1. Principle of Least Action:

$$\delta \int L \, dt = 0$$

By $L = p\dot{q} - H$,

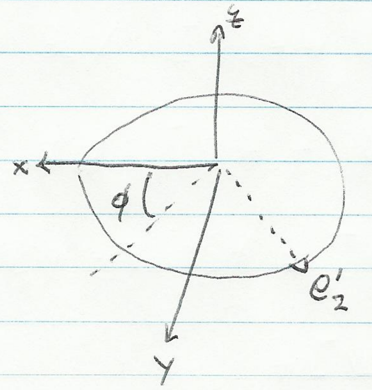
we have $0 = \delta \int p \, dq - \delta \int H \, dt$

$$= \int \delta p \frac{dq}{dt} dt + \int p \, d\delta q - \int \delta H \, dt - \int H \, d\delta t$$

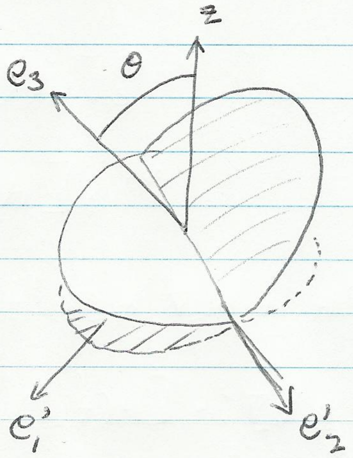
$$= \int dt \left[\delta p \dot{q} - \delta q \dot{p} - \frac{\partial H}{\partial p} \delta p - \frac{\partial H}{\partial q} \delta q - \frac{\partial H}{\partial t} \delta t + \frac{\partial H}{\partial t} \delta t \right]$$

$\Rightarrow \dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$ by matching δp, δq terms.

2 Euler Angles



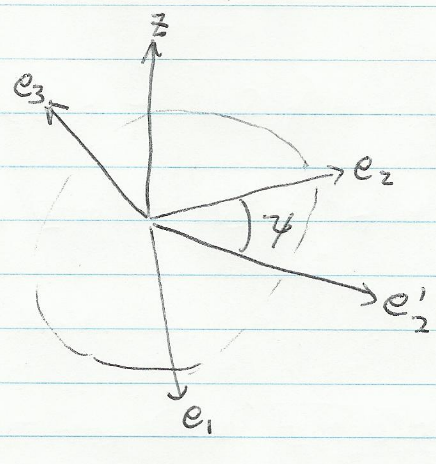
a)



b)

rotate along e_2'

c)



rotate along e_3

$$\Rightarrow \vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$$

Since $\hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1'$ (refer to fig b)

$$\Rightarrow \vec{\omega} = -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3$$

As angular momentum L is $(\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$ for set of principle axes,

$$\vec{L} = -\lambda_1\dot{\phi}\sin\theta\,\hat{e}_1' + \lambda_2\dot{\theta}\hat{e}_2' + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3$$

where $\lambda_1 = \lambda_2$ for symmetric top.

2 Cont'd

Thus,

$$T = \frac{1}{2}\lambda_1(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\psi + \dot\phi\cos\theta)^2$$

$$U = M_g R\cos\theta$$

$$p_\theta = \frac{\partial T}{\partial\dot\theta} = \lambda_1\dot\theta \quad , \quad p_\psi = \frac{\partial T}{\partial\dot\psi} = \lambda_3(\dot\psi + \dot\phi\cos\theta)$$

$$p_\phi = \frac{\partial T}{\partial\dot\phi} = \lambda_1\dot\phi\sin^2\theta + \lambda_3(\dot\psi + \dot\phi\cos\theta)\cos\theta$$

$$= \lambda_1\dot\phi\sin^2\theta + p_\psi\cos\theta$$

$$\Rightarrow T = \frac{(\lambda_1\dot\phi\sin^2\theta)^2}{2\lambda_1\sin^2\theta} + \frac{p_\theta^2}{2\lambda_1} + \frac{p_\psi^2}{2\lambda_3}$$

$$= \frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{p_\theta^2}{2\lambda_1} + \frac{p_\psi^2}{2\lambda_3}$$

$$\Rightarrow H = \frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{p_\theta^2}{2\lambda_1} + \frac{p_\psi^2}{2\lambda_3} + M_g R\cos\theta$$

Hamilton Equations:

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi - p_\psi\cos\theta}{2\lambda_1\sin^2\theta} \quad , \quad \dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\lambda_1}$$

$$\dot\psi = \frac{\partial H}{\partial p_\psi} = -\frac{p_\phi - p_\psi\cos\theta}{2\lambda_1\sin^2\theta}\cos\theta + \frac{p_\psi}{\lambda_3}$$

$$\dot p_\phi = \dot p_\psi = 0$$

$$\dot p_\theta = -\frac{\partial H}{\partial\theta} = M g R\sin\theta$$

4 a) $\nabla^2 \psi - \frac{1}{c(\underline{x})^2}\frac{\partial^2 \psi}{\partial t^2} = 0$

As the equation has no explicit t dependence, we let

$$\psi \rightarrow \psi_0 e^{-i\omega t}$$

$$\Rightarrow \nabla^2 \psi_0 + \frac{\omega^2}{c^2}\psi_0 = 0$$

We further let $\psi_0 = A e^{i\Phi(x)}$

where $\left|\frac{\nabla \Phi}{\Phi}\right| \gg \left|\frac{\nabla A}{A}\right|$ (limit of geometric optics)

Since $\nabla \psi_0 = (\nabla A + i \nabla \Phi A) e^{i\Phi}$

$$\nabla^2 \psi_0 = \nabla \cdot \nabla \Phi = i\nabla\Phi \cdot (\nabla A + i\nabla \Phi A) e^{i\Phi} + e^{i\Phi}(\nabla^2 A + i\nabla^2 \Phi A + i\nabla\Phi \cdot \nabla A)$$

$$\simeq -|\nabla \Phi|^2 \psi_0$$ by dropping ∇A terms and imaginary terms

$$\Rightarrow -|\nabla\Phi|^2 \psi_0 + \frac{\omega^2}{c^2}\psi_0$$

$$\Rightarrow |\nabla \Phi|^2 = \frac{\omega^2}{c^2} = \frac{\omega^2}{c_0^2} n^2(\underline{x})$$

Define $\underline{k} \equiv \nabla \Phi$

We have $\psi = A e^{i[\int \underline{k} \cdot d\underline{x} - \omega t]} = A e^{i\Phi_{total}(\underline{x}, t)}$

So the incremental total phase,

$$d\Phi = \underline{k} \cdot d\underline{x} - \omega dt$$

4 a) Cont'd : $\Phi_{tot} = \int_a^b [\underline{k} \cdot d\underline{x} - \omega dt]$

As $\omega = \omega(\underline{k}, \underline{x})$,

$$\delta\Phi = \int_a^b \left[\delta\underline{k} \cdot d\underline{x} + \underline{k} \cdot d\delta\underline{x} - \left(\frac{\partial\omega}{\partial\underline{k}} \cdot \delta\underline{k} + \frac{\partial\omega}{\partial\underline{x}} \cdot \delta\underline{x}\right) dt\right] = 0$$

Second term $= \int \underline{k} \cdot d\delta\underline{x}$

$$= \underline{k} \cdot \delta\underline{x} \Big|_a^b - \int \delta\underline{x} \cdot d\underline{k}$$

By matching coef. of $\delta\underline{x}$ and $\delta\underline{k}$ separately, we have

$$\frac{d\underline{x}}{dt} = \frac{\partial\omega}{\partial\underline{k}} \quad \text{and} \quad \frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial\underline{x}}$$

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b) Abbreviated phase $\Phi_0 = \int_a^b k \, dl$

$$\delta\Phi_0 = 0 \Rightarrow \delta\int_a^b n \, dl = 0$$

$$\Rightarrow 0 = \int_a^b \delta n \, dl + \int_a^b n(\underline{x}) \, d\delta l, \qquad \delta n = \frac{\partial n}{\partial\underline{x}} \cdot \delta\underline{x}$$

For 2^{nd} term, we want $\delta\underline{x}$:

Consider $dl^2 = d\underline{x} \cdot d\underline{x}$

$$dl \, \delta dl = dl \, d\delta l = d\underline{x} \cdot d\delta\underline{x}$$

$$\Rightarrow d\delta l = \frac{d\underline{x}}{dl} \, d\delta\underline{x}$$

$$\Rightarrow \int_a^b n(\underline{x}) \, d\delta l = \int_a^b n(\underline{x}) \frac{d\underline{x}}{dl} \cdot d\delta\underline{x}$$

$$= -\int_a^b \frac{d}{dl}\left[n(\underline{x}) \frac{d\underline{x}}{dl}\right] \cdot \delta\underline{x} \, dl$$

4b) Cont'd :

By match $\delta \underline{x} \, dl$: $\dfrac{\partial n(\underline{x})}{\partial \underline{x}} = \dfrac{d}{dl}\left[n(\underline{x}) \dfrac{dx}{dl} \right]$

Difference from a) :

Eq. b) doesn't involve frequency or time. So there is no information on how the wave propagate as time goes.

c) $\dfrac{\partial n}{\partial \underline{x}} = \dfrac{d}{dl}\left[n(\underline{x}) \dfrac{d\underline{x}}{dl}\right] \Rightarrow \nabla n = \left(\nabla n \cdot \dfrac{d\underline{x}}{dl}\right)\dfrac{d\underline{x}}{dl} + n(\underline{x}) \dfrac{d^2\underline{x}}{dl^2}$

By identifying tangent vector $\dfrac{dx}{dt} = \underline{t}$ and $\dfrac{1}{R} = \dfrac{d^2\underline{x}}{dl^2}$

with R being radius of curvature,

$$\Rightarrow \frac{1}{R} = \frac{1}{n(\underline{x})}\left[\nabla n - (\nabla n \cdot \underline{t})\underline{t}\right]$$

$\nabla n \cdot \hat{n}$ where $\hat{n} \equiv$ unit normal vector to path.

5. a) $\nabla^2 \psi + \frac{\omega^2}{c_0^2} n^2(\underline{x}) \psi = 0$

For $n^2(\underline{x}) = 1 + \delta(\underline{x})$ and assuming the sound beamed in $\hat{z}$ direction, we can write

$$\psi = \psi_0 e^{ik_z z} \quad \text{where} \quad \left|\frac{\partial_z \psi_0}{\psi_0}\right| \ll |k_z|$$

So $\nabla \psi = (\nabla \psi_0 + i k_z \psi_0 \hat{z}) e^{ik_z z}$

$$\nabla^2 \psi = \nabla \cdot \nabla \psi = (\nabla^2 \psi_0 + i k_z \hat{z} \cdot \nabla \psi_0) e^{ik_z z} + i k_z \hat{z} e^{ik_z z} \cdot (\nabla \psi_0 + i k_z \psi_0 \hat{z})$$

$$= [\nabla^2 \psi_0 + 2 i k_z \partial_z \psi_0 - k_z^2 \psi_0] e^{ik_z z}$$

$$\simeq [(\nabla_\perp^2 + 2 i k_z \partial_z - k_z^2) \psi_0] e^{ik_z z} \quad \text{by } k_z^2 \gg \frac{\partial_z^2 \psi_0}{\psi_0}$$

Recall for zeroth order : $k_z^2 = \frac{\omega^2}{c_0^2}$

$$\Rightarrow \quad 2 i k_z \partial_z \psi_0 + \nabla_\perp^2 \psi_0 + \frac{\omega^2}{c_0^2} \delta(\underline{x}) \psi_0 = 0 \quad //$$

b) k_z is defined as $\frac{1}{\psi}\frac{\partial \psi}{\partial z}$

i) Approximation : $|k_z| \gg \left|\frac{\partial_z \psi_0}{\psi_0}\right|$

change in phase >> that of amp. in z-direction.

ii) The $\delta(\underline{x})$ term corresponds to the spread in $\perp$ direction (i.e. $\nabla_\perp^2 \psi$) and the change in amplitude in z direction (i.e. $\partial_z \psi_0$)

iii) $\partial_z \psi$, $\delta(\underline{x}) \psi$ should be in same order

5. i) Let $\psi = A(\underline{x})\, e^{i\phi(\underline{x})}$

$$\partial_z \psi = [\, \partial_z A + i(\partial_z \phi) A \,] e^{i\phi}$$

$$\nabla_\perp \psi = [\, \nabla_\perp A + i(\nabla_\perp \phi) A \,] e^{i\phi}$$

$$\nabla_\perp^2 \psi = \nabla_\perp \cdot \nabla_\perp \psi = [\, \nabla_\perp^2 A + i\, \nabla_\perp \phi \cdot \nabla_\perp A + i A(\nabla_\perp^2 \phi) \,] e^{i\phi}$$

$$+ [\, \nabla_\perp A + i(\nabla_\perp \phi) A \,] \cdot i \nabla_\perp \phi\, e^{i\phi}$$

$$= \nabla_\perp^2 A + 2i(\nabla_\perp \phi)(\nabla_\perp A) + iA(\nabla_\perp^2 \phi) - |\nabla_\perp \phi|^2 A$$

$$\Rightarrow 2ik_z \partial_z A - 2k_z(\partial_z \phi) A + \nabla_\perp^2 A + 2i(\nabla_\perp \phi) \cdot (\nabla_\perp A)$$

$$+ iA(\nabla_\perp^2 \phi) - |\nabla_\perp \phi|^2 A + \frac{\omega^2}{c_0^2} \delta(\underline{x}) A = 0$$

Real Part:

$$-2k_z (\partial_z \phi) A + \nabla_\perp^2 A - |\nabla_\perp \phi|^2 A + \frac{\omega^2}{c_0^2} \delta(\underline{x}) A = 0$$

Except for the 2^{nd} term, $\delta(\underline{x})$ gives rise to change in ϕ as in eikonal theory : $|\nabla \phi|^2 = \frac{\omega^2}{c_0^2} n^2(\underline{x})$

Imaginary Part:

$$2k_z \partial_z A + 2(\nabla_\perp \phi) \cdot (\nabla_\perp A) + A(\nabla_\perp^2 \phi) = 0$$

6). Hamiltonian for 3D SHO:

$$H = \frac{1}{2m}\left[p_1^2 + p_2^2 + p_3^2 + m^2\omega_1^2 q_1^2 + m^2\omega_2^2 q_2^2 + m^2\omega_3^2 q_3^2 \right]$$

where $\omega_i^2 = k_i/m$ $\quad q_1, q_2, q_3 \Leftrightarrow x, y, z$

To get Hamilton-Jacobi Eq, we first let $p_i = \frac{\partial S}{\partial q_i}$

$$\Rightarrow \frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0$$

$$= \frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q_1}\right)^2 + \left(\frac{\partial S}{\partial q_2}\right)^2 + \left(\frac{\partial S}{\partial q_3}\right)^2 + m^2\omega_1^2 q_1^2 + m^2\omega_2^2 q_2^2 + m^2\omega_3^2 q_3^2\right]$$

No explicit time-dependence on S, except $\frac{\partial S}{\partial t}$

$\Rightarrow S = S_0(q_i) - Et$ $\qquad$ (OR $S = W - Et$)
$\Rightarrow H = E$

We can see both H and S_0 are seperable:

$$H = \sum_i \frac{1}{2m}\left[\left(\frac{\partial S_i}{\partial q_i}\right)^2 + m^2\omega_i^2 q_i^2\right], \quad S_0 = \sum_i S_i(q_i)$$

each [ ] is function of q_i only

So, we need to solve

$$\left(\frac{\partial S_i}{\partial q_i}\right)^2 + m^2\omega_i^2 q_i^2 = 2mE_i$$

for $i = 1, 2, 3$. Also $E = E_1 + E_2 + E_3$

$$S_i(q_i) = \sqrt{2mE_i}\int dq_i \sqrt{1 - \frac{m\omega_i^2 q_i^2}{2E_i}} = S_i(q_i; E_i)$$

6) Cont'd,

Note $S = S_1(q_1) + S_2(q_2) + S_3(q_3) - Et$

To get $q(t)$, we take

$$Q_i = \beta_i = \frac{\partial S}{\partial E_i} = \frac{\partial S_i}{\partial E_i} - t$$

$$\Rightarrow \quad t + \beta_i = \frac{\partial}{\partial E_i}\int dq_i \sqrt{2mE_i - m^2\omega_i^2 q_i^2}$$

$$= \sqrt{\frac{m}{2E_i}}\int dq_i \frac{1}{\sqrt{1 - \frac{m\omega_i^2 q_i^2}{2E_i}}} = \frac{1}{\omega_i}\sin^{-1}\left(q_i\sqrt{\frac{m\omega_i^2}{2E_i}}\right)$$

$$\Rightarrow \begin{cases} q_i(t) = \sqrt{\frac{2E_i}{m\omega_i^2}}\sin[\omega_i(t+\beta_i)] \\ p_i(t) = \frac{\partial S}{\partial q_i} = \frac{\partial S_i}{\partial q_i} = \sqrt{2mE_i - m^2\omega_i^2 q_i^2} \end{cases} \quad \text{for } i = 1, 2, 3$$

$$= \sqrt{2mE_i}\sqrt{1 - \frac{m\omega_i^2 q_i^2}{2E_i}} = \sqrt{2mE_i}\cos[\omega_i(t+\beta_i)]$$

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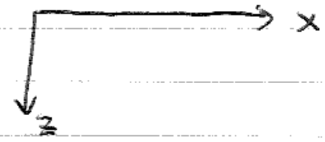
7.



By Fermat's Principle, we want $\delta T = \delta \int dt = 0$

$$dt = \frac{dl}{c(z)} \qquad \text{with} \quad dl = \sqrt{dx^2 + dz^2}$$
$$= \sqrt{\left(\frac{dx}{dz}\right)^2 + 1}\; dz$$

$$\Rightarrow \delta T = \delta \int \frac{1}{c(z)} \sqrt{(x')^2 + 1}\; dz \qquad x' \equiv \frac{dx}{dz}$$

As $\frac{\partial}{\partial x}(\text{integrand}) = 0$,

$$\Rightarrow \frac{d}{dz} \frac{\partial}{\partial x'} \left[\frac{1}{c(z)} \sqrt{(x')^2 + 1} \right] = 0$$

$$\Rightarrow \frac{1}{c(z)} \frac{x'}{\sqrt{(x')^2 + 1}} = \text{const.} = \frac{1}{c_0} \qquad \text{where } c(z) < c_0 \left[= c(z_0)\right]$$

$$\Rightarrow \frac{dx}{dz} = \pm \frac{c/c_0}{\sqrt{1 - c^2/c_0^2}} \qquad \text{OR} \qquad \frac{dz}{dx} = \pm \frac{\sqrt{1 - c^2/c_0}}{c/c_0}$$

$$\Downarrow$$

$$\Rightarrow x(z) = \pm \int \frac{c(z)}{\sqrt{c_0 - c(z)}}\, dz \qquad\qquad \frac{dz}{dx} = 0 \quad \text{at} \quad z = z_0,\ c = c_0$$

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