Poisson Brackets
and
Canonical Transformations.

c.)

H-J. Equation: Another Perspective

→ Recall, thrust of discussion is integrability

⟺

① H.-J. equation relevant, as separability of H-J. equation ⇒ integrability (converse not equivalent)

② more generally, integrability can render
1) all coordinates cyclic
2) all conjugate momenta constant

i.e. integrability if can find transformation such that:

$$p_i, q_i \rightarrow \alpha_i, \beta_i = \alpha_i t + t_0$$

(α_i: gen. momentum; β_i: position)

p_i, q_i: arbitrary g.c.'s

such that:

$$\frac{d}{dt}\alpha_i = 0$$

Then, will show that H-J. equation is generating function of canonical transformation

$$p_i, q_i \rightarrow \alpha_i, \beta_i$$

i.e. clearly, for conservative system, such a transformation must leave:

$$H(p_i; q_i) = H'(\alpha_i)$$

but : $H(p_i; q_i) = E = E(\alpha_i)$

$\Rightarrow$ $\boxed{H(p_i; q_i) = E(\alpha_i)}$ $\rightarrow$ but this is just time-independent H-J equation

$\Rightarrow$

Technical Preliminaries :

a.) Poisson Brackets

b.) Canonical Transformations and Generating Fctns.

a.) Poisson Brackets

Recall :

- fundamental notion/concept/fact of Hamiltonian mechanics is
  - $\rightarrow$ incompressibility of phase space flow (Liouville's Thm.)

i.e.

$$\underline{V}_\Gamma = (\dot{q}_i, \dot{p}_i)$$

$$\underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = \frac{\partial}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial p_i}\dot{p}_i$$

$$= \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \underbrace{\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right)}_{\text{common operator structure}} A$$

$$\equiv \frac{\partial A}{\partial t} + \{A, H\}$$

$$\boxed{\begin{gathered}\{A, H\} \to \{A, B\} \equiv \text{Poisson Bracket} \\ \{A, B\} = \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}\end{gathered}}$$

→ Poisson Bracket is operator rel'n.

→ Bracket properties:

① $\{f, g\} = -\{g, f\}$ (non-commutativ-ity)

② $\{f+g, h\} = \{f, h\} + \{g, h\}$ (distributive)

③ $\{fg, h\} = f\{g, h\} + g\{f, h\}$ (associative)

④ $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

(a la' X)

①-③ ⇒ trivially follow from calculus

④ ⇒ Jacobi identity (see L+L for proof)

Brackets ⇒ non-commutative Lie Algebra

Some key points:

a) if $\{A, H\} = 0 \Rightarrow dA/dt = 0$ for $A_t = 0$

$A = A(q, p) \Rightarrow A$ is COM

b) $\left.\begin{array}{l}\{A, H\} = 0 \\ \{B, H\} = 0\end{array}\right\} \Rightarrow \begin{cases}\{A, B\} \text{ is COM} \\ \Rightarrow \{\{A, B\}, H\} = 0\end{cases}$

from Jacobi identity, with:

$f \to A \quad , \quad g \to B \quad , \quad h \to H$

c) in particular:

$\{q_i, q_j\} = 0 \quad , \quad \{p_i, p_j\} = 0$

$\{p_i, q_j\} = \delta_{ij} \quad \to \begin{cases}[p_i, q_j] = -i\hbar\delta_{ij} \\ \text{in QM}\end{cases}$

(b) Canonical Transformations

→ in general, seek how transform:

$$\left.\begin{matrix} p_i \\ q_i \end{matrix}\right\} \rightarrow \left.\begin{matrix} P_i \\ Q_i \end{matrix}\right\}$$

"old" "new"

such that Hamiltonian structure preserved

i.e.

if $\dot{p}_i = -\dfrac{\partial H}{\partial q_i}$; $\dot{q}_i = \dfrac{\partial H}{\partial p_i}$

then: $\dot{P}_i = -\dfrac{\partial H'}{\partial Q_i}$; $\dot{Q}_i = +\dfrac{\partial H'}{\partial P_i}$

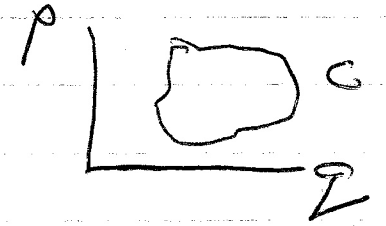
where $H' \leftrightarrow$ new Hamiltonian.

Clearly, such a transformation must:

1. preserve phase space volume
2. preserve bracket relations

$\Rightarrow$ equivalently, phase space volume conserved

i.e. $\int dp_i\,dq_i = \text{const}$
within C



but: $\int dp_i\,dq_i = \oint_C p_i\,dq_i$, (Stokes Thm.)

circulation about C in phase space.

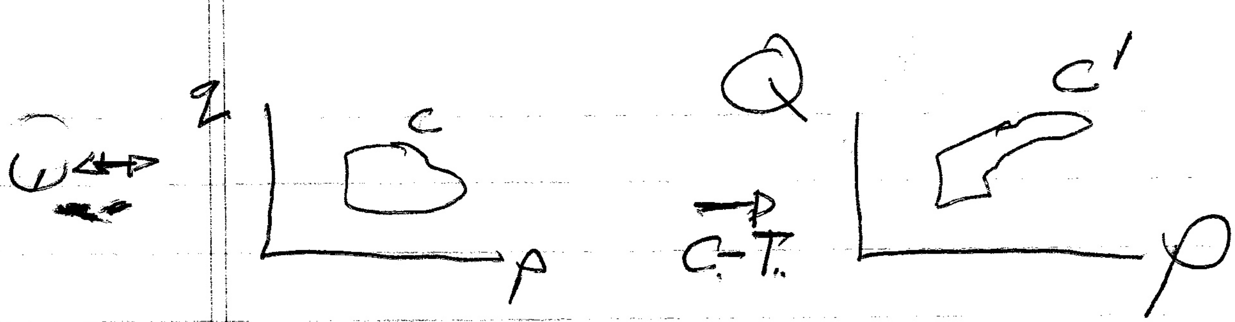
$\Rightarrow \oint_C p_i\,dq_i = \text{const.}$

N.B.: Liouville Thm. analogous to Kelvin circulation theorem for incompressible fluids. i.e. $\Gamma = \oint_C \underline{v}\cdot d\underline{l}$

Now, Liouville Thm $\Rightarrow$ for <u>any</u> $A(q,p)$

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial}{\partial q_i}\left(\frac{dq}{dt}A\right) + \frac{\partial}{\partial p_i}\left(\frac{dp}{dt}A\right)$$

$$= \frac{\partial A}{\partial t} + \dot{q}_i\frac{\partial A}{\partial q_i} + \dot{p}_i\frac{\partial A}{\partial p_i}$$



but areas must be equal!

$$\int_{A_C} dp_i\, dq_i = \int_{A_{C'}} dP_i\, dQ_i$$

$\Rightarrow$ must have:
$$\boxed{\frac{\partial(P_i, Q_i)}{\partial(p_i, q_i)} = 1}$$

A trivial example: Hamiltonian evolution

i.e. how transform $q, p \to P, Q$ s/t

$$P = p(t+\delta t)$$
$$Q = q(t+\delta t) \quad ?$$

Obviously, use Hamilton E-O-M's:

$$P = p(t+\delta t) = p(t) + \delta t\, dp/dt$$
$$= p(t) - \delta t\, \partial H/\partial q$$

$$Q = q(t+\delta t) = q(t) + \delta t\, dq/dt$$
$$= q(t) + \delta t\, \partial H/\partial p$$

i.e. clearly H generates transformation.

Is it canonical?

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} 1-\delta t\,\partial^2 H/\partial p\,\partial q, & -\delta t\,\partial^2 H/\partial q^2 \\ \delta t\,\dfrac{\partial^2 H}{\partial p^2}, & 1+\delta t\,\partial^2 H/\partial p\,\partial q \end{vmatrix}$$

$$= 1 + \delta t\left(\frac{\partial^2 H}{\partial p\,\partial q} - \frac{\partial^2 H}{\partial q\,\partial p}\right) - \delta t^2\left(\frac{\partial^2 H}{\partial p\,\partial q}\right)^2 + \delta t^{2}\left(\frac{\partial^2 H}{\partial q^2}\right)\left(\frac{\partial^2 H}{\partial p^2}\right)$$

$$= 1 + \delta t^3 \det\left[\text{Hessian}\,(H(q,p))\right]$$

$$= 1 + \delta t^3\,(\text{gaussian curvature } H).$$

so

$$\frac{\partial(P,Q)}{\partial(p,q)} = 1 \quad \text{to } O(\delta t^3) \quad \checkmark$$

Transformation is canonical $\rightarrow$ preserves phase volume!

i.e. H generates 'canonical transformation' to $q(t+dt)$, $p(t+dt)$

∴ {can view H.E.M.'s as sequence of canonical transformations

→ How to Transform Canonically ?

- generally, seek transformation

$$p, q \longrightarrow P, Q$$

- 'old' 'new'

{where have :→ 2 independent variables + gen. fctn.
{but seek :→ 2 dependent variables

i.e. point of C-T is that preservation of Hamiltonian structure immediately defines dependent variables from independent

- have 4 cases :

	independent	dependent
1)	q, Q	p, P
2)	q, P	p, Q
3)	p, Q	q, P
4)	p, P	q, Q

see 26a., 28

Case 1.) q, Q independent
p, P ? $\Rightarrow$ what constrains them?

Observe: Liouville Thm $\Rightarrow$

$$\int dp\,dq = \int dP\,dQ$$

$$\Rightarrow \quad \oint_C p\,dq = \oint_{C'} P\,dQ \qquad \text{(Stokes Thm.)}$$

but as $C' = C'(C)$ can write $\Rightarrow$

$$\oint \left[p\,dq - P\,dQ\right] = 0$$

but $p = p(q, Q)$
$P = P(q, Q)$

$$\therefore \quad \oint \left[p(q,Q)\,dq - P(q,Q)\,dQ\right] = \underset{\text{gen. fctn.}}{0}$$

Now
$$0 = \oint dF_1 \quad ; \quad F = F_1(q, Q)$$

$$= \oint \left(\frac{\partial F_1}{\partial q}\,dq + \frac{\partial F_1}{\partial Q}\,dQ\right)$$

$$\therefore \quad \oint \left[p(q,Q)\,dq - P(q,Q)\,dQ\right] = \oint \left[\frac{\partial F_1}{\partial q}\,dq + \frac{\partial F_1}{\partial Q}\,dQ\right]$$

$$\Rightarrow \begin{cases} p(q,Q) \equiv \partial F_1/\partial q \\ P(q,Q) \equiv -\partial F_1/\partial Q \end{cases}$$

there.
$\Rightarrow$ specify dependent variables

To transform:

① $\rightarrow q, Q \Rightarrow$ guess form F_1
② $\rightarrow p(Q,q) = \partial F_1/\partial q$ relation $\Rightarrow$ solve for $Q(p,q)$
③ $\rightarrow P$ from $-\partial F_1/\partial Q$.

Case 2.) q, P independent
Q, p ?

Note: This is case of Hamilton-Jacobi equation
i.e. $q \Rightarrow \alpha$ s/t $\dot{\alpha}=0$; β cyclic.
$p \qquad \beta$

$$\begin{pmatrix} q \rightarrow P = \alpha \\ p \qquad Q = \beta \end{pmatrix}.$$

Now, conservation phase volume $\Rightarrow$

$$\oint_c p\,dq = \oint_{c'(c)} P\,dQ \Rightarrow \oint_c (p\,dq - P\,dQ) = 0$$

but, q, P independent here (not q, Q)
$\Rightarrow$ transform to q, P via Legendre transform

i.e. $\oint d(PQ) = 0$

$\Rightarrow \oint P\,dQ + \oint Q\,dP = 0$

$\therefore \oint P\,dQ = -\oint Q\,dP$

$$\Rightarrow \oint (p\,dq + Q\,dP) = 0 = \oint dF_2(q, P)$$
$$= \oint \left(\frac{\partial F_2}{\partial q} dq + \frac{\partial F_2}{\partial P} dP \right)$$

$$\therefore \left\{ \begin{array}{l} p = \partial F_2 / \partial q \\ Q = \partial F_2 / \partial P \end{array} \right\}$$

3rd case) p, Q indep.

$\oint d(qp) = 0 \Rightarrow \oint p\,dq = -\oint q\,dp$

$$\Rightarrow \oint (+q\,dp + P\,dQ) = \oint \frac{\partial F_3}{\partial p} dp + \frac{\partial F_3}{\partial Q} dQ = 0$$

$+q = \partial F_3 / \partial p$ (overall sign irrelevant)

$+P = \partial F_3 / \partial Q$

4th case) p, P indep $\Rightarrow$ need 2 Legendre transforms!

$\oint p\,dq = -\oint q\,dp$

$\oint P\,dQ = -\oint Q\,dP$

$$\therefore \oint [q\,dp - Q\,dP] = \oint \left[\frac{\partial F_4}{\partial p} dp + \frac{\partial F_4}{\partial P} dP \right] = 0.$$

$$\begin{cases} \partial F_4/\partial p = q \\ \partial F_4/\partial P = -Q. \end{cases}$$

Examples of Canonical Transformations and C-T. Problems:

1) $F = qP$

Now, $F = F(q,P) \Rightarrow$ type 2
for $p, Q \Rightarrow p = \partial F/\partial q$
$Q = +\partial F/\partial P$

$$\Rightarrow \left.\begin{matrix} p = P \\ Q = q \end{matrix}\right\} \rightarrow \text{(identity)}.$$

2) $F = qQ$

$F = F(q, Q) \Rightarrow$ type 1

$$\left.\begin{matrix} p = \partial F/\partial q \\ P = -\partial F/\partial Q \end{matrix}\right\} \Rightarrow \underbrace{\begin{cases} p = Q \\ P = -q \end{cases}}_{\text{interchange of momenta, position}}$$

$\Rightarrow$ illustrates equivalence of momenta, position in Ham. dyn. $\Leftrightarrow$ distinction is semantical; only.

3) Now, try: $F = f(q)\,P = F(q, P)$

Type 2

$$\begin{cases} p = \dfrac{\partial F}{\partial q} = P\dfrac{\partial f}{\partial q} \\ Q = f(q) = \partial F/\partial P \end{cases}$$

Trivially: $\oint p\,dq = \oint P\,dQ$ but $Q = f(q)$ $\Rightarrow \oint p\,dq = \oint P\frac{\partial f}{\partial q}dq$

$P = p/\partial f/\partial q$

→ "Point transformation": (useful for coord change) $\Rightarrow \begin{cases} Q = f(q) \\ P = p/\partial f/\partial q \end{cases}$

4) Orthogonal transformation ↔ special case of point transformation

i.e. seek $Q_i = \sum_k a_{ik}\,q_k$ (rotation)

$$F = \sum_i P_i \sum_k a_{ik} q_k = \sum_{i,k} P_i\, q_k\, a_{ik}$$

(int. $i \leftrightarrow k$)

$$\therefore \begin{cases} p_i = \dfrac{\partial F}{\partial q_i} = \sum_k P_k\, a_{ki} \Rightarrow \text{invert for } P_k \\ Q_i = \sum_k q_k\, a_{ik} \end{cases}$$

observe: expect rotation ! ? $q \to Q$ rotation $\Rightarrow p \to P$

Check: $p_k = \sum_i P_i a_{ik}$

(mult)

$\Rightarrow \quad a_{jk} p_k = \sum_i P_i a_{jk} a_{ik}$

$$\sum_k a_{jk} p_k = \sum_{ik} P_i a_{jk} a_{ik}$$

$$= \sum_i \delta_{ij} P_i$$

(as a_{ik} is rotation matrix)

$\therefore \quad P_i = \sum_k p_k a_{ik}$ ✓

(rotation in p, also)

5) Harmonic Oscillator

Take: $F = \frac{m}{2} \omega q^2 \cot Q \quad = F(q, Q)$

Type 1, so $\quad p = \frac{\partial F}{\partial q}$

$$P = -\partial F / \partial Q$$

$$p = m\omega q \cot Q$$
$$P = m\omega q^2 / 2 \sin^2 Q$$

{difficult to invert $\Rightarrow$ $q, p = q, p(P, Q)$}

$$\Rightarrow \quad \text{so} \quad \begin{cases} q = \sqrt{2P/m\omega} \, \sin Q \\ p = \sqrt{2m\omega P} \, \cos Q \end{cases}$$

inverts a/c old (new)

Observe: $H = H(q,p) = H\left(q(\mathcal{P},Q), p(\mathcal{P},Q)\right)$

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

$$= \frac{1}{\cancel{2m}} \cancel{2m}\,\omega\, \mathcal{P}\cos^2 Q + \frac{1}{\cancel{2}}\cancel{m}\omega^{\cancel{2}}\frac{\cancel{2}\mathcal{P}}{\cancel{m\omega}}\sin^2 Q$$

$$= \omega \mathcal{P}$$

$\therefore \quad H = \omega \mathcal{P} \quad \Rightarrow \quad \mathcal{P} = E/\omega$

action is new momenta

$\Rightarrow$ What of Hamiltonian, upon C.T. ?

Consider 2 cases: (a) time independent
(b) time dependent

(a) for time independent, simply change variables, so:

$$H'(\mathcal{P}, Q) = H\left(p(\mathcal{P},Q), q(\mathcal{P},Q)\right)$$

need verify: $\left.\begin{aligned} \dot{Q} &= \partial H'/\partial \mathcal{P} \\ \dot{\mathcal{P}} &= -\partial H'/\partial Q \end{aligned}\right\}$ given $\dot{p} = -\partial H/\partial q$, $\dot{q} = \partial H/\partial p$

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p}$$

$$= \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q} \qquad \text{(HEMs)}$$

$$= \frac{\partial Q}{\partial q}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\right)$$

$$- \frac{\partial Q}{\partial p}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right)$$

(old → new)

Re-grouping ⇒

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{\partial H}{\partial P}\left(\underbrace{\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}}_{1 \to \text{Jacobian}}\right)$$

$$\cancel{+ \frac{\partial H}{\partial Q}\left(\frac{\partial Q}{\partial q}\underbrace{\frac{\partial Q}{\partial p}}_{\to 0} - \frac{\partial Q}{\partial p}\underbrace{\frac{\partial Q}{\partial q}}_{\to 0}\right)} \to \text{Cancel.}$$

$$= \partial H/\partial P \quad \checkmark$$

Time-dependent problems ?

Observe : - Trajectory in both old and new variables must satisfy Hamilton's Principle

$$- \quad \delta \int (p\,dq - H\,dt) = 0$$

$$\delta \int (P\,dQ - H'\,dt) = 0$$

so $\int [p\,dq - H\,dt] = \int [P\,dQ - H'dt + \underline{\underline{d}}F]$

but Liouville thm $\Rightarrow$

$$\int p\,dq = \int P\,dQ$$

$$\therefore \boxed{H' = H + \partial F/\partial t}$$

where choose F to be generating function.

Summary of C.T.-ology Hand Out

→ Purpose: identify structural constraints on variable transformations for Hamiltonian systems to preserve Hamiltonian structure.

i.e. $\{q, p\} = -1 \Rightarrow \{Q, P\} = -1$

$\{q,p\}$: old; $\{Q,P\}$: new

$\Leftrightarrow \dot{q} = \partial H/\partial p; \; \dot{p} = -\dfrac{\partial H}{\partial q} \Rightarrow \dot{Q} = \dfrac{\partial H'}{\partial P}; \; \dot{P} = -\dfrac{\partial H'}{\partial Q}$

etc.

→ Motivation: ① avoiding mutilating Hamiltonian formulation upon transformation
② Hamilton-Jacobi theory as "optimal transform" $\Leftrightarrow$ Q cyclic.

→ Leverage: Liouville Thm. (Phase Volume Conserved)
$\int dp\, dq = \int dP\, dQ \Rightarrow \oint_c p\, dq = \oint_{c'} P\, dQ$

∴

→ Formulation: C.-T. problems formulated as given 2 of q, p, Q, P and generating function of given 2, obtain other 2.

so have cases:

1) q, Q given; p, P ?

$$\oint_c (p\,dq - P\,dQ) = \int \left(\frac{\partial F_1}{\partial q} dq + \frac{\partial F_1}{\partial Q} dQ\right)$$

$\Rightarrow p = \partial F_1/\partial q$, $P = -\partial F_1/\partial Q$

2) q, P given; p, Q ? (H-J case)

$$\oint (p\,dq - P\,dQ) = \int (\partial F_2/\partial q\, dq + \partial F_2/\partial P\, dP)$$

need $\oint P\,dQ = -\oint Q\,dP$ from $\oint d(PQ) = 0$

$\Rightarrow p = \partial F_2/\partial q$; $Q = \partial F_2/\partial P$

3) p, Q given; q, P ?

$$\oint (p\,dq - P\,dQ) = \int ((\partial F_3/\partial p)\,dp + (\partial F_3/\partial Q))\,dQ$$

need $\oint p\,dq = -\int q\,dp$, from $\oint d(pq) = 0$

$\therefore q = \partial F_3/\partial p$; $P = \partial F_3/\partial Q$

4) p, P given; q, Q ?

need 2 Legendre transform $\Rightarrow \begin{cases} \oint d(pq) = 0 \Rightarrow \oint p\,dq = -\oint q\,dp \\ \oint d(PQ) = 0 \Rightarrow \oint P\,dQ = -\oint Q\,dP \end{cases}$

$$\oint (q\,dp - Q\,dP) = \oint \left(\frac{\partial F_4}{\partial p} dp + \frac{\partial F_4}{\partial P} dP\right)$$

$\therefore q = \partial F_4/\partial p$

$Q = -\partial F_4/\partial P$

- Transformation of Hamiltonian:

(a) $\partial H/\partial t = 0 \Rightarrow H(p, q) = H'(P, Q)$

(b) $\partial H/\partial t \neq 0 \Rightarrow H' = H + \partial F/\partial t$

→ Interesting and Useful C.-T.'s:

a) Identity: $F = Pq$ (Case 2)

$p = P, \; Q = q$

b) Interchange: $F = qQ$ (Case 1)

$p = Q; \; P = -q$

⇒ graphically demonstrates interchangeability of "p, q" labels in Ham. mechanics

c) Point Transform: $F = f(q)P$ (Case 2)

(Case 2)

$p = P\, \partial f/\partial q \Rightarrow (P = p/\partial f/\partial q)$

$Q = f(q)$

⇒ basic coordinate transformation

d) Orthogonal Transformations: (Case 2)

$F = \sum_{ik} P_i a_{ik} q_k$ (special case of point transform)

$p_i = \sum_k a_{ki} P_k \, ; \; Q = \sum_k q_k a_{ik}$

Recall:

→ thrust of discussion is integrability

→ H.-J. equation is generating function for C.-T. :

$$p, q \to \alpha, \beta \qquad \begin{cases} \dot{\alpha} = 0 \\ \beta \text{ cyclic} \end{cases}$$

(motivated brackets, C-T.'s)

∴

→ Return to H.-J. Theory

Recall H.-J. equation (time independent)

$$H(q, p) = H\left(q, \frac{\partial S}{\partial q}\right) = H'(\alpha) = E \qquad (1\text{A})$$

$$\therefore \quad p = \frac{\partial S}{\partial q}, \quad \text{but} \begin{cases} \text{Type 2; } q, P \text{ given} \\ p = \partial F_2/\partial q \\ Q = \partial F_2/\partial P \end{cases} \quad P \to \text{const.}$$

$$\therefore \quad p = \partial F_2/\partial q$$

Thus: $p = \partial F_2/\partial q = \partial S/\partial q$

⇒ $\boxed{S(q) \text{ (abbreviated action) is Generating Function of C.-T. to cyclic coordinates, const. momenta}}$

example:

→ 1D motion:

$H(q, \partial S/\partial q) = E = H_1'(\alpha)$

$q, p \to \alpha, \beta$ transform.

$S = S(q, \alpha)$ is generating function F_2 for C. T. to cyclic variables

⇒ immediately write:

$$\begin{cases} Q = \partial S/\partial P = \partial S/\partial \alpha \\ p = \partial S/\partial q \end{cases} \Rightarrow \begin{cases} \beta = \partial S/\partial \alpha \\ p = \partial S/\partial q \end{cases}$$

and

$$\left.\begin{cases} \dot{\alpha} = -\partial H_1/\partial \beta = 0 \\ \dot{\beta} = \partial H_1/\partial \alpha = H_1'(\alpha) \end{cases}\right\} \underline{\text{general}}$$

$$\Rightarrow \quad \alpha = \text{const} \; ; \quad \beta = H_1'(\alpha)(t - t_0) + c.$$

i.e. consider $H(q, p) = \frac{p^2}{2m} + V(q)$

$\Rightarrow$ and $\underset{(H_1' = 1)}{H = E \equiv \alpha}$ ($E = \alpha$ is optimal, logical choice)

$$p(q, \alpha) = \sqrt{2m}\,(\alpha - V(q))^{1/2}$$

so

$$S = \int p\, dq$$

$$= \int \sqrt{2m}\,(\alpha - V(q))^{1/2}\, dq$$

$$\gamma = \partial S / \partial \alpha = \int \sqrt{\tfrac{m}{2}} \, (\alpha - V(q))^{-1/2} \, dq$$

$\beta = t - t_0 \Rightarrow$
$(\alpha = E)$

$$\Rightarrow \boxed{t - t_0 = \sqrt{\tfrac{m}{2}} \int dq \Big/ (\alpha - V(q))^{1/2}}$$

→ Now observe structure of solution:

i.e. α, β
$$\begin{cases} \dot{\alpha} = 0, & \alpha = \alpha_0 \\ \dot{\beta} = \partial H_1 / \partial \alpha \big|_{\alpha_0} & \beta = (\partial H_1 / \partial \alpha)(t + t_0) \end{cases}$$

$\Leftrightarrow$ for closed orbits, suggest view:
$\alpha \Leftrightarrow$ generalized radius in phase space
$\beta \Leftrightarrow$ generalized angle in phase space

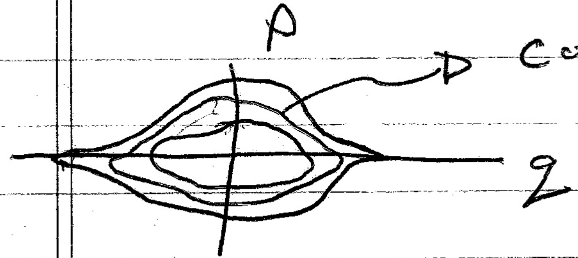
$\Rightarrow$ special canonical variables: Action-Angle Variables

why?

$\Leftrightarrow$ simplify description of motion.

i.e.
$$\begin{cases} H = p^2/2m + V(q) \\ \text{closed orbit} \end{cases}$$

i.e.



→ contours constant energy
orbits stay on contour

∴ to each contour $E = \text{const}$, assign Action variable. → enclosed area → closed contour.

$$I = \oint p\,dq = \int dp\,dq$$

↓ action for 1 circuit — ↓ phase area enclosed by one circuit

as particle stays on contour ⇒

$\dot{I} = 0$, so

$$p, q \to \alpha, \beta \to I, \Theta$$

↓ angle variable
(goes with Action Variable)

∴ natural to seek $\left\{ \begin{matrix} I \\ \Theta \end{matrix} \right\}$ such that Θ increases by 2π in 1 circuit.

Constructing Action–Angle Variables:

Now, $S = F_2$, generating function

$$S = S(q, I)$$

↓ P

$\therefore$ can immediately write:

$p = \partial S/\partial q$

$\theta = \partial S/\partial I$

so $$\frac{d\theta}{dq} = \frac{\partial}{\partial q}\frac{\partial S}{\partial I} = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}$$

$\therefore$ $$d\theta = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}\,dq$$

$\Rightarrow$ $$2\pi = \frac{\partial}{\partial I}\oint \frac{\partial S}{\partial q}\,dq = \frac{\partial}{\partial I}\oint p\,dq$$

$\therefore$ $$\boxed{I \equiv \frac{1}{2\pi}\oint_{\text{associated (path!)}} p\,dq \quad \text{is Action Variable}}$$

$H = H(I, \theta)$
$$\begin{cases} \dot{I} = -\dfrac{\partial H}{\partial \theta} = 0 \\[2ex] \dot{\theta} = \dfrac{\partial H}{\partial I} = \omega(I) \end{cases}$$

$\omega'(I) = 0$ linear, no shear

$\omega'(I) \neq 0$, NL

$\Rightarrow$ $I = I_0$

$\theta = \omega(I_0)\,t + \theta_0$

specifies angle and winding rate $\omega = \partial H/\partial I$

Examples of Action - Angle Variables

1) 1D Harmonic Oscillator

$$H = (p^2 + \omega^2 q^2)$$

$\Rightarrow$

$$\left(\frac{\partial S}{\partial q}\right)^2 + \omega^2 q^2 = E \quad \text{is H.-J. eqn.}$$

$$\therefore \quad I = \frac{1}{2\pi} \oint \left[(E - \omega^2 q^2) \right]^{1/2} dq$$

$$\oint = 2 \int_{q_-}^{q_+}$$

$E = \omega^2 q^2$

$\Rightarrow \begin{cases} q_\pm = \pm \sqrt{E}/\omega \\ \text{are turning pts} \end{cases}$

$$\Rightarrow \quad I = \frac{1}{\pi} \int_{q_-}^{q_+} \left[(E - \omega^2 q^2) \right]^{1/2} dq$$

$$= \frac{\omega}{\pi} \int_{q_-}^{q_+} \left(\frac{E}{\omega^2} - q^2 \right)^{1/2} dq$$

$$q = \frac{\sqrt{E}}{\omega} \sin\theta$$

$$dq = \frac{\sqrt{E}}{\omega} \cos\theta$$

$$\Rightarrow \quad I = \frac{2}{\pi} \cancel{\omega} \frac{E}{\omega \cancel{^2}} \int_{-\pi/2}^{\pi/2} d\theta \cos^2\theta = E/\omega$$

$\therefore$ $I = E/\omega$ $\Leftrightarrow$ Action Variable ($\alpha = I$, "new momentum")

$H = I\omega \Rightarrow \dot{\theta} = \frac{\partial H}{\partial I} = \omega$

$\theta = \omega t + \theta_0 \Leftrightarrow$ Angle Variable

$\Rightarrow S' = S(q, I) = \int_{q_0}^{q} (I\omega - \omega^2 q^2)^{1/2} dq$

2) $H = p^2 + V(q)$ general 1D

$I = \frac{1}{2\pi}\oint p\,dq$

$\left([I^{-1}(E)] = E(I) \right)$

$= \frac{1}{2\pi}\oint [E - V(q)]^{1/2} dq = I(E)$; $\omega = \frac{\partial}{\partial I} E(I)$

i.e.

$V = \beta q^4 \Rightarrow I = \underset{\text{coeff. \#}}{c}\, E^{3/4}$

$\therefore H(I) = c' I^{4/3}$

and $\omega(I) = H'(I) \sim c' I^{1/3} \Rightarrow$ nonlinear.

aside: Can observe correspondance

Classical

$I = E/\omega$

$H = I\omega$

Quantum

$E = (N + 1/2)\hbar\omega$

$\Downarrow$

quanta $\Leftrightarrow$ occupation
(quantum #)

$\therefore$ { suggests view I as classical # of excitations/waves $\Leftrightarrow$ exciton density }

$\Rightarrow$ straightfoward to generalize: (linear wave)

wave energy density

$I = E/\omega \quad \Leftrightarrow \quad N(\underline{k}, \underline{x}, \omega) = E(\underline{k}, \underline{x}, \omega)/\omega_k$

I: linear H.O.

N: Action Density as Wave Density, # waves

ω_k: wave frequency

3) 2D Harmonic Oscillator

$$H = p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2$$

- obviously separable, i.e.

i.e. $H = f(1) + f(2) = E$

where:

$$f(1) = p_1^2 + \omega_1^2 q_1^2 = E_1$$

$$f(2) = p_2^2 + \omega_2^2 q_2^2 = E_2$$

for action variables I_1, I_2:

$$I_1 = \frac{1}{2\pi}\oint p_1 \, dq = \frac{1}{2\pi}\oint p(q_1)\, dq_1 = \frac{E_1}{\omega_1}$$

$$I_2 = \frac{1}{2\pi}\oint p(q_2)\, dq_2 = \frac{E_2}{\omega_2}$$

{additive form of H in A-A $\Leftrightarrow$ separability}

$$\Rightarrow H(I_1, I_2) = I_1\omega_1 + I_2\omega_2 = E_1 + E_2$$

4) Free Particle in 2D $\begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$ (hard wall)

$$H = \frac{1}{2m}(p_x^2 + p_y^2)$$

Now,

- 2 degs. freedom $\Rightarrow$ 2 I's, 2 θ's

- $\therefore$ $I_1 = \frac{1}{2\pi}\oint p_x\, dx$

$$I_2 = \frac{1}{2\pi}\oint p_y\, dy$$

Now $\oint p_x\, dx = \int_0^a p_{x_+} dx + \int_a^0 p_{x_-} dx$

but $p_{x_-} = -p_{x_+}$ (bounces off wall!)

$\therefore$

$$\oint p_x\, dx = 2a|p_x| \qquad |p_x| = p_{x_+}$$

$\Rightarrow$ $I_1 = \frac{a}{\pi}|p_x|$

$$I_2 = \frac{b}{\pi}|p_y|$$

$\therefore$ $H = \frac{1}{2m}p^2 = \frac{1}{2m}\left(I_1^2\frac{\pi^2}{a^2} + I_2^2\frac{\pi^2}{b^2}\right)$

$$= \frac{\pi^2}{2m}\left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2}\right)$$

Observe!

a) classically, $H = E = \frac{\pi^2}{2m}\left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b}\right)$

if $\left.\begin{array}{l} I_1 \to n\hbar \\ I_2 \to m\hbar \end{array}\right\}$ quantize action variables

$E_{n,m} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$ → eigenstates of free QM particle in box

b) $H(I_1, I_2) = \frac{\pi^2}{2m}\left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2}\right)$

$\omega(I_{1,2}) = \frac{\partial H}{\partial I_1} = \frac{\pi^2}{ma^2} I_{1,2}$

i.e. $\frac{\partial \omega(I_{1,2})}{\partial I_{1,2}} \neq 0$ (unlike harmonic oscillator)

⇒ nonlinear problem!

Yet, particle is free ?? → Origin of NL?

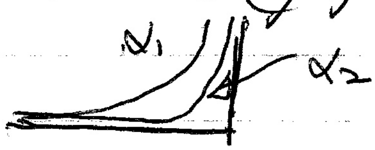
Recall: $V(q) = \frac{1}{2}kq^2 \Rightarrow I = E/\omega$

$V(q) = \beta q^4 \Rightarrow I = c\,E^{3/4}/\beta^{1/4}$

i.e. ① Nonlinearity develops from $V \propto q^\alpha$ potential for $\alpha > 2$

② view hard wall at a as limiting case

i.e. $V = V_0 (x/a)^\alpha$, $\alpha \gg 1$

$\Rightarrow$ 

$\therefore$ hard wall boundary condition appears as nonlinearity due high powers implicit in piecewise potential.

Kepler Problem (General)

i.e. $$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

$\Rightarrow$ 2 degs. freedom, 2 COM's $\left\{ \begin{matrix} p_\phi \\ E \end{matrix} \right.$

$\therefore$ $$I_1 = I_\phi = \frac{1}{2\pi}\oint p_\phi \, d\phi$$
$$= p_\phi$$

$$I_2 = \frac{1}{2\pi}\oint p_r \, dr \qquad r_{1,2} \leftrightarrow \text{t.p.}$$
$$= \frac{1}{2\pi}\, 2\int_{r_1}^{r_2} \left[2m\left(E - V(r) - I_1^2/2mr^2\right)\right]^{1/2}$$

etc.