Ponderomotive Force

Ponderomotive Force

Can consider two complementary limits:

$$U(\underline{x}) = U(x, \lambda(t))$$

- $\dot{\lambda}/\lambda < \omega_{nat}$ (orbit frequency fast) $\Rightarrow$ adiabatic limit
- $\dot{\lambda}/\lambda \gg \omega_{nat}$ (orbit frequency slow) $\Rightarrow$ ponderomotive; parametric ($\dot{\lambda}/\lambda \sim 2\omega_{nat}$)

i.e. 2 extremes of time scale ordering

$\Rightarrow$ Ponderomotive Problem:

- consider 1D problem:

$$m\ddot{x} = -\frac{dU}{dx} + f \qquad U = U(x)$$

f → high frequency drive → imposes jitter on potential

f (small)

in absence f,

$$U \cong U_0 + \frac{1}{2} m \Omega_0^2 x^2 \text{ [redacted] } + \ldots$$

i.e. $U'' \simeq \frac{1}{2} m \Omega_0^2$

$\Rightarrow$ non-driven motion has natural frequency Ω.

Now, $f = f_1 \cos\omega t + f_2 \sin\omega t$

$\left.\begin{matrix} f_1 \\ f_2 \end{matrix}\right\} = F_1(x)$, F_2 — position dependent

$\omega \gg \Omega_0$ — fast variation on time; spatial envelope.

$\therefore$ Physical idea is that f_1, f_2 induce oscillation on top of motion in U.

→ oscillation is rapid, i.e. $\omega \gg \Omega_0$

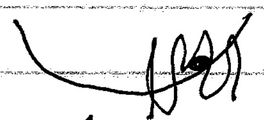
→ only care about centroid motion ⟺ slow

- key point here, then, to understand motion is to:

→ write $x(t) = \underbrace{X(t)}_{\substack{\text{slow} \to \Omega \\ \text{(quiver center)}}} + \underbrace{\mathcal{E}(t)}_{\substack{\text{fast} \to \omega \\ \text{quiver}}}$

→ separate (motion) equation into fast and slow components, respectively

→ look for beats, i.e. high + high → low. ⟹ modify mean, low ω eqn.



Now, $x(t) = X(t) + \mathcal{E}(t)$

$$\frac{1}{T}\int_0^{2\pi/\omega} dt\, \mathcal{E}(t) = 0$$

period avg. of high freq. oscillation vanishes.

so, plugging in:

$$m(\ddot{x}) = m(\ddot{X} + \ddot{\varepsilon})$$

$$= -\frac{d}{dx}\left(U(X+\varepsilon)\right) + f(X+\varepsilon)$$

$$= -\frac{d}{dx}U(X) - \varepsilon \frac{d^2U}{dx^2}\Big|_X + f(X) + \varepsilon \frac{\partial f}{\partial x}\Big|_X$$

$\Rightarrow$

$$m(\ddot{X}) + m\ddot{\varepsilon} = -\frac{d}{dx}U(X) - \varepsilon \frac{d^2U}{dX^2}\Big|_X$$

$$+ f(X) + \varepsilon \frac{\partial f}{\partial x}\Big|_X$$

avg. $\langle\ \rangle = \frac{1}{\tau}\int_0^{2\pi/\omega} dt$:

fast

$$\langle m\ddot{X}\rangle + m\langle\ddot{\varepsilon}\rangle = -\frac{d}{dx}\langle U(X)\rangle - \left\langle \varepsilon \frac{d^2U}{dX^2}\right\rangle\Big|_X$$

$$+ \langle \cancel{f(X)}\rangle + \left\langle \varepsilon \frac{\partial f}{\partial x}\Big|_X \right\rangle$$

fast

survives !

$\therefore$ for slow,

non-trivial survivor $\Rightarrow$ beat effect

$$m\ddot{X} = -\frac{d}{dx}\langle U(X)\rangle + \underbrace{\left\langle \xi \left.\frac{\partial f}{\partial x}\right|_X \right\rangle}$$

for fast;

as $\omega \gg \Omega_0$

$$\underbrace{m\ddot{\xi}}_{O(\xi m \omega^2)} = f(X) - \underbrace{\cancel{\xi \left.\frac{d^2U}{dx^2}\right|_X}}_{O(\xi m \Omega_0^2)}$$

$$m\ddot{\xi} = f(X) = f_1(X)\cos\omega t + f_2(X)\sin\omega t$$

$$\Rightarrow \boxed{\xi = \frac{-f(X)}{m\omega^2}}$$

$\Rightarrow$ fast quiver variation (f is space, time dependent)

Now, to calculate beat term:

$$\text{beat} = \left\langle \xi \left.\frac{\partial f}{\partial x}\right|_X \right\rangle = \left\langle \left(\frac{-f(X)}{m\omega^2}\right) \left.\frac{\partial f}{\partial x}\right|_X \right\rangle$$

so

$$m\ddot{X} = -\frac{d}{dx}\langle U(X)\rangle - \frac{1}{m\omega^2}\left\langle f(X)\frac{\partial f}{\partial x}\bigg|_{X}\right\rangle$$

$$f = f_1 \cos\omega t + f_2 \sin\omega t$$

$$\langle\ \rangle = \frac{1}{\tau}\int_0^{2\pi/\omega} dt$$

$\Rightarrow$

$$m\ddot{X} = -\frac{d}{dx}\langle U(X)\rangle - \frac{1}{2m\omega^2}\frac{d}{dx}\langle f(X)^2\rangle\bigg|_{X}$$

$$\langle f^2\rangle = \frac{1}{2}\left(f_1^2 + f_2^2\right)$$

$\therefore$ finally:

$$\boxed{m\ddot{X} = -\frac{d}{dx}\left[U(x)\Big|_{X} + \frac{1}{4m\omega^2}\left(f_1(X)^2 + f_2(X)^2\right)\right]}$$

i.e. effective potential becomes:

$$U_{eff} = U(x)\Big|_{X} + \frac{1}{4m\omega^2}\left(f_1(X)^2 + f_2(X)^2\right)$$

where pondermotive potential

$$\boxed{U_{\substack{\text{pondero.}\\ \text{motive}}} = \frac{1}{4m\omega^2}\left(f_1(\underline{X})^2 + f_2(\underline{X})^2\right)}$$

is piece induced by high frequency quiver

obviously:

$$\boxed{F_{\text{pond.}} = -\nabla U_{\text{pond}} = \frac{-1}{4m\omega^2}\nabla_{\underline{X}}\left(f_1(\underline{X})^2 + f_2(\underline{X})^2\right)}$$

- pondermotive force

- ~ gradient of (spatial) envelope of high frequency energy field

- observe:

$$\frac{1}{4m\omega^2}\left(f_1^2 + f_2^2\right) = \frac{1}{2}m\langle\dot{\mathcal{E}}^2\rangle$$

quiver kinetic energy

alternatively, can write

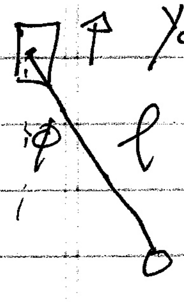
$$\boxed{U_{\text{ponderomotive}} = \frac{1}{2} m \langle \dot{\xi}^2 \rangle}$$

→ effective kinetic energy field $(= U_p(X))$ of quiver kinetic energy

→ gradient in kinetic energy field induces mean force.

→ Example — Inverted Pendulum

Consider 'the usual' pendulum with vertically oscillating point of support:



$\uparrow y_0 = y_0 \cos\omega t$ $\qquad\qquad \omega \gg g/\ell$

$$y = y_0 \cos\omega t + \ell \cos\phi$$
$$x = \ell \sin\phi$$

$$L = T - U$$

$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - mg\ell(1 - \cos\phi)$$

$$L = \frac{1}{2} m \left(\ell^2 \cos^2\phi\, \dot\phi^2 + \left[\omega y_0 \sin\omega t + \ell \sin\phi \dot\phi\right]^2 \right) - mg\ell\,(1-\cos\phi)$$

$$= \frac{1}{2} m \left(\ell^2 \cos^2\phi\, \dot\phi^2 + \ell^2 \sin^2\phi\, \dot\phi^2 + \cancel{\omega^2 y_0^2 \sin^2\omega t} + 2\omega y_0 \ell \dot\phi \sin\omega t \sin\phi \right) - mg\ell\,(1-\cos\phi)$$

$$= \frac{1}{2} m \left(\ell^2 \dot\phi^2 + 2\omega y_0 \ell \dot\phi \sin\omega t \sin\phi \right) + mg\,\ell\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi}$$

$\Rightarrow$

$$\frac{d}{dt}\left(m\ell^2 \dot\phi + m\omega y_0 \ell \sin\omega t \sin\phi \right) = -mg\,\ell \sin\phi + m\omega y_0 \ell \dot\phi \sin\omega t \cancel{\sin}\cos\phi$$

$\Rightarrow$

$$m\ell^2 \ddot\phi + m\omega y_0 \ell \left(\omega \cos\omega t \sin\phi + \sin\omega t \cos\phi \dot\phi \right) = -mg\ell \sin\phi + m\omega y_0 \ell \dot\phi \sin\omega t \cos\phi$$

$\underline{\infty}$

$$m\ell^2\ddot{\phi} = -mg\ell\sin\phi - m\omega^2 y_0 \ell \sin\phi\cos\omega t$$

$$\ddot{\phi} = -\frac{g}{\ell}\sin\phi - \frac{\omega^2 y_0}{\ell}\sin\phi\cos\omega t$$

$\Rightarrow$

$$\ddot{\phi} = -\frac{d}{d\phi}U(\phi) + f_1(\phi)\cos\omega t$$

$$U(\phi) = -\frac{g}{\ell}\cos\phi \quad , \quad f_1(\phi) = -\frac{\omega^2 y_0}{\ell}\sin\phi$$

$$\phi = \langle\phi\rangle + \tilde{\phi}$$

$$\langle\ddot{\phi}\rangle + \ddot{\tilde{\phi}} = -\frac{d}{d\phi}U(\phi)\Big|_{\langle\phi\rangle} + f_1(\langle\phi\rangle)\cos\omega t$$

$$+ \frac{\partial f_1}{\partial\phi}\Big|_{\langle\phi\rangle}\left\langle\tilde{\phi}\cos\omega t\right\rangle$$

$$\ddot{\tilde{\phi}} = f_1(\langle\phi\rangle)\cos\omega t$$

$$\tilde{\phi} = \frac{-1}{\omega^2} f_1(\langle\phi\rangle)\cos\omega t$$

$\Rightarrow$

$$\left.\frac{\partial f_1}{\partial\phi}\right|_{\langle\phi\rangle} \langle\tilde{\phi}\cos\omega t\rangle = \frac{\partial f_1}{\partial\phi}\left\langle -\frac{f_1\cos^2\omega t}{\omega^2}\right\rangle$$

$$\equiv -\frac{1}{4}\frac{d}{d\phi}\frac{f_1^2}{\omega^2}$$

$\therefore$

$$\langle\ddot{\phi}\rangle = -\frac{d}{d\phi}\left(u(\phi) + \frac{f_1^2}{4\omega^2}\right)$$

$$u(\phi) = \frac{-g}{\ell}\cos\phi$$

$$f_1^2/4\omega^2 = \frac{\omega^4 y_0^2 \sin^2\phi}{4\omega^2\ell^2} = \frac{\omega^2 y_0^2\sin^2\phi}{4\ell^2}$$

$\therefore$

$$\boxed{U_{eff} \cong \left(\frac{-g}{\ell}\cos\phi + \underbrace{\frac{\omega^2 y_0^2\sin^2\phi}{4\ell^2}}_{\text{ponderomotive potential}}\right)} \rightarrow \text{effective potential}$$

$$\boxed{U_{eff} = -\frac{g}{l}\left(\cos\phi - \frac{\omega^2 y_0^2 \sin^2\phi}{4gl}\right)}$$ effective potential

$$U_{eff} = -\frac{g}{l}\left(\cos\phi - \frac{\omega^2 y_0^2}{4gl}\left(\frac{1}{2} - \frac{1}{2}\cos 2\phi\right)\right)$$

minima $\phi = 0$
$\phi = \pi$

observe:

$$U''_{eff} = \frac{g}{l}\left(\cos\phi + \frac{\cancel{4}\omega^2 y_0^2}{2(\cancel{4}gl)}\cos 2\phi\right)$$

$\phi = 0$ is stable minimum

if $\frac{\omega^2 y_0^2}{2gl} > 1$, $\phi = \pi$ is stable minimum.

$\Rightarrow$ can stabilize inverted pendulum

if $\omega^2 y_0^2 / 2gl > 1$.

Comments:

⇒ spatially varying envelope of high frequency energy field ⇒ ponderomotive force

i.e. $U_{eff} = U + \frac{1}{2} m \dot{\xi}^2$

⇒ can think of ponderomotive potential as effective radiation pressure.

i.e. e.m waves