Physics 200B **Mechanics II** **Winter 2013**

Problem Set IV: Due by last class

1.) Consider a string of length L and mass-per-length μ which is, as usual, clamped at both ends. Assume the tension is T.

a.) Express the Hamiltonian density in terms of the Fourier coefficients, thereby converting the problem to one of particle dynamics. (Hint: Expand the displacement in terms of the *spatial* eigenfunctions.) Derive the Hamiltonian EOMs.

b.) Use your experience from a previous homework problem to simplify the Hamiltonian to one which has a "creation and annihilation operator" form. Write the new EOMs.

c.) Now, let $L = L(t)$ where $\dot{L}/L << \omega$, and ω corresponds to the frequency of the fundamental mode. Assume the tension is fixed, how does the amplitude of vibrations on the string change with L?

d.) Reconsider problem c.) from the viewpont of wave action density as discussed in class. Compare your result to that of c.).

2.) Problem 7.16, Fetter and Walecka

3a.) Generalize the derivation of the nonlinear wave equation for a string to that for a 2D membrane (i.e. drum head), with clamped boundary. Show that you recover the wave equation in the linear limit.

b.) Derive the energy-momentum conservation equations for linear waves on this membrane.

4.) Show that if ψ and ψ^* are taken as two independent field variables, the Lagrangian density

$$\mathfrak{L} = \frac{h^2}{8\pi^2 m}\nabla\psi\cdot\nabla\psi^* + V\psi^*\psi + \frac{h}{2\pi i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*),$$

leads to the Schrödinger equation

$$-\frac{h^2}{8\pi^2 m}\nabla^2\psi + V\psi = \frac{ih}{2\pi}\frac{\partial\psi}{\partial t},$$

and its complex conjugate. What are the canonical momenta? Obtain the Hamiltonian density corresponding to $\mathfrak{L}$.

5.) Consider a free nonlinear oscillator which satisfied the equation

$$\ddot{x}+\omega_0^2 x=-\alpha x^2-\beta x^3 .$$

Use Poincare-Linstedt perturbation theory to calculate the non-linear frequency shift and lowest order non-trivial solution.

6.) *Liénard's construction.* Show that the equation

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=0$$

is equivalent to the system

$$\frac{dx}{dt}=y-F(x), \qquad \frac{dy}{dt}=-g(x),$$

where $F(x)=\int_0^x f(u)\,du$. Then the (x, y)-plane is called the *Liénard plane.*

Suppose next that $g(x)=x$ for all x. Then show that the direction of the orbit at a point P in the Liénard plane may be constructed as follows. Draw the line through P parallel to the y-axis, and let Q be the point where the line cuts the curve with equation $y=F(x)$. Draw the line through Q parallel to the x-axis, and let R be the point where this line cuts the y-axis. Then the orbit through P is in the direction perpendicular to $\overrightarrow{RP}$.

What is the lesson we learn from this problem?

7.) *Epidemics.* A simple model of an epidemic of a disease is the system,

$$\frac{dx}{dt}=-cxy, \qquad \frac{dy}{dt}=cxy-by,$$

where $x(t)$ represents the number of individuals in a population who are liable to infection, $y(t)$ is the number who are infectious at time t, $b>0$ is the rate of recovery (or death) from the disease, and $c>0$ is a rate of infection.

Show that if $x(0)<b/c$ then the number y of infectious individuals will decrease monotonically to zero, but if $x(0)>b/c$ then the number will increase monotonically until the number x of susceptible individuals decreases to b/c.

8.)

The Lotka–Volterra equations. Given that the growth of a population of x individuals of a species of prey and y individuals of a species of predator is governed by the equations

$$\frac{dx}{dt} = x(a - cy), \qquad \frac{dy}{dt} = -y(b - cx),$$

for constants $a, b, c > 0$, show that $\mathbf{X} = (0, 0)$ is a saddle point and $\mathbf{X} = (b/c, a/c)$ is a centre. Show further that small oscillations about the centre have period $2\pi/(ab)^{1/2}$. Prove that $dy/dx = -y(b - cx)/x(a - cy)$, and integrate this equation. Hence or otherwise sketch the phase portrait in the first quadrant of the (x, y)-plane.

[Lotka (1920) used these equations to model the chemical reactions $D + X \xrightarrow[a]{} 2X$, $E + Y \xrightarrow[b]{} E + F$, $X + Y \xrightarrow[c]{} 2Y$ in a well-stirred reaction vessel, where x denotes the concentration of molecule X and y of Y, where D, E are abundant molecules, and where a, b, c are the reaction coefficients. Volterra (1926) used these prey-predator equations to model the population of fish in the Adriatic Sea.]

9.)

Fishing. Suppose that a population of superpredators, e.g. fishermen, preys with equal intensity f on both species such that a is replaced by $a - f$ and b by $b + f$ in the above model (Q6.5). Then deduce that in stable equilibrium the predator species y is decreased but the prey species x is increased by the superpredators.

10.) Problem 9.4, Fetter and Walecka

11.) Problem 9.5, Fetter and Walecka

12.) Problem 9.27, Fetter and Walecka