HW 2

2a) $H = \frac{1}{2m}\left[p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2 \right] + V(r, \phi, z)$

[p_ϕ, p_z are not COM as $V = V(r, \phi, z)$]

Construct S such that $p_r = \frac{\partial S}{\partial r}$, $p_\phi = \frac{\partial S}{\partial \phi}$, $p_z = \frac{\partial S}{\partial z}$

and $H\left(q_i, \frac{\partial S}{\partial q_i}\right) + \frac{\partial S}{\partial t} = 0$

Let $S = S_0(q_i) - Et$ $\quad (H - E = 0)$

$$\Rightarrow E = \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + V(r, \phi, z)$$

If $V = f_1(r) + \frac{f_2(\phi)}{r^2} + f_3(z)$ and $S = S_1(r) + S_2(\phi) + S_3(z)$

we have

$$\begin{cases} \frac{1}{2m}(S_1')^2 + f_1(r) = \alpha_1 \\ \frac{1}{2m}(S_2')^2 + f_2(\phi) = \alpha_2 \\ \frac{1}{2m}(S_3')^2 + f_3(z) = \alpha_3 \end{cases}$$

b) $S_1(r) = \sqrt{2m} \int dr \sqrt{\alpha_1 - f_1(r)}$

$S_2(\phi) = \sqrt{2m} \int d\phi \sqrt{\alpha_2 - f_2(\phi)}$

$S_3(z) = \sqrt{2m} \int dz \sqrt{\alpha_3 - f_3(z)}$

3a) ① $\frac{\partial \tilde{\rho}}{\partial t} + \underline{v} \cdot \nabla \tilde{\rho} = -\rho_0 \, \underline{\nabla} \cdot \tilde{v}$ (continuity)

② $\rho_0 \left(\frac{\partial \tilde{v}}{\partial t} + \underline{v} \cdot \nabla \tilde{v} \right) = -c_s^2 \, \nabla \tilde{\rho}$

$\cancel{\frac{\partial}{\partial t} ① : \frac{\partial^2}{\partial t^2} \tilde{\rho} + \underline{v} \cdot \nabla \frac{\partial \tilde{\rho}}{\partial t} = \rho_0 \underline{\nabla} \cdot \frac{\partial \tilde{v}}{\partial t}}$

$\nabla \cdot ②$: $\cancel{\rho_0 \left(\frac{\partial}{\partial t} \nabla \cdot \tilde{v} \right) + \rho_0 \nabla \cdot [\underline{v} \cdot \nabla \tilde{v}] = -c_s^2 \nabla^2 \tilde{\rho}}$ [dropped ∇c and $\nabla \cdot \underline{v}$]

$\Rightarrow \rho_0 \left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla \right) \nabla \cdot \tilde{v} = -c_s^2 \nabla^2 \tilde{\rho}$

$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla \right)$ ① : $\left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla \right)^2 \tilde{\rho} = -\rho_0 \left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla \right) \nabla \cdot \tilde{v}$

$= c_s^2 \nabla^2 \tilde{\rho}$ ⊛

<u>Eikonal:</u> $\tilde{\rho} = \rho_1 e^{i\phi}$

$\nabla \tilde{\rho} = i \nabla \phi \, \tilde{\rho}$ $\qquad \nabla^2 \tilde{\rho} \simeq -|\nabla \phi|^2$

⊛ $\Rightarrow$ $\left(\frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla \phi \right)^2 = c_s^2 |\nabla \phi|^2$ "−" sign cancels

Plane Wave: $\tilde{\rho} = \rho_1 e^{i(\underline{k} \cdot \underline{x} - \omega t)}$ $\qquad \nabla^2 \tilde{\rho} = -k^2 \tilde{\rho}$

$\Rightarrow -(\omega - \underline{k} \cdot \underline{x})^2 \tilde{\rho} = -k^2 c_s^2 \tilde{\rho}$

$\Rightarrow (\omega - \underline{k} \cdot \underline{x})^2 = k^2 c_s^2$ //

3b) Consider $(\omega - \underline{k}\cdot\underline{v})^2 = k^2c^2$ $\qquad c = c(\underline{x})$

$$2(\omega - \underline{k}\cdot\underline{v})(\partial\omega - \partial\underline{k}\cdot\underline{v}) = 2\underline{k}\cdot\partial\underline{k}\,c^2 \qquad \omega - \underline{k}\cdot\underline{v} = \pm kc$$

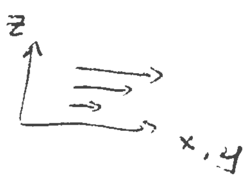
$$\Rightarrow \quad \partial\omega = \partial\underline{k}\cdot\underline{v} \pm \partial k\, c\hat{k}$$

$$\Rightarrow \quad \frac{\partial\omega}{\partial\underline{k}} = \underline{v} \pm c\hat{k} \qquad \Rightarrow \quad \frac{dx}{dt} = \frac{\partial\omega}{\partial\underline{k}} = \underline{v} \pm c\hat{k}$$

By $\omega = \underline{k}\cdot\underline{v} \pm kc$

$$\frac{dk}{dt} = -\frac{\partial\omega}{\partial\underline{x}} = -\frac{\partial}{\partial\underline{x}}\left(\underline{k}\cdot\underline{v} \pm kc\right) \quad //$$

c) $v(z)$



sheared flow

term $-\frac{\partial}{\partial\underline{x}}(\underline{k}\cdot\underline{v}) = -\hat{z}\,k_z\frac{dv}{dz}$ $\qquad \Rightarrow \quad \frac{dk_z}{dt} \sim -k_z v'$

d) $\cancel{k = \nabla\phi \quad \omega = -\dot{\phi} \quad \dot{k} = \nabla\dot{\phi} = \nabla\omega \Rightarrow \frac{dx}{dt} = \frac{\partial\omega}{\partial k}}$

$\omega = \omega(k, x)$

$$\dot{\underline{x}} = \frac{\partial\omega}{\partial\underline{k}} \quad \text{and} \quad \dot{\underline{k}} = -\frac{\partial\omega}{\partial\underline{x}} \qquad \Leftrightarrow \qquad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

HW6.6 $\quad H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \qquad \omega = \sqrt{k/m}$

$$\Rightarrow \dot{q} = \frac{\partial H}{\partial p} = p/m \,, \quad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q \quad [= -kq]$$

$$\begin{cases} Q = C(p + im\omega q) \\ P = C(p - im\omega q) \end{cases} \qquad PQ = QP = C^2(p^2 + m^2\omega^2 q^2)$$

$$\begin{cases} \dot{Q} = C(\dot{p} + im\omega \dot{q}) = C(-m\omega^2 q + i\omega p) = i\omega Q = \frac{\partial \tilde{H}}{\partial P} \\ \dot{P} = C(\dot{p} - im\omega\dot{q}) = C(-m\omega^2 q - i\omega p) = -i\omega P = -\frac{\partial \tilde{H}}{\partial Q} \end{cases}$$

$$\Rightarrow \tilde{H} = iQP\omega = i\omega C^2(p^2 + m^2\omega^2 q^2)$$

Setting $\tilde{H} = H$, we have

$$i\omega C^2 = \frac{1}{2m} \qquad C = \sqrt{\frac{-i}{2m\omega}} = \frac{e^{-i\pi/2}}{\sqrt{2m\omega}} \,,$$

b) Find S : $\quad S = S(q, P) \quad$ and $\quad Cp = P + Cim\omega q$

$$\Rightarrow Q = \frac{\partial S}{\partial P} = C(p + im\omega q)$$

$$= Cp + Cim\omega q = P + 2im\omega q C = P + q/C$$

~~$p = \frac{\partial S}{}$~~

$$\Rightarrow S = \frac{1}{2}P^2 + qP/C + f(q) \quad \leftarrow \text{unknown } f = f(q)$$

FW6.6

b) Cont'd

Since $p = \left.\frac{\partial S}{\partial q}\right|_P \Rightarrow \frac{P}{c} + i m \omega q = \frac{P}{c} + f'(q)$

$\Rightarrow f(q) = \frac{1}{2} i m \omega q^2 = \frac{q^2}{4c^2}$

$\Rightarrow S(q, P) = \frac{1}{2} P^2 + qP/c + q^2/(4c^2)$ //

c) $\begin{cases} \dot{Q} = i\omega Q \\ \dot{P} = -i\omega P \end{cases} \Rightarrow \begin{array}{l} \ddot{Q} = -\omega^2 Q \\ \ddot{P} = -\omega^2 P \end{array} \Rightarrow Q = A\cos(\omega t + \delta)$

FW 6.7

a) $S_0 = \sum_\sigma q_\sigma P_\sigma$

$$\frac{dS_0}{dt} = \sum_\sigma \left[\frac{\partial S}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma} \dot{P}_\sigma \right]$$

Consider $F = S_0 - PQ$,

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H} + \frac{dF}{dt}$$

$$= -\sum_\sigma \dot{P}_\sigma Q_\sigma - \tilde{H} + \frac{dS_0}{dt}$$

$$\Rightarrow \sum_\sigma \left(p_\sigma - \frac{\partial S_0}{\partial q_\sigma} \right) \dot{q}_\sigma - H = -\sum_\sigma \dot{P}_\sigma \left(Q_\sigma - \frac{\partial S_0}{\partial P_\sigma} \right) - \tilde{H}$$

Set ( )'s to zero such that $H = \tilde{H}$:

$$p_\sigma = \frac{\partial S_0}{\partial q_\sigma} = P_\sigma \quad \text{and} \quad Q_\sigma = \frac{\partial S}{\partial P_\sigma} = q_\sigma \quad //$$

b) $S_0 = \sum_\sigma q_\sigma P_\sigma + H\Delta t$

$$\frac{dS_0}{dt} = \sum_\sigma \left(\dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma \right) + \sum_\sigma \left[\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right] \Delta t$$

Similarly, set $F = S_0 - PQ$ and ( )'s to zero

$$\Rightarrow \begin{cases} p_\sigma = P_\sigma + \dfrac{\partial H}{\partial q_\sigma} \Delta t \Rightarrow p_\sigma = P_\sigma - \dot{p}_\sigma \Delta t \Rightarrow P_\sigma = p_\sigma + \dot{p}_\sigma \Delta t \simeq p_\sigma(t+\Delta t) \\ Q_\sigma \dot{P}_\sigma - q_\sigma \dot{P}_\sigma - \dfrac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \Delta t = 0 \end{cases}$$

$$\Rightarrow \left(Q_\sigma - q_\sigma - \frac{\partial H}{\partial p_\sigma} \Delta t \right) \dot{P}_\sigma + \mathcal{O}(\Delta t^2) = 0 \quad (\text{by } \dot{p}_\sigma \simeq \dot{P}_\sigma)$$

$$\Rightarrow Q_\sigma = q_\sigma + \dot{q}_\sigma \Delta t \simeq q_\sigma(t+\Delta t) \quad //$$

FW 6.8

a) $\cancel{S = S_0 +}\; S = \sum q_\sigma P_\sigma + \underline{P} \cdot \Delta r$ $\qquad \underline{P} = \sum_\sigma p_\sigma \hat{e}_\sigma$ (total linear momentum)

$$\frac{dS}{dt} = \sum_\sigma \left\{ \dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma + \dot{p}_\sigma \Delta r_\sigma \right\}$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H} + \frac{d}{dt}(S - PQ)$$

$$= -\sum_\sigma \dot{P}_\sigma Q_\sigma - \tilde{H} + \sum_\sigma \left(\dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma + \dot{p}_\sigma \Delta r_\sigma \right)$$

$$\Rightarrow \sum_\sigma \left(p_\sigma - P_\sigma \right) \dot{q}_\sigma - H = \sum_\sigma \left(-\dot{P}_\sigma Q_\sigma + q_\sigma \dot{P}_\sigma + \dot{p}_\sigma \Delta r_\sigma \right) - \tilde{H}$$

Set $p_\sigma = P_\sigma$ $\quad [\, 1^{st} (\;) = 0 \,]$

$\Rightarrow \quad \dot{p}_\sigma = \dot{P}_\sigma$

$\Rightarrow \quad Q_\sigma = q_\sigma + \Delta r_\sigma$ $\quad$ [translation]

FW 6.8

b) $S = \sum_\sigma q_\sigma P_\sigma + \hat{n}\cdot\underline{L}\,\Delta\phi$

$\underline{L} = \sum_\sigma L_\sigma \hat{e}_\sigma$

$L_\sigma = \varepsilon_{\sigma ij}\, q_i P_j \quad (\vec{q}\times\vec{p})$

$\hat{n}\cdot\underline{L} = \varepsilon_{\sigma ij}\, q_i P_j n_\sigma$

$= \varepsilon_{\sigma jk}\, q_\sigma P_j n_k$

$$\frac{dS}{dt} = \sum_\sigma \left(q_\sigma \dot{P}_\sigma + \dot{q}_\sigma P_\sigma\right)$$

$$+ \sum_\sigma \varepsilon_{\sigma jk}\left(\dot{q}_\sigma P_j + q_\sigma \dot{P}_j\right) n_k \Delta\phi$$

to change $P\dot{Q} \to \dot{P}Q$

$$\Rightarrow \sum_\sigma p \dot{q}_\sigma - H = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H} + \frac{d}{dt}(S - PQ)$$

set $\Rightarrow \begin{cases} (p_\sigma - P_\sigma - \varepsilon_{\sigma jk} P_j n_k \Delta\phi)\,\dot{q}_\sigma = 0 \\ q_\sigma \dot{P}_\sigma - Q_\sigma \dot{P}_\sigma + \varepsilon_{\sigma jk}\, q_\sigma \dot{P}_j n_k \Delta\phi = 0 \end{cases}$

$\Rightarrow P_\sigma = p_\sigma - \varepsilon_{\sigma jk} P_j n_k \Delta\phi$

$\Downarrow$

$\underline{P} = \underline{p} - \hat{n}\times\underline{p}\,\Delta\phi$

$\Rightarrow \quad Q_\sigma = q_\sigma - \varepsilon_{\sigma jk}\, q_j n_k \Delta\phi \qquad$ ["−" from $\varepsilon_{\sigma jk} = -\varepsilon_{j\sigma k}$]

$\underline{Q} = \underline{q} + \hat{n}\times\underline{q}\,\Delta\phi$

6. FW 6.13 Symmetric Top

Hamiltonian:

$$H = \frac{p_\beta^2}{2I_1} + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$

Constants of Motions:

$$\frac{\partial H}{\partial \alpha} = 0 \Rightarrow p_\alpha = \text{const.} = c_1 \quad , \quad \frac{\partial H}{\partial \gamma} = 0 \Rightarrow p_\gamma = \text{const.} = c_2 \quad , \quad \frac{\partial H}{\partial t} = 0 \Rightarrow H = E = \text{const.}$$

Expression for S:

$$p_\alpha = \frac{\partial S}{\partial \alpha} = c_1 \quad , \quad p_\gamma = \frac{\partial S}{\partial \gamma} = c_2 \quad , \quad -E = \frac{\partial S}{\partial t} \qquad \left(\text{s.t. } H + \frac{\partial S}{\partial t} = H - E = 0\right)$$

$$\Rightarrow S = S_0(\beta, E) + c_1\alpha + c_2\gamma - Et$$

Hamilton-Jacobi Equation:

$$H\left(q_i, \frac{\partial S}{\partial q_i}\right) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow \frac{1}{2I_1}\left(\frac{\partial S_0}{\partial \beta}\right)^2 + \frac{(c_1 - c_2 \cos\beta)^2}{2I_1 \sin^2\beta} + \frac{c_2^2}{2I_3} + Mgl\cos\beta - E = 0$$

$$\left(\frac{\partial S_0}{\partial \beta}\right)^2 = \left[E - Mgl\cos\beta - \frac{c_2^2}{2I_3} - \frac{1}{2I_1\sin^2\beta}(c_1 - c_2\cos\beta)^2\right](2I_1)$$

$$S_0 = \int d\beta \sqrt{2I_1}\left[E - Mgl\cos\beta - \frac{c_2^2}{2I_3} - \frac{1}{2I_1\sin^2\beta}(c_1 - c_2\cos\beta)^2\right]^{1/2}$$

Other const. of motion:

$$\delta = \frac{\partial S}{\partial E} = \sqrt{\frac{I_1}{2}} \int d\beta \left[E - Mgl\cos\beta - \frac{c_2^2}{2I_3} - \frac{(c_1 - c_2\cos\beta)^2}{2I_1\sin^2\beta}\right]^{-1/2} - t$$

$$= \text{const.}$$

6 Cont'd

$$\Rightarrow \quad t = t(\beta, E) = \sqrt{\frac{I_1}{2}} \int d\beta \left[E - Mgl\cos\beta - \frac{c_1^2}{2I_3} - \frac{(c_1 - c_2\cos\beta)^2}{2I_1\sin^2\beta} \right]^{-1/2} - \delta$$

By evaluating the integral and invert it, we get the solution $\beta = \beta(t)$. //

8 a) Equation for quantum harmonic oscillator :

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}m\omega^2x^2\psi = E\psi \qquad \omega^2 = k/m$$

Let $\psi = \psi_0 e^{i\phi(x)/\hbar}$ ($\hbar$ is a small parameter)

$$\frac{\partial}{\partial x}\psi = \frac{i\phi'}{\hbar}\psi_0 e^{i\phi(x)/\hbar} \;, \quad \nabla^2\psi = \frac{\partial^2}{\partial x^2}\psi = \left[-\frac{(\phi')^2}{\hbar^2} + \frac{i\phi''}{\hbar}\right]\psi_0 e^{i\phi(x)/\hbar}$$

Expand $\phi = \phi_0 + \hbar\phi_1$.

$$\Rightarrow (\phi')^2 - i\hbar\phi'' - (2mE - m^2\omega^2x^2) = 0$$

$$\Rightarrow (\phi_0' + \hbar\phi_1')^2 - i\hbar(\phi_0'' + \hbar\phi_1'') = 2mE - m^2\omega^2x^2$$

Matching orders :

$$\mathcal{O}(\hbar^0) . \quad (\phi_0')^2 = 2mE - m^2\omega^2x^2 \Rightarrow \phi_0(x) = \int\sqrt{2mE - m^2\omega^2x^2}\,dx$$

$$\mathcal{O}(\hbar^1) : \quad 2\phi_0'\phi_1' - i\phi_0'' = 0 \quad \Rightarrow \phi_1' = \frac{i}{2}\frac{\phi_0''}{\phi_0'} = \frac{i}{2}\frac{d}{dx}\ln\phi_0'$$

$$\phi_1 = i\ln\sqrt{\phi_0'}$$

Since $\frac{\phi}{\hbar} = \frac{\phi_0}{\hbar} + \frac{\hbar\phi_1}{\hbar} = \frac{\phi_0}{\hbar} + \phi_1$

$$\psi = \psi_0 e^{i\phi} = \psi_0 e^{\frac{i}{\hbar}\phi_0 - \ln\sqrt{\phi_0'}} = \frac{\psi_0}{(2mE - m^2\omega^2x^2)^{1/4}} e^{\frac{i}{\hbar}\int\sqrt{2mE - m^2\omega^2x^2}\,dx}$$

//