_z \frac{\partial}{\partial z} V_\varphi + \frac{V_\varpi V_\varphi}{\varpi} + 2\Omega V_\varpi$$

$$z: \quad LHS = \left(\frac{\partial}{\partial t} V_{\cancel{\varphi} z}\right)_\varphi + V_\varpi \frac{\partial}{\partial \varpi} V_z + \frac{V_\varphi}{\varpi}\frac{\partial}{\partial \varphi} V_z + V_z \frac{\partial}{\partial z} V_z$$

Putting these back into vector form: $\underline{v}' = (\varpi, \varphi, z)$

$$\frac{\partial}{\partial t}\underline{v}' + \underline{v}' \cdot \nabla \underline{v}' - \varpi\Omega^2 + 2\underline{\Omega} \times \underline{v}' = -\frac{1}{\rho}\nabla P + \underline{f}$$

$$\Rightarrow \frac{\partial}{\partial t}\underline{v}' + \underline{v}' \cdot \nabla \underline{v}' = -\nabla\left(\frac{P}{\rho} - \frac{1}{2}\varpi^2\Omega^2\right) - 2\underline{\Omega} \times \underline{v}' + \underline{f}$$

$$= -\nabla\left(\frac{P}{\rho} - \frac{1}{2}|\underline{\Omega} \times \vec{r}|^2\right) - 2\underline{\Omega} \times \underline{v}' + \underline{f}$$

$\underline{r} = \varpi \hat{e}_\varpi + z\hat{e}_z$ (position)

9.27

b) $\underline{f} = -\nabla U \quad \Rightarrow \quad \nabla\times\underline{f} = 0$

$\underline{\zeta} \equiv \nabla\times\underline{v}$

Write: $\frac{\partial}{\partial t}\underline{v} + \nabla\frac{1}{2}|\underline{v}|^2 - \underline{v}\times\nabla\times\underline{v} = -\nabla\left(\rho^{-1}p - \frac{1}{2}|\underline{\Omega}\times\underline{r}|\right)$

$- 2\underline{\Omega}\times\underline{v} + \underline{f}$

Take curl :

$\frac{\partial}{\partial t}\underline{\zeta} - \nabla\times(\underline{v}\times\underline{\zeta}) = -2\nabla\times(\underline{\Omega}\times\underline{v})$

$\Rightarrow \frac{\partial}{\partial t}\underline{\zeta} = \nabla\times[\underline{v}\times(\underline{\zeta}+2\underline{\Omega})]$ //

For steady flow, $|\zeta| \ll \Omega \quad \Rightarrow \quad \underline{\zeta}+2\underline{\Omega} \simeq 2\underline{\Omega}$

Also, $\dot{\zeta} = 0 \quad \Rightarrow \quad \nabla\times(\underline{v}\times\underline{\Omega}) = 0$

$= -\underline{v}\cdot\underbrace{\cancel{\nabla\underline{\Omega}}}_{\Omega\text{ const.}} + \underline{\Omega}\cdot\nabla\underline{v} + \underline{v}\,\underbrace{\cancel{\nabla\cdot\underline{\Omega}}}_{\Omega\text{ const.}} - \underline{\Omega}\,\underbrace{\cancel{\nabla\cdot\underline{v}}}_{\text{incompressible}}$

$\Rightarrow \underline{\Omega}\cdot\nabla\underline{v} = 0$ //

Recall $\underline{\Omega} = \Omega\hat{z}$, $\hat{z}$ is the rotation axis,

$\underline{\Omega}\cdot\nabla\underline{v} = \Omega\frac{\partial}{\partial z}\underline{v} = 0 \quad \Rightarrow$ $\underline{v}$ has no variation along the rot. axis

$\Rightarrow \underline{v} = \underline{v}(x, y, t)$ hence "2D" //

c) Suppose $\underline{v} = \underline{v}_0 e^{i(\underline{k}\cdot\underline{r}-\omega t)}$ ← non-steady $\Rightarrow \frac{\partial\zeta}{\partial t} \neq 0$

$\underline{\zeta} = \nabla\times\underline{v} = i\underline{k}\times\underline{v} \quad \frac{\partial}{\partial t}\underline{\zeta} = +\omega\underline{k}\times\underline{v} = 2\underline{\Omega}\cdot\nabla\underline{v} = 2i\Omega k_z\underline{v}$

By $\underline{k}\times\underline{k}\times\underline{v} = \underline{k}(\underbrace{\cancel{\underline{k}\cdot\underline{v}}}_{\nabla\cdot\underline{v}=0}) - k^2\underline{v}$, $-\omega k^2\underline{v} = 2i\Omega k_z\underline{k}\times\underline{v}$

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c) Cont'd, $-\omega k^2 \underline{v} = 2i\Omega k_z \underline{k} \times \underline{v}$

$$= 2i\Omega k_z \left(\frac{2i\Omega k_z}{\omega}\right) \underline{v}$$

$$\Rightarrow \quad \omega^2 = 4\Omega^2 k_z^2 / k^2$$

$$\omega = \pm 2\Omega k_z / k = \pm 2 \underline{\Omega} \cdot \hat{k}$$

Polarization: Since $\omega \underline{k} \times \underline{v} = 2i\Omega k_z \underline{v}$

$$\hat{k} \times \underline{v} = i \underline{v}$$ → circularly polarized

see p.41 of Landau's Fluid Book.

Phase velocity: $v_p = \frac{\omega}{k}$

$$= \pm 2 \frac{\underline{\Omega} \cdot \underline{k}}{k^2}$$

Group velocity: $\underline{v_g} = \frac{\partial \omega}{\partial \underline{k}} = \cancel{\hat{z} \frac{\partial 2\Omega k_z}{\partial k_z}}$

$\omega = 2\Omega k_z / k$

$k = \sqrt{k_x^2 + k_y^2 + k_z^2}$

$$v_{g,z} = \frac{\partial \omega}{\partial k_z} = 2\Omega \left[\frac{1}{k} - \frac{1}{2} \frac{2k_z^2}{k^3} \right]$$

$$= \frac{2\Omega}{k} \left(1 - \frac{k_z^2}{k^2} \right)$$

$$v_{g,x} = \frac{\partial \omega}{\partial k_x} = 2\Omega \cdot - \frac{k_x k_z}{k^3} = - \frac{2\Omega}{k} \left(\frac{k_x k_z}{k^2} \right)$$

$$v_{g,y} = \frac{\partial \omega}{\partial k_y} = 2\Omega \left(- \frac{k_y k_z}{k^3} \right) = - \frac{2\Omega}{k} \left(\frac{k_y k_z}{k^2} \right)$$