→ A different look at adiabatic theory . . .

One might forego canonical formalism, and simply investigate an oscillator with slowly varying frequency

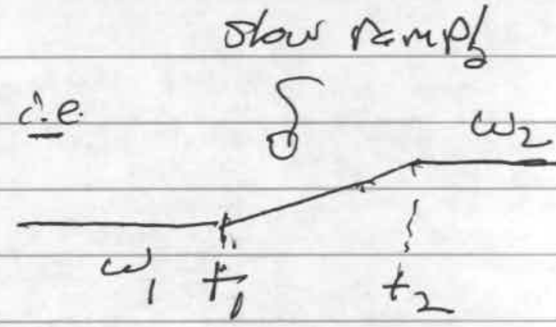
i.e.

$$\ddot{x} + \omega^2 x = 0 \quad \Rightarrow$$

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

slowly varying frequency

i.e.



i.e.

$$\frac{1}{\omega}\frac{d\omega}{dt} \sim O(\epsilon\omega) \ll 1$$

⇒ expect, on basis of previous discussion,

$I \leftrightarrow$ adiabatic invariant

i.e. if $a \equiv$ oscillator amplitude, then

$$I = E/\omega = \tfrac{1}{2} m\omega^2 a^2/\omega \simeq m\omega a^2$$

as const

Now, for slowly varying ω, can solve by WKB !

now, $\epsilon t = \tau$

$$\frac{d^2 X}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2} X = 0$$

$$X(\tau) = a_0 e^{i\phi(t)/\epsilon}$$

where : $\phi = \phi_0 + \epsilon \phi_1 + \ldots$

eikonal correction $\Rightarrow$ ampl.

$$\frac{d}{dt}\left(a_0 \frac{i\dot{\phi}(t)}{\epsilon} e^{i\phi(t)} \right) + \frac{\omega(t)^2}{\epsilon^2} a_0 e^{i\phi(t)} = 0$$

$$\left(-\frac{\dot{\phi}^2}{\epsilon^2} + \frac{i\ddot{\phi}(t)}{\epsilon} \right) a_0 e^{i\phi} + \frac{\omega(t)^2}{\epsilon^2} a_0 e^{i\phi(t)} = 0$$

need to $O(1/\epsilon)$

$\Rightarrow$

$$\left(-\frac{(\dot{\phi}_0 + \epsilon \dot{\phi}_1)^2}{\epsilon^2} + \frac{i\ddot{\phi}_0(t)}{\epsilon} \right) + \frac{\omega(t)^2}{\epsilon^2} = 0$$

$$-\frac{\dot{\phi}_0^2}{\epsilon^2} + \frac{\omega(t)^2}{\epsilon^2} = 0$$

$$\dot{\phi}_0(t) = \omega(t)$$

$$\phi_0(t) = \int \omega(t)\, dt$$

For next order correction.

$$-\underline{2\dot{\phi}_0\dot{\phi}_1}_{\epsilon} + \underline{i\ddot{\phi}_0}_{\epsilon} = 0$$

$$\dot{\phi}_1 = i\ddot{\phi}_0 / 2\dot{\phi}_0$$

$$= \frac{i}{2}\frac{d}{dt} \ln\left(\dot{\phi}_0(t)\right)$$

$$\phi_1 = \frac{i}{2} \ln\left(\dot{\phi}_0(t)\right)$$

$$= \frac{i}{2} \ln\left(\omega(t)\right)$$

$\underline{\underline{so}}$

$$x(t) = q_0\, e^{i\phi(t)/\epsilon}$$

$$= q_0\, e^{i\int \frac{\omega(t)}{\epsilon} dt}\, e^{i\frac{i}{2}\ln(\omega(t))}$$

$$= \underline{q_0}\, e^{i\int \frac{\omega(t)}{\epsilon} dt}\, e^{-\frac{\ln}{2}\omega(t)}$$

$$\Rightarrow \quad x(t) = \underline{a_0}\, e^{i\int \frac{\omega(t)}{\epsilon}\, dt}\; e^{-\frac{1}{2}\ln(\omega)}$$

$$= \frac{a_0}{\sqrt{\omega}}\, e^{i\int \frac{\omega(t)}{\epsilon}\, dt}$$

re-scaling, $t = T/\epsilon$:

$$\boxed{x(t) = \frac{a_0}{\sqrt{\omega}}\, e^{i\int \omega(t)\, dt}} \qquad \text{WKB soln.}$$

and can observe:

$$\underbrace{\overline{\omega x^2}}_{\text{(cycle) action}} = \omega\, \overline{x^2} = \omega \frac{a_0^2}{\omega} = \text{const}!$$

$\Rightarrow$ Action is invariant, due to frequency modulation of amplitude!

check:
$$I = \frac{1}{2\pi}\oint p\, dq$$
$$= \frac{1}{2\pi}\oint p\, dx$$
$$= \frac{1}{2\pi}\oint m\dot{x}\, dx = \frac{1}{2\pi}\oint m\dot{x}\,\dot{x}\, dt$$

$$I = \frac{1}{2\pi} \oint_{\omega^{-1}} m\dot{x}^2 \, dt$$

$$x(t) = \frac{a_0}{\sqrt{\omega}} \cos(\omega t + \delta)$$

$$\dot{x} = -a_0 \sqrt{\omega} \sin(\omega t + \delta)$$

$\theta = \omega t$ $\qquad \delta \to 0$ $\qquad \omega^2/(\sqrt{\omega})^2$

$d\theta = \omega dt$

$$I = \frac{1}{2\pi} \oint p \, dq = \frac{1}{2\pi} \oint d\theta \, a_0^2 \, \omega \sin^2\theta \frac{d\theta}{\omega}$$

$$= \frac{1}{2} a_0^2 \to \text{real const.!}$$

⇒ the message:

- adiabatic invariants basically as consequence of WKB approximation (time scale sep!)
- WKB would lead one to adiabatic invariance of action, even if did not realize it.
- need retain WKB correction beyond pure eikonal for freq. modulation of amplitude ⇒ essential.